NEARLY SEMIPRIME IDEALS

Nuhad S Al-Mothafar, Mohammed B.H.Alhakeem
Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq
nuhad_math@yahoo.com

ABSTRACT: *Let R be a commutative ring with unity 1 an ideal I of a ring R is called semiprime ideal if $\sqrt{I} = I$. In this paper, we say that an ideal I of a ring R is nearly semiprime ideal, if $\sqrt{I} = I + J(R) \cap \sqrt{I}$, where $\sqrt{I} = \{r \in R: r^n \in I, n \in Z^+\}$, where $J(R)$ is the Jacobson radical of R. We give many results of this type of ideals.*

Keywords: semiprime submodule, nearly semiprime submodule, nearly semiprime ideal, nearly regular ring.

§ 1.INTRODUCTION:

A submodule of an R- module M which Dauns[3] was named semiprime submodules that they are generalized of semiprime ideals, which get significant importance at last years, many studies and searches are published about semiprime submodules by many people who care with the subject of commutative algebra and some of them are J.Dauns, R.L.McCasland, C.P.LU, P.F.Smith, M.E.Moore. A proper submodule N of an R-module M is called nearly semiprime if whenever〚 r〛^n x∈N,r∈R,x∈M,n∈Z^+, implies rx∈N+J(M), where J(M) is the Jacobson radical of M, [6]. It is clear that every semiprime submodule is nearly semiprime submodule, but the converse is not true in general. In section two, we introduced the concept of nearly semiprime ideals. An ideal I of a ring R is called nearly semiprime ideal if √I = I + J(R)∩√I where √I = {r∈R:r^n∈I, n∈Z^+}. We show that every semiprime ideal is nearly semiprime ideal, but the converse is not true. Also, we give another characterization for nearly semiprime ideals, see theorem (2.10). In Section three, we will offer some properties of nearly semiprime ideals; we show that if N is a submodule of an R-module M with J(M)⊆N., Then N is nearly semiprime submodule of M iff [N: M] is nearly semiprime ideal of R.**§ 2.**1 Nearly Semiprime Ideals:

In this section, we introduce the concept of nearly semiprime ideals. We give some basic properties and characterization of nearly semiprime ideals.

Definition(2.1):
An ideal I of a ring R is said to be a nearly semiprime ideal if $\sqrt{I} = I + J(R) \cap \sqrt{I}$, where $\sqrt{I} = \{r \in R: r^n \in I, n \in Z^+\}$

Remarks and examples (2.2) :

1. Every semiprime ideal is nearly semiprime ideal, where I is an ideal of a ring R, $I \subseteq \sqrt{I}$ and $J(R) \cap \sqrt{I} \subseteq \sqrt{I}$, then $I + J(R) \cap \sqrt{I} \subseteq \sqrt{I}$, but I is semiprime ideal, then $\sqrt{I} \subseteq I \subseteq I + J(R) \cap \sqrt{I}$, then $\sqrt{I} = I + J(R) \cap \sqrt{I}$. Which implies that I is nearly semiprime ideal, but the converse is not true in general for example, if $I = \langle \bar{4} \rangle$ is an ideal of Z_8 as Z- module, $J(Z_8) = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$. $\sqrt{\langle \bar{4} \rangle} = \langle \bar{2} \rangle \neq \langle \bar{4} \rangle$, then I is not semiprime ideal. But $\langle \bar{4} \rangle + \sqrt{\langle \bar{4} \rangle} \cap J(Z_8) = \sqrt{\langle \bar{4} \rangle}$. Which implies that I is a nearly semiprime ideal.
2. If J(R)=0, then every nearly semiprime ideal is semiprime ideal.
3. In a ring Z, J(Z)=0, the ideal $N = \langle 4 \rangle$ in the ring Z is not nearly semiprime ideal because $\sqrt{\langle of\ 4 \rangle} \neq \langle 4 \rangle + J(Z) \cap \sqrt{\langle 4 \rangle}$.
4. If I is nearly semiprime ideal of a ring R and $J(R) = L(R)$, then $\sqrt{I} = I + L(R)$, where $L(R) = \{r \in R : r^n = 0, n \in N\} = \sqrt{0}$, where I is a nearly semiprime ideal of R, then $\sqrt{I} = I + J(R) \cap \sqrt{I}$, but $J(R) = L(R)$, then $\sqrt{I} = I + L(R) \cap \sqrt{I}$, then $\sqrt{I} = I + \sqrt{0} \cap \sqrt{I}$, thus $\sqrt{I} = I + \sqrt{0 \cap I}$, then $\sqrt{I} = I + \sqrt{0}$ .Which implies that $\sqrt{I} = I + L(R)$.

