CONVERGENCE OF SCHRÖDINGER OPERATOR WITH ELECTROMAGNETIC POTENTIAL

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ABSTRACT: *We consider the Schrödinger operator with electromagnetic potentials*
$H=\sum_{j=1}^{n}\frac{1}{2}\left(i\partial_j+b_j(x)\right)^2+V(x)$ *in* $L^2(R^n)$ *where* $b_j(x), j=1,2,\dots n$ *and* $V(x)$ *are real-valued functions on* R^n, $V\in L^1_{loc}(R^n)$, $b\in C^2(R^n)$, $\partial_j=\frac{\partial}{\partial x_j}$, *and* $i=\sqrt{-1}$. *We investigate the convergence of the function* $\Psi(t,x)$ *in* $L^2(R^n)$ *which is defined by*

$$\Psi(t,x)=\int d\mu_x^t(\omega)\,\{exp[-i\int_0^t b(\omega(s))\,d\omega-\frac{1}{2}i\int_0^t div\, b(\omega(s))\,ds-\int_0^t V(\omega(s))\,ds]\}\varphi(\omega(t)) \quad ...(1)$$

,and we research its analytic in the space $L^2(R^n)$

Keywords: Schrödinger operator, electric potential, magnetic potential, Feynman-Kac- Itô formula**.**

1. INTRODUCTION

The study of self-adjoint differential operators on Hilbert spaces is a central problem in the theory of partial differential operators.

Kato [4] showed on the basis of his elegant inequality that, if $V(x)\geq 0$ and $V\in L^2_{loc}$, then the Schrödinger operator is essentially self-adjoint on the set of infinitely differentiable finite functions.

Gaysinsky, Goldstein[3] proved theorems of self-adjointness of the operator $H=-\Delta+V$ and its powers H^p.

Aliev and Eyvazov [1]they showed that the Schrödinger operator under certain conditions Stummel type, imposed on the magnetic and electric potentials is an essential self-adjoint operator. K. U.Noor, H. S.Yahea in 2015[10]: they proved that the function $\Psi(t,x)$ in $L^2(R^n)$ which is defined by

$$\Psi(t,x)=\int d\mu_x^t(\omega)\{\exp[-\int_0^t V(\omega(s))\,ds]\}\varphi(\omega(t))$$

is converges for almost every V where V is any real-valued function and discuss analytic of this function.

The first steps to proving that an operator H essential self-adjoint by using Feynman -Kac- Itô formula which the form (1) converges, analytic and smoothness. Many papers interested in the self-adjoint operators, we refer to[6-9].

In our work, we consider Schrödinger operator with electromagnetic potential

$H=\sum_{j=1}^{n}\frac{1}{2}(i\partial_j+b_j(x))^2+V(x)$,

$V:R^n\to R$ is the electric (scalar) potential, $b:R^n\to R^n$ is the magnetic (vector) potential where $x\in R^n$, $\partial_j=\frac{\partial}{\partial x_j}$, $i=\sqrt{-1}$, and the domain of H denoted by $D(H)$ which is a dense subset in $L^2(R^n)$. We will show the necessary conditions which make the equation (1) function is convergent and discuss its analytic in the space $L^2(R^n)$, and we used some important theories such as Faynman-Kac-Itô formula [2] also used the Wiener integral and Itô stochastic integral to simplify some of the integrals i.e

$$exp(-tH)\,(x,y)=\int d\mu_{x,y}^t(\omega)\left\{\exp\left[-i\int_0^t b(\omega(s))\,dx-\frac{1}{2}i\int_0^t \mathrm{div}b(\omega(s))\,ds-\int_0^t V(\omega(s))\,ds\right]\right\},$$

$$exp(-tH)\,\varphi(x)=\int d\mu_x^t(\omega)\left\{\exp\left[-i\int_0^t b(\omega(s))\,d\omega-\frac{1}{2}i\int_0^t \mathrm{div}b(\omega(s))\,ds-\int_0^t V(\omega(s))\,ds\right]\right\}\varphi(\omega(t)),$$

for every $\varphi\in D(H)$, where $b(\omega)$ is twice continuously differentiable vector field on R^n, and $\mathrm{div}\, b=\sum_{j=1}^n\partial_j b_j$. $\int_0^t b(\omega(s))\,d\omega$ denote the Itô stochastic integral of $b(\omega)$ with respect to n-dimensional Wiener process $\omega(s), 0<s<\infty$.

2. Statement of the problem and the main result: Convergence of the basic integral (the function $\Psi(t,x)$)

We will consider a random function $V(x)$ of the following form: let $\prod_{d=1}^n(a_{1d},a_{0d})\,,\prod_{d=1}^n(a_{2d},a_{1d})\,,\dots$ be a system of intervals, $\lim_{m\to\infty}a_{md}=0$, $1\leq d\leq n$; suppose that every interval $\prod_{d=1}^n\left(a_{md},a_{(m-1)d}\right)$ is divided into $N(m)$ equal intervals $\prod_{d=1}^n(a_{j,md}\,,a_{j-1,md})$; let $v_{j,m}(x)$ be an infinitely differentiable function which is equal to zero outside the interval $\prod_{d=1}^n\left(-\ell_{md}\,N(m)^{-1},\ell_{md}\,N(m)^{-1}\right)$ where $\ell_{md}=a_{(m-1)d}-a_{md}$, and let $V(x)$ be a random function which is equal to

$$v_{j-1,m}\left(x-(a_{j-1,m1},a_{j-1,m2},\dots.a_{j-1,mn})\right)\xi_{j-1,m}+v_{j,m}\left(x-(a_{j,m1},a_{j,m2},\dots.a_{j,mn})\right)\xi_{j,m} \quad (2)$$

on the interval $\prod_{d=1}^n\left(a_{j,md},a_{j-1,md}\right)$, where $\xi_{j,m}$ is the system of independent random variables. We will show in this section that, under definite restriction of the values $\xi_{j,m}$ and under the condition of sufficiently quick tending of $N(m)$ to ∞.

