COMBIND LAPLACE TRANSFORM – VARIATIONAL ITERATION METHOD FOR SINE-GORDON EQUATION

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ABSTRACT: *In this paper, the combined Laplace transform –Variation iteration method presented and used to solve the initial value problem for the sine-Gordon partial differential equation to obtain the approximate analytical solution. The obtained results show the liability and the efficiency of this method.*

Keywords: Laplace transforms (LT), variation iteration transform method, sine–Gordon equation, and nonlinear partial differential equation (NLpde).

1. INTRODUCTION

Modelling several physical systems using mathematical modeling, which resulted in discovering the nonlinear partial differential equations (NLpde) and used later in different discipline of science and engineering.

One of the leading equations of all partial differential equations in the field of mathematics and physics is the sine-Gordon equation. It is a nonlinear hyperbolic partial differential equation that has a vast range of applications in differential geometry, the propagation of fluxons in Josephson junctions [1,2] between two superconductors, relativistic field theory, the motion of a rigid pendulum attached to a stretched wire [3], solid state physics, and fluid motions stability.

The sine-Gordon equation can be considered as below:

$$u_{tt}- u_{xx} + sin(u) = 0 \quad (1)$$

Subject to initial conditions:

$$u(x,0) = f(x)\,, u_t(x,0)=g(x) \quad (2)$$

Where u is a function of x and t which represents the amplitude or elevation of the solitary wave generated at position x and time t; $f(x)$ and $g(x)$ represent known analytic functions.

Many numerical methods were used sine-Gordon equation in calculating the exact and approximate solutions. Wazwaz [4] achieved the exact solution of sine-Gordon equations by using a suggested method called tanh method. Kaya [5] attained the numerical solution of sine-Gordon equations by applying the modified decomposition method(MDM). Batiha et al. [6] used variation iteration method(VIM) to find the approximate analytic solution of sine-Gordon equation. Yucel [7] investigated the exact solution of sine-Gordon equation by means of homotopy analysis method(HAM). Both Mohyud-Din et al. [8] and Saeed [9] used the modified variation iteration method(MVIM) to solve the sine-Gordon equations. Biazar et al. [10] suggested the differential transform method to attain the semi analytical solution of sine-Gordon equations. Xinping shao et al. [11] presented variation iteration method coupled with adomian decomposition and homotopy perturbation method to find the exact solution of sine-Gordon equations. Hassan et al. [12] proposed a new technique to solving nonlinear sine-Gordon equation through homotopy analysis method (HAM). Jin [13] suggested a method to obtain the numerical solutions or analytical solutions of sine-Gordon equation by using Homotopy Perturbation Method(HPM). Lu [14] proposed an analytical approach to solve the sine-Gordon equation using Modified Homotopy Perturbation Method(MHPM). Rao [15] proposed reduced differential transform method(RDTM) to obtain the exact and semi analytic solution of sine-Gordon equations. Also, recently Shukla *et al.* [16] proposed a numerical solution for solving sin-Gordon equations using Modified Cubic B-spline differential quadrature method(MCB-DQM).

Similarly, about the methods that are used in solving NLpde, Meiappane et al. [17] and AL-Fayadh et al [18] proposed an analytical method called wavelet method, which is successfully resulted in solving NLpde. Also, AL-Fayadh et al. [19] proposed a new approach in obtaining the exact solution of a NLpde. The new approach is consisting of combining two methods, Laplace Transform and VIM. The VIM can solve a large class of nonlinear problems, including both the systems expressed as either ordinary or partial differential equations [20]. The VIM is an analytical asymptotic approach [21], where the initial guess function is corrected step by step, and finally reaches the true solution. By combining the Laplace Transform method with VIM will be more suitable and powerful in solving NLpde.

In this paper the new approach that proposed by AL-Fayadh et al. [19]; Laplace Transform – Variation Iteration Method (LT-VIM); will be used for solving and obtaining the exact and approximate solution of sine-Gordon equation.

The main objective of this paper is to use the LT-VIM for solving the initial value problem for the sine-Gordon equation. The effectiveness of this method will be demonstrated by two examples. The first example will compare the obtained results from the proposed method with the exact solutions and other numerical methods. The second example illustrates the approximate solutions obtained by the suggested method and other existing methods.

2. COMBIND LAPLACE TRANSFORM-VARIATIONAL ITERATION METHOD

The combined Laplace transform – Variation iteration method (LT-VIM**)** is a combination of the well-known Laplace transform and the variation iteration method. The general form of inhomogeneous NLpde has been considered with initial conditions as given below:

$$Lu(x,t)+Ru(x,t)+N\,u(x,t)=h(x,t) \quad (3)$$

$$u(x,0)=f(x), \quad u_t(x,0)=g(x) \quad (4)$$

Where L is the second order differential operator $L = \frac{\partial^2 u}{\partial t^2}$, N represents a general nonlinear differential operator, R represents the remaining linear operator and $h(x,t)$ is a source term.