Compare the following propositions with (2.14), (2.15) in [6].

Proposition (2.3):
If R and R' are rings and $\emptyset: R \to R'$ is an epimorphism and $ker\emptyset \subseteq I$. If I is a nearly semiprime ideal in R, then $\emptyset(I)$ is nearly a semiprime ideal in R'.

Proof: We have to show that $\sqrt{\phi(I)} = \phi(I) + J(R') \cap \sqrt{\phi(I)}$

Clearly that $\phi(I) + J(R') \cap \sqrt{\phi(I)} \subseteq \sqrt{\phi(\ )}I$. We want to show that $\sqrt{\phi(I)} \subseteq \phi(I) + J(R') \cap \sqrt{\phi(I)}$. Let $a \in \sqrt{\phi(I)}$, then $a^n \in \phi(I)$ for some $n \in Z^+$, since ϕ is onto, then $a^n = \phi(t)$; $t \in I$ and $a \in R'$, $\exists\, x \in R$ such that $\phi(x) = a$, since ϕ is homomorphism, $\phi(t) = a^n = \phi(x)^n = \phi(x^n)$, $\phi(t) - \phi(x^n) = 0$, then $t - x^n \in ker\emptyset \subseteq I$ and $t \in I$, then $x^n \in I$, since I is nearly semiprime ideal in R, then $x \in \sqrt{I} = I + J(R) \cap \sqrt{I}$, then $\phi(x) \in \phi(\, I + J(R) \cap \sqrt{I}\,)$)), since ϕ is homomorphism, then $\phi(x) \in \phi(I) + \phi(J(R)) \cap \phi(\sqrt{I})$, hence by [2. P. 242,233], $\phi(x) \in \phi(I) + J(R') \cap \sqrt{\phi(\ )}I$, then $a \in \phi(I) + J(R') \cap \sqrt{\phi(\ )}I$, which implies that $\emptyset(I)$ is nearly semiprime in R'.

Proposition (2.4):
If R and R' are rings and $\emptyset: R \to R'$ is an epimorphism. If J' is nearly semiprime ideal in R', then $\emptyset^{-1}(J')$ is nearly semiprime ideal in R.

Proof: We have to show that $\sqrt{\emptyset^{-1}(J')} = \emptyset^{-1}(J') + J(R) \cap \sqrt{\emptyset^{-1}(J')}$. It's clear that $\emptyset^{-1}(J') + J(R) \cap \sqrt{\emptyset^{-1}(J')} \subseteq \sqrt{\emptyset^{-1}(J')}$. We want to show that $\sqrt{\emptyset^{-1}(J')} \subseteq \emptyset^{-1}(J') + J(R) \cap \sqrt{\emptyset^{-1}(J')}$. Let $a \in \sqrt{\emptyset^{-1}(J')}$, then $a \in \emptyset^{-1}(J')$ for some $n \in Z^+$, thus $\phi(a^n) \in J'$, then $(\phi(a))^n \in J'$, then $\phi(a) \in \sqrt{J'}$, but J' is nearly semiprime ideal in R' , then $\sqrt{J'} = J' + J(R') \cap \sqrt{J'}$, then $\phi(a) \in J' + J(R') \cap \sqrt{J'}$, then $a \in \emptyset^{-1}(J') + \emptyset^{-1}(J(R')) \cap \emptyset^{-1}(\sqrt{(J')})$, by [2. P. 242,233], then $a \in \emptyset^{-1}(J') + J(R) \cap \sqrt{\emptyset^{-1}(J')}$, then $\sqrt{\emptyset^{-1}(J')} = \emptyset^{-1}(J') + J(R) \cap \sqrt{\emptyset^{-1}(J')}$. Which implies that $\emptyset^{-1}(J')$ is nearly semiprime ideal in R'.

Recall that a ring R is said to be a nearly regular ring (for short N −regular) if $\frac{R}{J(R)}$ is a regular ring. Equivalently R is supposed to be a nearly regular ring if for each $a \in R$, $\exists\, b \in R$ such that, $a - a^2b \in J(R)$, [5].