We will suppose that the values $\xi_{j,m}$have Gauss distributions with the densities

$\rho_{j,md}(x)=\frac{e^{\frac{-x^2}{2\beta_{j,md}}}}{\sqrt{2\pi\beta_{j,md}}}$, where $\beta_{j,md}$ are constant, $\beta_{j,md}\leq\beta$.

We will suppose the function $v_{j,m}$ to be smooth, $M_{j,m}$=max$|v_{j,m}|$, $M(m)=\max_j M_{j,m}$.

The basic assumption on the value $N(m)$:

$N(m)\geq exp(\delta m^2M(m)^2)$, where $\delta>0$ is constant.

Also consider a random function $\int_0^t b(\omega(s))\,dx$ denotes the Itô stochastic integral of $b(\omega)$ with respect to n-dimensional Wiener process $\omega(s), 0<s<\infty$ of the following form: Let $P=\{0=t_0<t_1<\cdots<t_l=T\}$ be a partition such that

$b(t)\equiv b_k$ for $t_k\leq t<t_{k+1}(k=0,\dots,l-1,$ is a step process then *Itô* stochastic integral of $b(\omega)$ with respect to n-dimensional Wiener process $\omega(s)$, $0<s<\infty$ on the interval (0, *T*)

$$\int_0^T b(\omega)d\omega:=\sum_{k=0}^{l-1}b_k(\,\omega(t_{k+1})-\omega(t_k))$$

Proposition 2.1:

The function

$\Psi(t,x)=\int d\mu_x^t(\omega)\left\{\exp\left[-i\int_0^t b(\omega(s))\,d\omega-\frac{1}{2}i\int_0^t \operatorname{div} b(\omega(s))\,ds-\int_0^t V(\omega(s))ds\right]\right\}\varphi(\omega)$,

converges for almost every V, b and represents a bounded intergrable function of a variable x.

In addition:-

$\left|\Psi(t,x)\right|\le\tilde{\beta}(t,V).\exp\left(-\frac{(x-\alpha)^2}{2t}\right),\ E\tilde{\beta}(t,V)<+\infty$

where $supp\ \varphi\subset[-\alpha,\alpha]^n$, where $\tilde{\beta}<+\infty$ is constant.

Proof:

$E_x[\int d\mu_x^t(\omega)\left\{\exp\left[-i\int_0^t b(\omega(s))\,d\omega-\frac{1}{2}i\int_0^t \operatorname{div} b(\omega(s))\,ds-\int_0^t V(\omega(s))ds\right]\right\}]\varphi(\omega)=$
$\int d\mu_x^t(\omega).E_x\left\{\exp\left[-i\int_0^t b(\omega(s))\,d\omega-\frac{1}{2}i\int_0^t \operatorname{div} b(\omega(s))\,ds-\int_0^t V(\omega(s))ds\right]\right\}]\varphi(\omega)$. (2.1.1)

Then from [5] we have

$W=V+\frac{1}{2}b.b+\frac{1}{2}i(\operatorname{div} b)$.

Thus we have

$-\frac{1}{2}i(\operatorname{div} b)-V=-W+\frac{1}{2}b.b$ (2.1.2)

Substitute the equation (2.1.2) in (2.1.1) we get

$\int d\mu_x^t(\omega).E_x\left\{\exp\left[-i\int_0^t b(\omega(s))\,d\omega+\frac{1}{2}\int_0^t b^2(\omega(s))\,ds-\int_0^t W(\omega(s))ds\right]\right\}\varphi(\omega)$
$=\int d\mu_x^t(\omega).E_x\left\{\exp\left(-i\int_0^t b(\omega(s))\right)d\omega\right\}$
$\times E_x\left\{\exp\left(\frac{1}{2}\int_0^t b^2(\omega(s))\,ds\right)\right\}$
$\times E_x\left\{\exp\left(-\int_0^t W(\omega(s))ds\right)\right\}\varphi(\omega)$ (2.1.3)
$=\int d\mu_{x,y}^t(\omega)\{I_1\times I_2\times I_3\}\varphi(\omega)$

Now

$I_1=E_x\left\{\exp\left(-i\int_0^t b(\omega(s))\,d\omega\right)\right\}=$
$E_x\left\{\exp\left(-i\sum_{k=0}^{l-1}b_k\times(\omega(t_{k+1})-\omega(t_k))\right)\right\}$,
$=\prod_{k=0}^{l-1}E_x\left\{\exp\left(-ib_k\times(\omega(t_{k+1})-\omega(t_k))\right)\right\}$,
$(\omega(t_{k+1})-\omega(t_k))\sim N(0,t_{k+1}-t_k)$

Write $E_x\left\{\exp\left(-ib_k\times(\omega(t_{k+1})-\omega(t_k))\right)\right\}$

$=\int_{-\infty}^{\infty}e^{\left(-ib_k\times(\omega(t_{k+1})-\omega(t_k))\right)}\frac{e^{\left(\frac{-\omega^2}{2(t_{k+1}-t_k)}\right)}}{\sqrt{2\pi(t_{k+1}-t_k)}}\,d\omega=$
$\exp\left(\frac{-(t_{k+1}-t_k)}{2}\times {b_k}^2\right)$,

$I_1=\exp\left(\sum_{k=0}^{l-1}\frac{-(t_{k+1}-t_k)}{2}\times {b_k}^2\right)$.

So

$I_2=E_x\left\{\exp\left(\frac{1}{2}\int_0^t b^2(\omega(s))\,ds\right)\right\}$
$I_2=E_x\left\{\exp\left(\sum_{k=0}^{l-1}b_k^2\times\frac{(t_{k+1}-t_k)}{2}\right)\right\}$.

And

$\int_0^t W(\omega(s))\,ds=\sum_m\sum_{j=1}^{N(m)}\int_{I(\omega,m,j)}W(\omega(s))ds$, (2.1.4)

where $I(\omega,m,j)=\{s\in[0,t]:\omega(s)\in\prod_{d=1}^n[a_{j,md},a_{j+1,md})\}$.

If $s\in I(\omega,m,j)$, then

$W(\omega(s))=\xi_{j,m}\ w_{j,m}\ (\omega(s)-(a_{j,m1},a_{j,m2},\dots,,a_{j,mn}))+\xi_{j+1,m}\ w_{j+1,m}\ (\omega(s)-(a_{j+1,m1},a_{j+1,m2},\dots,,a_{j+1,mn}))$,

hence

$\int_{I(\omega,m,j)}W(\omega(s))\,ds=\mu_{j,m}^I(\omega)\,\xi_{j,m}+\mu_{j+1,m}^{II}(\omega)\,\xi_{j+1,m}$, (2.1.5)

Where

$\mu_{j,m}^I=\int_{I(\omega,m,j)}w_{j,m}\ (\omega(s)-(a_{j,m1},a_{j,m2},\dots,,a_{j,mn}))\,ds$,

$\mu_{j+1,m}^{II}=\int_{I(\omega,m,j)}w_{j+1,m}(\omega(s)-(a_{j+1,m1},a_{j+1,m2},\dots,,a_{j+1,mn}))\,ds$,

$$\int_0^t W(\omega(s))\,ds=\sum_m\sum_j\mu_{j,m}\,\xi_{j,m},$$

$$\mu_{j,m}=\mu_{j,m}^I+\mu_{j+1,m}^{II}.$$

We now calculate the mean value of the expression (I_3) over potentials W

$I_3=E_x\left(exp(-\int_0^t W(\omega(s))\,ds)\right)$
$=\prod_m\prod_{j=1}^{N(m)}E_x exp\left(-\mu_{j,m}\,\xi_{j,m}\right)$.

Write $E_x(e^{-\mu\xi})=\int_{-\infty}^{\infty}e^{-\mu x}\,e^{\frac{-x^2}{2\beta}}\Big/\sqrt{2\pi\beta}\ dx=exp(\frac{\beta}{2}\mu^2)$,

if ξ has the density of distribution $e^{\frac{-x^2}{2\beta}}\Big/\sqrt{2\pi\beta}$.

Hence

$I_3=E_x\left(exp(-\int_0^t W(\omega(s))\,ds)\right)$
$\exp\left(\beta\sum_m\sum_{j=1}^{N(m)}\mu_{j,m}^2(\omega)\right)$

we estimate the value $\mu_{j,m}^I(\omega)$ as follows:

$\mu_{j,m}^I=\int_{I(\omega,m,j)}w_{j,m}(\omega(s)-(a_{j,m1},a_{j,m2},\dots,,a_{j,mn})))\,ds\le M(m)\tau_{j,m}(\omega)$,

where

$$M(m)=\max_{x,j}|w_{j,m}(x)|,$$

$$\tau_{j,m}(\omega)=\int_{I(\omega,m,j)}ds=\lambda I(\omega,m,j),$$

λ is the Lebesgue measure. Similarly, the estimate of the values $\mu_{j+1,m}^{II}(\omega)$. Further,

$$\sum_m\sum_j\tau_{j,m}(\omega)=t.$$

Now we can write:-

$E_x\left(exp(-\int_0^t W(\omega(s))\,ds\right)\le$
$exp\left(\beta_1\sum_m M(m)^2\sum_j\tau_{j,m}^2(\omega)\right)$, ($\beta_1=const$); (2.1.6)

$\int\tau_{j,m}^2(\omega)\,d\mu_x(\omega)=\int[\int_0^t\chi_{j,m}(\omega(s_1))ds_1.\int_0^t\chi_{j,m}(\omega(s_2))ds_2]\,d\mu_x(\omega)$
$=2\int_0^t\int_{s_1}^t\int ds_2\,ds_1\,\chi_{j,m}(\omega(s_1))\,\chi_{j,m}(\omega(s_2))\,d\mu_x(\omega)$
$=2\int_0^t\int_{s_1}^t[\iint p(x,x_1,s_1)\,p(x_1,x_2,s_2-s_1)\,\chi_{j,m}(x_1).\chi_{j,m}(x_2)dx_1dx_2]ds_1ds_2$
$=\int_{\prod_{d=1}^n[a_{j,md},a_{j+1,md})}\int_{\prod_{d=1}^n[a_{j,md},a_{j+1,md})}dx_1dx_2$
$\times\left\{\int_0^t ds_1\int_{s_1}^t ds_2\frac{e^{-(x-x_1)^2/2s_1}}{\sqrt{2\pi s_1}}.\frac{e^{-(x_1-x_2)^2/2s_2}}{\sqrt{2\pi(s_2-s_1)}}\right\}$
$\le const\prod_{d=1}^n(a_{j+1,md}-a_{j,md})=\frac{c}{N(m)^n}$,

where $\chi_{j,m}(\ )$ is the indicator of the set $\prod_{d=1}^n[a_{j,md},a_{j+1,md})$, $c=const$, Let $\Omega_{j,m}=\{\omega:\tau_{j,m}^2(\omega)>\varepsilon\}$, where $\varepsilon=\varepsilon(m)$.Then, by the Čebyšev inequality,

$\mu\Omega_{j,m}\le\varepsilon^{-1}\int\tau_{j,m}^2(\omega)\,d\omega\le\varepsilon^{-1}N(m)^{-n}c$.