If R is a regular ring, then every ideal in R is semiprime ideal, see[$2.p.243$]. For nearly semiprime ideals we have the following.

Theorem(2. 5) :
Let R be a ring, then R is nearly regular iff every ideal in R is nearly semiprime ideal of R.
Proof: Let R be a nearly regular ring and let I be an ideal in R $I \subseteq \sqrt{I}$ and $J(R) \cap \sqrt{I} \subseteq \sqrt{I}$, therefore $I + J(R) \cap \sqrt{I} \subseteq \sqrt{I}$
Now, let $a \in \sqrt{I}$, then $a^n \in I$, for some $n \in Z^+$. Since R is a nearly regular ring, then $\exists\, r \in R$ such that $a - ra^2 \in J(R)$, but $a - ra^2 \in \sqrt{I}$, then $a - ra^2 \in J(R) \cap \sqrt{I}$, put $a - ra^2 = s$, $s \in J(R) \cap \sqrt{I}$, $a = ra^2 + s$, then $a = ra(ra^2 + s) + s = r^2a^3 + ras + s$, then $a = r^2a^3 + t$, $t = ras + s = (ra + 1)s$, continue $a = r^{n-1}a^n + w$, where $w \in R$. It's clear that $r^{n-1}a^n \in I$ and $w \in J(R) \cap \sqrt{I}$, thus $a \in I + J(R) \cap \sqrt{I}$, then $I = I + J(R) \cap \sqrt{I}$, which implies that I is nearly semiprime ideal in R.
For the converse, suppose that $I = I + J(R) \cap \sqrt{I}$ for each ideal I of R, let $a \in R$, Put $I = \langle a^2 \rangle$, since $a^3 \in \langle a^2 \rangle$, then an $\in \sqrt{\langle a^2 \rangle} = \langle a^2 \rangle + J(R) \cap \sqrt{\langle a^2 \rangle}$. Thus an $\in \langle a^2 \rangle + J(R) \cap \sqrt{\langle a^2 \rangle}$, $a = ra^2 + s$, $s \in J(R)$, $a - ra^2 \in J(R)$, th erefore $a + J(R) = ra^2 + J(R)$, then $\frac{R}{J(R)}$ is a regular ring, which implies that R is a nearly regular ring.
R is a regular ring if and only if $\sqrt{I} = I$, for each ideal I of a ring R.
By using the same prove of a theorem(2. 5).
Theorem(2.6) :
Let R be a ring, then R is a nearly regular if and only if $\sqrt{I} = I + J(R) \cap \sqrt{I}$, for all ideal I of R.
Corollary(2.7):
A ring R is a nearly regular if and only if $I = I^2 + J(R) \cap I$, for each ideal I of R.
Corollary(2.8):
A ring R is a nearly regular if and only if $\sqrt{\langle a \rangle} = \langle a \rangle + J(R) \cap \sqrt{\langle a \rangle}$, for every $an \in R$
Corollary(2.9) :
Let R be a ring such that $J(R) = L(R)$, then R is a nearly regular ring if and only if $\sqrt{I} = I + L(R)$, for each ideal I of R.
Proof:Let R be a nearly regular ring, and I be an ideal of R by theorem(2.6), $\sqrt{I} = I + J(R) \cap \sqrt{I}$, but $J(R) = L(R)$, $\sqrt{I} = I + L(R) \cap \sqrt{I} = I + \sqrt{0} \cap \sqrt{I}$, by [2, P 233], then $\sqrt{I} = I + \sqrt{0 \cap I}$, then $\sqrt{I} = I + \sqrt{0}$, which implies that $\sqrt{I} = I + L(R)$. For the converse, suppose that $\sqrt{I} = I + L(R)$ and let $a \in R$, put $I = \langle a^2 \rangle$ since $a^3 \in \langle a^2 \rangle$, then $a \in \sqrt{\langle a^2 \rangle} = \langle a^2 \rangle + L(R)$, $a = ra^2 + s$, $s \in L(R)$, $r \in R$, $a - ra^2 \in L(R) = J(R)$, thus $a - ra^2 \in J(R)$, then $a + J(R) = ra^2 + J(R)$, thus $\frac{R}{J(R)}$ is a regular ring, which implies that R is a nearly regular ring.
The following theorem gives another characterization for nearly semiprime ideals.
Theorem(2.10) :
An ideal I of a ring R is nearly semiprime ideal if and only if $\sqrt{I} \subseteq I + J(R)$.