Put $\Omega_m=\bigcup_j\Omega_{j,m}$ and write

$\mu\Omega_m\le N(m)\max_j\mu(\Omega_{j,m})=c\varepsilon^{-1}N(m)^{-(n-1)}$.

To estimate the integral at the right side of (2.1.6), we first consider finite summation by m at the exponent. Let $m=1,2,\dots,g$. By the Hölder inequality,

$\int exp(\sum_{m=1}^{g} \theta_m(\omega))\, d\mu(\omega) \leq$

$\prod_{m=1}^{g}[\int exp(\alpha_m \theta_m(\omega))\, d\mu(\omega)]^{1/\alpha_m}$ (2.1.7)

where $\alpha_1^{-1}+\ldots+\alpha_g^{-1}=1$, $\alpha_j > 0$. Put $\theta_m(\omega)= M_m^2 \sum_j \tau_{j,m}^2(\omega)$, $\alpha_m=$ c m^2,

where $c = c(g)=\sum_{m=1}^{g} m^{-2}$. Write

$\int exp(\alpha_m\theta_m(\omega))\, d\mu(\omega)=\int_{\Omega_m} exp(\alpha_m\theta_m(\omega))\, d\mu(\omega) + \int_{c\Omega_m} exp(\alpha_m\theta_m(\omega))\, d\mu(\omega);$

$\int_{c\Omega_m} exp(\alpha_m\theta_m(\omega))\, d\mu(\omega) \leq$

$\int_{c\Omega_m} exp(\beta_1\, cm^2\, M(m)^2 \sum_{j=1}^{N(m)} \tau_{j,m}{}^2(\omega))\, d\mu(\omega)$, (2.1.8)

if $x \in c\Omega_m$, then $\tau_{j,m}^2 < \varepsilon$, therefore, $\tau_{j,m}(\omega) < \varepsilon^{\frac{1}{2}}$, $\tau_{j,m}^2(\omega) < \varepsilon^{\frac{1}{2}}\, \tau_{j,m}(\omega)$,

$$\sum \tau_{j,m}^2(\omega) < \varepsilon^{\frac{1}{2}} \sum \tau_{j,m}(\omega) \leq \varepsilon^{\frac{1}{2}}\, t. \quad (2.1.9)$$

So, if we put $\varepsilon(m)=M(m)^{-4}m^{-4}$, then we get from (2.1.7), (2.1.8) that

$$\int_{c\Omega_m} exp(\alpha_m\theta_m(\omega))\, d\mu(\omega)\leq exp\ (\beta_1 ct) \quad (2.1.10)$$

further, if $t < 1$, then $\sum_j \tau_{j,m}(\omega) \leq 1$, $\sum \tau_{j,m}^2(\omega) < \sum \tau_{j,m}(\omega) \leq t$,

$\int_{\Omega_m} exp(\alpha_m\theta_m(\omega))\, d\mu(\omega)\leq \mu(\Omega_m)$

$exp(\beta_1\, cm^2\, M(m)^2 t)$

$\leq c\, (\varepsilon(n))^{-1}(N(m))^{-(n-1)} exp(\beta_1\, C\, M(m)^2 t).$ (2.1.11)

According to our assumption, $N(m)\geq exp(\delta m^2 M(m)^2)$. Hence, it follows from the estimate (2.1.11) that for t small enough

$$\int_{\Omega_m} exp(\alpha_m\theta_m(\omega))\, d\mu(\omega) \leq C_1, \quad (2.1.12)$$

where C_1 is a constant. We now conclude from (2.1.9), (2.1.11) that

$[\int exp(\alpha_m\theta_m(x))\, d\mu(x)]^{1/\alpha_m} \leq exp\ (C_1\, m^{-2})$, (2.1.13)

where C_1 is constant. Substitute the estimate (2.1.13) into (2.1.7) and pass to the limit as g→ ∞. We get

$E_x\{\int d\mu_x^t\, (x)\{I_1.I_2.I_3\}\} < +\infty.$

In similar way we can show that, if $|y| \leq \alpha$, then

$E_x\{\int d\mu_x^t\, (x)\{I_1.I_2.I_3\}\} \leq \beta.$

$\exp\left(-\left(\sum_{k=0}^{l-1}\frac{(t_{k+1}-t_k)}{2}(b_k)^2\right)\right).\exp\left(\left(\sum_{k=0}^{l-1}\frac{(t_{k+1}-t_k)}{2}(b_k)^2\right)\right).$

$\exp\left(-\frac{(x-\alpha)^2}{2t}\right) = \beta.\exp\left(-\frac{(x-\alpha)^2}{2t}\right)$

, where $\beta < +\infty$ is a constant.

Hence, if $\varphi \in C_0^\infty$ and $\varphi(y)$=0 for $|y| > \alpha$, then we have

$E_x\{\int d\mu_x^t\, (\omega)\{I_1.I_2.I_3\}\}\varphi(\omega(t))$

$=E_x[\int_{R^n} dy \left\{\int d\mu_{x,y}^t\, (x)\{I_1.I_2.I_3\}\right\}$

$.\varphi(y)] \leq \tilde{\beta}.\exp\left(-\frac{(x-\alpha)^2}{2t}\right),$

where $\tilde{\beta} < +\infty$ is constant. Now the Fubini theorem implies the above assertion. □

Corollary2.2: Let us consider the function $\Psi(t,x)$ which define in equation (1) then we have the estimate

$E_x(|\Psi(t,x)|^2) \leq$

$M.\exp\left(\sum_{k=0}^{l-1} -(t_{k+1}-t_k)(b_k)^2\right).\exp\left(-\frac{(x-\alpha)^2}{2t}\right)$

where $M < +\infty$ is constant.

Proof :

By using the equation (2.1.3) we have

$E_x(\Psi(t,x)^2) =$

$[\int d\mu_x^t\, (x).E_x\left\{\exp\left(-i\int_0^t b(\omega(s))\right)d\omega\right\} \times$

$E_x\left\{\exp\left(\frac{1}{2}\int_0^t\, b^2(\omega(s))\, ds\right)\right\}$

$\times E_x\left\{\exp\left(-\int_0^t W(\omega(s))ds\right)\right\}\varphi(\omega)]^2$

$= \int d\mu_{x,y}^{2t}\, (\gamma).E_x\left\{\exp\left(-2i\int_0^{2t} b(\gamma(s))\right)d\gamma\right\} \times$

$E_x\left\{\exp\left(\int_0^{2t}\, b^2(\gamma(s))\, ds\right)\right\}$

$\times E_x\left\{\exp\left(-2\int_0^{2t} W(\gamma(s))ds\right)\right\}\varphi(y)$

$= \int d\mu_{x,y}^{2t}\, (\gamma)\{I_1.I_2.I_3\}\varphi(y),$

by the same method in proposition (2.1) we get the following

$E_x\{\int d\mu_x^{2t}\, (\omega)\{I_1.I_2.I_3\}\} < +\infty.$

In similar way we can show that, if $|y| \leq \alpha$, then

$E_x\{\int d\mu_x^{2t}\, (\gamma)\{I_1.I_2.I_3\}\} \leq M.\exp\left(\sum_{k=1}^{l-1} -2(t_{k+1}-t_k)(b_k)^2\right).$

$\exp\left(\sum_{k=0}^{l-1}(t_{k+1}-t_k)(b_k)^2\right).\exp\left(-\frac{(x-\alpha)^2}{2t}\right) = M$

$.\exp\left(\sum_{k=0}^{l-1} -(t_{k+1}-t_k)(b_k)^2\right)\exp\left(-\frac{(x-\alpha)^2}{2t}\right),$

where $M < +\infty$ is a constant.

Hence, if $\varphi \in C_0^\infty$ and $\varphi(y)$=0 for $|y| > \alpha$, then we have

$E_x\{\int d\mu_x^{2t}\, (\gamma)\{I_1.I_2.I_3\}\}\varphi(\gamma(t))$

$=E_x[\int_{-\infty}^{\infty} dy\left\{\int d\mu_{x,y}^{2t}\, (\gamma)\{I_1.I_2.I_3\}\right\}.\varphi(y)] \leq$

$M.\exp\left(\sum_{k=0}^{l-1} -(t_{k+1}-t_k)(b_k)^2\right).\exp\left(-\frac{(x-\alpha)^2}{2t}\right),$

where $M < +\infty$ is constant.

This mean that

$E_x(|\Psi(t,x)|^2) \leq$

$M.\exp\left(\sum_{k=0}^{l-1} -(t_{k+1}-t_k)(b_k)^2\right).\exp\left(-\frac{(x-\alpha)^2}{2t}\right)$

where $M < +\infty$ is constant.

3.Analytic Extension of $\Psi(t,x)$ by the Parameter t into the Domain $Ret > 0$

To investigate properties of the function $\Psi(t,x)$, we define its analytic extension into a certain complex domain of the variable t. First, we consider the Schrödinger operator $H = \sum_{j=1}^{n}\frac{1}{2}(i\partial_j + b_j(x))^2 + V(x)$ where $x \in R^n$, $b_j(x), j = 1,2,\ldots,n$ and $V(x)$ are real-valued functions on R^n, $\partial_j = \frac{\partial}{\partial x_j}$, $i = \sqrt{-1}$ defined on a finite interval $[\alpha,\beta]^n$ with zero boundary conditions, where $V(x)$ is some continuous function defined on $[\alpha,\beta]^n$. By the Feynman-Kac formula, if $\varphi(x)$ is a continuous function on $[\alpha,\beta]^n$, then

$exp(-tH)\ \varphi(x)=$

$\int\left[\int d\mu_{x,y}^t\, (\omega)\chi_{[\alpha,\beta]^n}(\omega)\left\{\exp\left(-i\int_0^t b(\omega(s))\, d\omega - \frac{1}{2}i\int_0^t \operatorname{div} b(\omega(s))\, ds - \int_0^t V(\omega(s))\, ds\right)\right\}\right]\ \varphi(y)dy.$