Proof: Suppose I is a nearly semiprime ideal in R, by definition(2.1) we have $\sqrt{I} = I + J(R) \cap \sqrt{I} \subseteq I + J(R)$, then $\sqrt{I} \subseteq I + J(R)$. To show that I is nearly semiprime ideal in R, we only have to prove $\sqrt{I} \subseteq I + J(R) \cap \sqrt{I}$. Let $a \in \sqrt{I}$, since $\sqrt{I} \subseteq I + J(R)$, then $a = b + s$; $b \in I, s \in J(R)$, then $a - b = s$, thus $s \in J(R) \cap \sqrt{I}$, then $a \in I + J(R) \cap \sqrt{I}$, th*erefore* $\sqrt{I} \subseteq I + J(R) \cap \sqrt{I}$, which implies that I is nearly semiprime ideal in R.
We know that an ideal I is semiprime if and only if $\frac{R}{I}$ has no non-zero nilpotent element. If I am a nearly semiprime ideal, they may have a non-zero nilpotent element.
Proposition (2.11):
If $R/_{I + J(R)}$ has no non-zero nilpotent element, then I is a nearly semiprime ideal in R. If $J(R) \subseteq I$, then the converse is true.
Proof: Let $0 \neq a \in \sqrt{I}$, then $\exists\, n \in Z^+$ such that $a^n \in I \subseteq I + J(R)$, then $a^n \in I + J(R)$, thus $(a + I + J(R))^n = I + J(R)$, then $a + I + J(R)$ is a nilpotent element in $R/_{I + J(R)}$, then $a \in I + J(R)$, then $\sqrt{I} \subseteq I + J(R)$, which implies that I is a nearly semiprime ideal in R.
For the converse, let $r + I + J(R)$ be a non-zero nilpotent element in $R/_{I + J(R)}$, then $\exists\, n \in Z^+$ such that $r^n \in I + J(R)$, but $J(R) \subseteq I$, then $r^n \in I$, then $r \in \sqrt{I}$, since I is a nearly semiprime ideal in R, then by Theorem (2.10) $\sqrt{I} \subseteq I + J(R)$, then $r \in I + J(R)$, which implies that $R/_{I + J(R)}$ has no non-zero nilpotent element.
§ 3. More About Nearly Semiprime Ideals:
Recall that a submodule N of an R- module M is called nearly semiprime if whenever $r^n x \in N, r \in R, x \in M$, $n \in Z^+$, implies $rx \in N + J(M)$, [6] where $J(M)$ is the Jacobson radical of M.
If N is semiprime submodule of an R- module M, then $[N: M]$ is a semiprime ideal of R [1]. For nearly semiprime submodule we have the following:
Proposition (3.1):
Let N be a submodule of an R −module M such that $J(M)$ contain in every submodule of M, then N is nearly semiprime submodule of M iff $[N: M]$ is a semiprime ideal in R.
Proof: Since $J(M) \subseteq N$, by [2, 243], N is semiprime submodule, then $[N: M]$ is a semiprime ideal in R.
For the converse, let N be a proper submodule of M. Let $r \in R$, $x \in M$, $n \in Z^+$ such that $r^n x \in N$, then $r^n \in [N: M]$ but $[N: M]$ is a semiprime ideal in R, thus $r \in [N: M]$, hence $rx \in N \subseteq N + J(M)$, which implies that N is nearly semiprime in M.
Proposition (3.2):
If $N + J(M)$ is a nearly semiprime submodule of an R −module M, then $[N + J(M): M]$ is a semiprime ideal of R
Proof: Let $a \in \sqrt{[N + J(M): M]}$, then $\exists\, n \in Z^+$ such that $a^n \in [N + J(M): M]$, thus $a^n M \subseteq N + J(M)$ but $N + J(M)$ is a nearly semiprime submodule of M**,** then $aM \subseteq N + J(M)$, then an $\in [N + J(M): M]$ which implies that $[N + J(M): M]$ is a semiprime ideal in R.
Proposition (3.3):
Let $[N: M]$ be a semiprime ideal of R, then $\sqrt{[N: M]}$ is a semiprime ideal of R.