Let us take the advantage of the following known arguments by the Hilbert-Schmidt theorem, $\exp(-tH)$ is an integral operator with the kernel

$$exp(-tH)\,(x,y) = \sum_m e^{-tE_m}\, \varphi_m(x)\varphi_m(y), \quad (3.1.1)$$

where $\varphi_1, \varphi_2, ...$, is a complete orthonormal system of eigenfunctions of the operator H,

$$H\varphi_m = E_m\varphi_m \ , \ m = 1,2, ...$$

on the other hand,

$exp(-tH)(x,y) = \int d\mu_{x,y}^t(\omega)$

$\chi_{[\alpha,\beta]^n}\exp[-i\int_0^t b(\omega(s))\,dx + \frac{1}{2}\int_0^t b^2(\omega(s))\,ds - \int_0^t W(\omega(s))\,ds]$,

the functions $\varphi_m(x)$are continuous, the series (3.1.1) converges uniformly and the correlation

$$\int_{[\alpha,\beta]^n} dx\, exp(-tH)(x,x) = \sum_m e^{-tE_m} \quad (3.1.2)$$

takes place; in particular, the series (3.1.2) converges.
Put now $t = \tau + i\theta$. Write

$$\sum_m |e^{-tE_m}\varphi_m(x)\varphi_m(y)| \le \sum_m e^{-\tau E_m}|\varphi_m(x)||\varphi_m(y)| \le [\sum_m e^{-\tau E_m}|\varphi_m(x)|^2]^{1/2} \times [\sum_m e^{-\tau E_m}|\varphi_m(y)|^2]^{1/2} = [exp(-\tau H)(x.x)]^{1/2} \times [exp(-\tau H)(y,y)]^{1/2}.$$

Hence the series (3.1.1) uniformly converges for $x,\ y$ in $[\alpha,\beta]^n$, $t = \tau + i\theta,\ \ \tau \ge \tau_0 > 0$. Thus $exp(-tH)(x,y)$ is extended up to an analytic function of the variable t in the indicated domain.

Let us return to the case of the potential V under consideration.

Lemma 3.1: Let $\varphi, h \in C_0^\infty$be given, h=0 in a neighborhood of the point x=0. Denoted by $L^2(R^n, dV)$, the set of square integrable functions (in the sense of the mean E).
Put

$$F(t,V) = \int_{R^n} \Psi(t,x)h(x)dx.$$

Then $F(t,V) \in L^2(R^n,)$f or every $t > 0$and mapping $t \to F(t,V) \in L^2(R^n, dV)$ can be extended up to an analytic function in the domain $t = \tau + i\theta, \tau \ge \tau_0 > 0$ with values in $L^2(R^n, dV)$.

Proof:

We check that $F \in L^2(R^n, dV)$ write
By using the estimate in (corollary 2.2) we can write

$E(\int_{R^n} dx\, \Psi(t,x)^2) \le \beta \int_{-\alpha}^{\alpha} dy \int_{-\alpha}^{\alpha} dz \exp\left(-\frac{(x-\alpha)^2}{2t}\right)$
$\times \exp\left(\sum_{k=0}^{l-1} -(t_{k+1}-t_k)(b_k)^2\right)|\varphi(y)| \times |\varphi(z)| < +\infty,$

$$\|F(t,V)\|^2_{L^2(R^n,dV)} = E|F(t,V)|^2 \le const < +\infty,$$

i.e. $F(t,V) \in L^2(R^n, dV)$.

Let $-\infty < \alpha < \beta < +\infty$. Consider the operator $H = \sum_{j=1}^n \frac{1}{2}(i\partial_j + b_j(x))^2 + V(x)$ where $x = (x_1, x_2, ..., x_n) \in R^n,\ b_j(x),\ j = 1,2, ..., n$ and $V(x)$ are real valued functions on R^n, $\partial_j = \frac{\partial}{\partial x_j}$,$i = \sqrt{-1}$ on the interval $[\alpha,\beta]^n$ with zero boundary conditions. Denote is by $H_{\alpha,\beta}$ and consider the following function: $\Psi_{\alpha,\beta}(t,x) = \int d\mu_x^t(\omega)\exp[-i\int_0^t b(\omega(s))\,dx + \frac{1}{2}\int_0^t b^2(\omega(s))\,ds - \int_0^t W(\omega(s))\,ds]\chi_{[\alpha,\beta]^n}(\omega)\varphi(\omega(t))$

$=\int dy\left[\int d\mu_{x,y}^t(\omega)\exp\left[-i\int_0^t b(\omega(s))\,dx + \frac{1}{2}\int_0^t b^2(\omega(s))\,ds - \int_0^t W(\omega(s))\,ds\right]\chi_{[\alpha,\beta]^n}(\omega)\right]\varphi(y),$

$$\Psi_{\alpha,\beta}(t,x) = \int dy\, exp(-tH_{\alpha,\beta})(x,y), \quad (3.1.3)$$

and also

$$F_{\alpha,\beta}(t,V) = \int_{R^n} \Psi_{\alpha,\beta}(t,x)h(x)dx, \quad (3.1.4)$$

write for $x, y \in [\alpha,\beta]^n$

$exp(-tH_{\alpha,\beta})(x,y) \le$
$\int d\mu_{x,y}^t(\omega)\exp\left[-i\int_0^t b(\omega(s))\,dx + \frac{1}{2}\int_0^t b^2(\omega(s))\,ds - \int_0^t W(\omega(s))\,ds\right] \le const. \quad (3.1.5)$

The function $exp(-tH_{\alpha,\beta})(x,y)$ is analytically extended in the domain $t = \tau + i\theta, \tau \ge \tau_0 > 0$. Besides,

$|exp(-tH_{\alpha,\beta})(x,y)| \le [exp(-\tau H_{\alpha,\beta})(x,x)]^{1/2} \times [exp(-\tau H_{\alpha,\beta})(y,y)]^{1/2},$
(3.1.6)

thus we have,

$E|exp(-tH_{\alpha,\beta})(x,y)|^2 \le E\{[exp(-\tau H_{\alpha,\beta})(x,x)][\,exp(-\tau H_{\alpha,\beta})(y,y)]\}$

$\le \{E[exp(-\tau H_{\alpha,\beta})(x,x)]^2\}^{\frac{1}{2}}\{E[exp(-\tau H_{\alpha,\beta})(y,y)]^2\}^{\frac{1}{2}}.$
(3.1.7)

It follows from (3.1.3)-(3.1.7) that $\Psi_{\alpha,\beta}(t,x)$ is analytically extended by t, $F_{\alpha,\beta}(t,V)$ is also analytically extended by t. In addition,

$E|F_{\alpha,\beta}(t,V)|^2 = E\left|\int_{[\alpha,\beta]^n} exp(-tH_{\alpha,\beta})(x,y)\varphi(y)dy\right|^2$

$= E\{\int_{[\alpha,\beta]^n} exp(-tH_{\alpha,\beta})(x,y)\varphi(y)dy \times \int_{[\alpha,\beta]^n} \overline{exp(-tH_{\alpha,\beta})(x,z)\varphi(z)dz}\}$
$= \int_{[\alpha,\beta]^n}\int_{[\alpha,\beta]^n} E\{exp(-tH_{\alpha,\beta})(x,y)\, exp(-tH_{\alpha,\beta}) \times (x,z)\}\varphi(y)\varphi(z)dydz$
$\le \int_{[\alpha,\beta]^n}\int_{[\alpha,\beta]^n}[E|exp(-tH_{\alpha,\beta})(x,y)|^2]^{\frac{1}{2}} \times [E|exp(-tH_{\alpha,\beta})(x,z)|^2]^{\frac{1}{2}}|\varphi(y)||\varphi(z)|\,dydz$
(3.1.8)

further, using the relation (3.1.5) we get

$E[exp(-\tau H_{\alpha,\beta})(x,x)]^2 \le$
$E\left[\int d\mu_{x,x}^t(\omega)\exp\left[-i\int_0^t b(\omega(s))\,d\omega + \frac{1}{2}\int_0^t b^2(\omega(s))\,ds - \int_0^t W(\omega(s))\,ds\right]^2\right] \le$
$E\left[\int d\mu_{x,x}^t(\omega)\exp\left(-2i\int_0^t b(\omega(s))\,d\omega + \int_0^t b^2(\omega(s))\,ds - 2\int_0^t W(\omega(s))\,ds\right)\right]^{1/2} \times$
$\left[\int d\mu_{x,x}^t(\omega)\right]^{1/2} \le const < +\infty. \quad (3.1.9)$

Now, it follows from (3.1.8), (3.1.9) that

$\|F_{\alpha,\beta}(t,V)\|^2_{L^2(R^n,dV)} = E|F_{\alpha,\beta}(t,V)|^2 \le$const$< +\infty.$
(3.1.10)

Let now $\alpha \to -\infty$ and $\beta \to +\infty$. By (3.1.10), $F_{\alpha,\beta}(t,V)$ satisfies the conditions of the Montel theorem on the compactness of families of analytic functions. Therefore, for a suitable choice of $\alpha_n \to -\infty$ and $\beta_n \to +\infty$, there exists the limit

$$\lim_{\substack{\alpha_n \to -\infty \\ \beta_n \to +\infty}} F_{\alpha_n,\beta_n}(t,V) = \tilde{F}(t,V)$$

uniformly in each compact subdomain $G \subset \{t = \tau + i\theta, \tau \ge \tau_0 > 0\}$, where $\tilde{F}(t,V)$is an analytic function with values in $L^2(R^n, dV)$.

If t is real, then it is possible to pass to the limit as $\alpha_n \to -\infty$, $\beta_n \to +\infty$ in the integral in (3.1.3), (3.1.4), using the Lebesgue dominated convergence theorem, i.e.

$$\lim_{\substack{\alpha\to-\infty \\ \beta\to+\infty}} F_{\alpha,\beta}(t,V) = F(t,V). \quad (3.1.11)$$

The assertion of the lemma follows from (3.1.10), (3.1.11) . In the general case, we may again repeat our arguments based on the Feynman-Kac Itô formula and the Montel theorem. In this connection, one must just consider domains of the form $(\alpha',\beta')^n \cup (\alpha'',\beta'')^n$, where $\alpha' < \beta' < 0 < \alpha'' < \beta''$, $\alpha' \to -\infty$, $\beta' \to 0$, $\alpha'' \to 0$, $\beta'' \to +\infty$, instead of the intervals $(\alpha,\beta)^n$, and take into account that $h(x) \equiv 0$ in a neighborhood of the point $x = 0$.

Let us consider now the values of $\Psi(t,x)$, $F_{\alpha,\beta}(t,V)$, $F(t,V)$ defined the functions $\varphi(x)$, $V(x)$ and $h(x)$, we have

$$\|F(t,V)\|^2_{L^2(R^n,dV)} \le \{E[\int_{R^n}\Psi(t,x)^2dx]^2\}^{\frac{1}{2}} \times \{E[\int_{[\alpha,\beta]^n} V(x)^2h(x)^2\,dx]^2\}^{\frac{1}{2}}$$

where $supp\ h \subset (\alpha,\beta)^n$;

$$E[\int_{R^n}\Psi(t,x)^2dx]^2 = E(\int_{R^n}\int_{R^n} dxdy\Psi(t,x)^2\Psi(t,y)^2) =$$