Proof: It is precisely that $\sqrt{[N: M]} \subseteq \sqrt{\sqrt{[N: M]}}$. We have to show that $\sqrt{\sqrt{[N: M]}} \subseteq \sqrt{[N: M]}$. Let $an \in \sqrt{\sqrt{[N: M]}}$, then $\exists\, n \in Z^+$ such that $a \in \sqrt{[N: M]}$, but $\sqrt{[N: M]} = [N: M]$ (since $[N: M]$ is a semiprime ideal in R), then

$a \in \sqrt{[N:M]}$, then $\sqrt{\sqrt{[N:M]}} = \sqrt{[N:M]}$, which implies that $\sqrt{[N:M]}$ is semiprime ideal of R.

Recall that M is called multiplication R −module if $N = IM$ [7], in particular for each submodule N of M, M is a multiplication R −module, if $[N:M]M = N$.

Recall that a ring R is said to be a good ring if $J(M) = J(R).M$ for each R −module M, [4].

Proposition (3.4):

Let R be a good ring and M be a multiplication R −module, then N is a nearly semiprime submodule of M iff $[N:M]$ is nearly semiprime ideal of R.

Proof: Suppose $[N:M]$ is a nearly semiprime ideal of R by theorem (2.10), we have $\sqrt{[N:M]} \subseteq [N:M] + J(R)$. We have to show that N is a nearly semiprime submodule of M. Let $r \in R$, $x \in M$, $n \in Z^+$ such that $r^n x \in N$, then $r^n M \in N$, then $r^n \in [N:M]$, hence $r \in \sqrt{[N:M]}$, but $[N:M]$ is a nearly semiprime ideal of R, then by thmeorm (2.10) $r \in [N:M] + J(R)$, then $rM \in [N:M]M + J(R)M$, since M is a multiplication R −module and $J(R).M \subseteq J(M)$, then $rM \in N + J(M)$, then $rx \in N + J(M)$, which implies that N is a nearly semiprime submodule of M.

For the converse, to show that $[N:M]$ is a nearly semiprime ideal, we have to prove that $\sqrt{[N:M]} \subseteq [N:M] + J(R)$. Let $a \in \sqrt{[N:M]}$, then $\exists\, n \in Z^+$ such that $a^n \in [N:M]$, then $a^n M \in N$, since N is nearly semiprime submodule of M, then $aM \subseteq N + J(M)$, since M is is a multiplication R −module and R is a good ring, then $aM \subseteq [N:M]M + J(R)M$, then $aM \subseteq ([N:M] + J(R)).M$, then $a \in [N:M] + J(R)$, which implies that $\sqrt{[N:M]} = [N:M] + J(R)$.

Recall that a proper submodule N of an R −module is called a primary submodule if for each $r \in R, x \in M$ such that $rx \in N$, then either $x \in N$ or $r^k \in [N:M]$, for some $k \in Z^+$, [1].

Proposition (3.5):

If N is a primary submodule of an R −module M and $J(R) \subseteq [N:M]$, then N is a prime submodule of M iff $[N:M]$ is a nearly semiprime ideal of R.

Proof: Suppose that N is a prime submodule of M, then by [1], $[N:M]$ is a prime ideal of R, which implies that $[N:M]$ is a nearly semiprime ideal of R.

For the converse, let $r \in R, x \in M$ such that $rx \in N$, we want to show that either $x \in N$ or $r \in [N:M]$. If $x \notin N$ and N is a primary submodule of an R −module M, then $r \in \sqrt{[N:M]}$, then $\exists\, n \in Z^+$ such that $r^n \in [N:M]$, but $[N:M]$ is a nearly semiprime ideal of R, then $r \in [N:M] + J(R)$, but $J(R) \subseteq [N:M]$, then $r \in [N:M]$. Which implies that N is a prime submodule of M.

Corollary (3.6):

If N is a primary submodule of an R −module M and $J(R) \subseteq [N:M]$, then N is semiprime submodule of M iff $[N:M]$ is nearly semiprime ideal of R.

Corollary (3.7.):

If N is a primary submodule of an R −module M and $J(R) \subseteq [N:M]$, then N is nearly semiprime submodule of M iff $[N:M]$ is nearly semiprime ideal of R.

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