$$\int_{R^n}dx\int_{R^n}dy\Big[E\int_{R^n}d\mu^t_{x,u}(\omega)\Big\{\exp\Big(-i\int_0^t b(\omega(s))\,d\omega + \frac{1}{2}\int_0^t b^2(\omega(s))\,ds - \int_0^t W(\omega(s))\,ds\Big)\Big\}\varphi(u)du \times \int_{R^n}d\mu^t_{x,z}(\eta)$$
$$\Big\{\exp\Big(-i\int_0^t b(\eta(s))\,d\eta + \frac{1}{2}\int_0^t b^2(\eta(s))\,ds - \int_0^t W(\eta(s))\,ds\Big)\Big\}\times\varphi(z)dz\times$$
$$\int_{R^n}d\mu^t_{y,w}(\xi)\Big\{\exp\Big(-i\int_0^t b(\xi(s))\,d\xi + \frac{1}{2}\int_0^t b^2(\xi(s))\,ds - \int_0^t W(\xi(s))\,ds\Big)\Big\}\times \varphi(w)dw \times$$
$$\int_{R^n}d\mu^t_{y,q}(\zeta)\Big\{\exp\Big(-i\int_0^t b(\zeta(s))\,d\zeta + \frac{1}{2}\int_0^t b^2(\zeta(s))\,ds - \int_0^t W(\zeta(s))\,ds\Big)\Big\}\times\varphi(q)dq\Big].$$

$$= \int_{R^n}\int_{R^n}dz\,du\int_{R^n}\int_{R^n}d\omega\,dq\,\times E\{\int d\mu^{2t}_{z,u}(\lambda)\exp\Big(-i\int_0^{2t}b(\lambda(s))\,d\lambda + \frac{1}{2}\int_0^{2t}b^2(\lambda(s))\,ds - \int_0^{2t}W(\lambda(s))\,ds\Big)\times$$
$$\int d\mu^{2t}_{\omega,q}(\curlyvee)\exp\Big(-i\int_0^{2t}b(\curlyvee(s))\,d\curlyvee + \frac{1}{2}\int_0^{2t}b^2(\curlyvee(s))\,ds - \int_0^{2t}W(\curlyvee(s))ds\Big)\varphi(z)\varphi(u)\varphi(\omega)\varphi(q); \quad (3.1.12)$$

$$E\{\int d\mu^{2t}_{z,u}(\lambda)\exp\Big(-i\int_0^{2t}b(\lambda(s))\,d\lambda + \frac{1}{2}\int_0^{2t}b^2(\lambda(s))\,ds - \int_0^{2t}W(\lambda(s))\,ds\Big)\times \int d\mu^{2t}_{\omega,q}(\curlyvee)\exp\Big(-i\int_0^{2t}b(\curlyvee(s))\,d\curlyvee + \frac{1}{2}\int_0^{2t}b^2(\curlyvee(s))\,ds - \int_0^{2t}W(\curlyvee(s))\,ds\Big)\} \le$$
$$\Big[E\Big\{\int d\mu^{2t}_{z,u}(\lambda)\exp\Big(-i\int_0^{2t}b(\lambda(s))\,d\lambda + \frac{1}{2}\int_0^{2t}b^2(\lambda(s))\,ds - \int_0^{2t}W(\lambda(s))\,ds\Big)\Big\}^2\Big]^{\frac{1}{2}} \times$$
$$\Big[E\Big\{\int d\mu^{2t}_{z,u}(\curlyvee)\exp\Big(-i\int_0^{2t}b(\curlyvee(s))\,d\curlyvee + \frac{1}{2}\int_0^{2t}b^2(\curlyvee(s))\,ds - \int_0^{2t}W(\curlyvee(s))\,ds\Big)\Big\}^2\Big]^{\frac{1}{2}};$$

$$E\Big\{\int d\mu^{2t}_{z,u}(\lambda)\exp\Big(-i\int_0^{2t}b(\lambda(s))\,d\lambda + \frac{1}{2}\int_0^{2t}b^2(\lambda(s))\,ds - \int_0^{2t}W(\lambda(s))\,ds\Big)\Big\}^2 \le$$
$$E\Big[\Big[\int d\mu^{2t}_{z,u}(\lambda)\exp\Big(-2i\int_0^{2t}b(\lambda(s))\,d\lambda + \int_0^{2t}b^2(\lambda(s))\,ds - 2\int_0^{2t}W(\lambda(s))\,ds\Big)\Big]\times\Big[\int d\mu^{2t}_{z,u}(\lambda)\times 1\Big]\Big] \quad (3.1.13)$$

Now the estimate

$E[\int_{R^n}\Psi(t,x)^2dx]^2 \le const < +\infty$,

follows from (3.1.12), (3.1.13) and from the estimate in (corollary 2.2).

Now, we write the expression for $V^2(x)$:

$$V^2(x) = \sum\sum \xi^2_{j,m}\, v^2_{j,m}(x - a_{j,m})$$

and take into account that here $x \in [\alpha,\beta]^n$ and, therefore, the number of summands remains bounded. Since $\xi_{j,m}$ has the Gaussian distribution, we have $E|\xi_{j,m}|^k < +\infty$. From this, it follows that $E[\int_{[\alpha,\beta]^n}V^2(x)h^2(x)dx]^2 \le const$, hence

$$\|F_{\alpha,\beta}(t,V)\|_{L^2(R^n,dV)} \le const.$$

Now, we can again apply the previous constructions and show that $F(t,V)$ is an analytic function in the mentioned domain.

Note that one can similarly get the following estimates:

$$\int_{R^n}E(\Psi(t,x))^2|V|^m)dx \le const. \quad (3.1.14)$$

where m =1,2,… and the constant depends on m . □

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