Taking Laplace transform on equation (3), we obtain:

$$L\{Lu(x,t)\} + L\{Ru(x,t)\} + L\{Nu(x,t)\} = L\{h(x,t)\} \quad (5)$$

By applying the Laplace transform differentiation property, we have

$$s^2L\{Lu(x,t)\} - sf(x) - g(x) + L\{Ru(x,t)\} + L\{Nu(x,t)\} = L\{h(x,t)\} \quad (6)$$

Or

$$\{Lu(x,t)\} = \frac{1}{s}f(x) + \frac{1}{s^2}g(x) + \frac{1}{s^2}L\{h(x,t)\} - \frac{1}{s^2}L\{Ru(x,t)\} - \frac{1}{s^2}L\{Nu(x,t)\} \quad (7)$$

Take Laplace inverse to eq. (7) and Derivative by $\frac{\partial}{\partial t}$ both sides, we have

$$u(x,t) = f(x) + tg(x) + L^{-1}\left(\frac{1}{s^2}L\{h(x,t)\}\right) - L^{-1}\left(\frac{1}{s^2}L\{Ru(x,t)\}\right) - L^{-1}\left(\frac{1}{s^2}L\{Nu(x,t)\}\right) \quad (8)$$

$$u_t(x,t) + \frac{\partial}{\partial t}L^{-1}\left(\frac{1}{s^2}L\{R\,u(x,t)\}\right) + \frac{\partial}{\partial t}L^{-1}\left(\frac{1}{s^2}L\{N\,u(x,t)\}\right) - \frac{\partial}{\partial t}L^{-1}\left(\frac{1}{s^2}L\{h(x,t)\}\right) - g(x) = 0 \quad (9)$$

By the correction function of the itrational method, $\lambda = s - t$

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (s-t)\left((u_n)_\xi(x,\xi) + \frac{\partial}{\partial \xi}L^{-1}\left(\frac{1}{s^2}L\{Ru_n(x,\xi)\}\right) + \frac{\partial}{\partial \xi}L^{-1}\left(\frac{1}{s^2}L\{Nu_n(x,\xi)\}\right) - \frac{\partial}{\partial \xi}L^{-1}\left(\frac{1}{s^2}L\{h(x,t)\}\right) - g(x)\right)d\xi \quad (10)$$

Finally the solution u(x, t) is given by:

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) \quad (11)$$

2.1 Example 1

we consider the following sine-Gordon equation:

$$u_{tt} - u_{xx} + \sin u = 0 \quad (12)$$

subject to the initial conditions:

$$u(x,0) = 0, \quad u_t(x,0) = 4sech(x) \quad (13)$$

the exact solution is:

$$u(x,t) = 4\,tan^{-1}\big(t\,sech(x)\big) \quad (14)$$

Taking Laplace transform on equation both sides:

$$L\{Lu_{tt}\} - L\{u_{xx}\} + L\{\sin u\} = 0 \quad (15)$$

$$s^2L\{u(x,t)\} - su(x,0) - u_t(x,0) - L\{u_{xx}\} + L\{\sin u\} = 0 \quad (16)$$

$$s^2\{Lu(x,t)\} - 4\,\mathrm{sech}(x) - L\{u_{xx}\} + L\{\sin u\} = 0 \quad (17)$$

$$s^2\{Lu(x,t)\} = 4\,\mathrm{sech}(x) + L\{u_{xx}\} - L\{\sin u\} \quad (18)$$

Divided eq. (18) on s^2

$$\{Lu(x,t)\} = \frac{1}{s^2}4\,\mathrm{sech}(x) + \frac{1}{s^2}L\{u_{xx}\} - \frac{1}{s^2}L\{\sin u\} \quad (19)$$

Applying the inverse Laplace transform on both sides of Eq. (19), we get:

$$u(x,t) = 4t\,\mathrm{sech}(x) + L^{-1}\left(\frac{1}{s^2}L\{u_{xx}\}\right) - L^{-1}\left(\frac{1}{s^2}L\{\sin u\}\right) \quad (20)$$

Derivative by $\frac{\partial}{\partial t}$ both sides of equation (20):

$$u_t(x,t) - 4sech(x) - \frac{\partial}{\partial t}L^{-1}\left(\frac{1}{s^2}L\{u_{xx}\}\right) + \frac{\partial}{\partial t}L^{-1}\left(\frac{1}{s^2}L\{\sin u\}\right) = 0 \quad (21)$$

By the correction function of the irrational method**:**

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (s-t)\begin{pmatrix}(u_n)_\xi(x,\xi) - 4sech(x) - \frac{\partial}{\partial \xi}L^{-1}\left(\frac{1}{s^2}L\{(u_n)_{xx}(x,\xi)\}\right)\\ +\frac{\partial}{\partial \xi}L^{-1}\left(\frac{1}{s^2}L\{\sin u_n(x,\xi)\}\right)\end{pmatrix}d\xi \quad (22)$$

$$u_1(x,t) = u_0(x,t) + \int_0^t (s-t)\begin{pmatrix}(u_0)_\xi(x,\xi) - 4sech(x) - \frac{\partial}{\partial \xi}L^{-1}\left(\frac{1}{s^2}L\{(u_0)_{xx}(x,\xi)\}\right)\\ +\frac{\partial}{\partial \xi}L^{-1}\left(\frac{1}{s^2}L\{\sin u_{n0}(x,\xi)\}\right)\end{pmatrix}d\xi \quad (23)$$

$$u_1(x,t) = 4tsech(x) + \int_0^t (s-t)\begin{pmatrix}4sech(x) - 4sech(x) -\\ \frac{\partial}{\partial \xi}L^{-1}\left(\frac{1}{s^2}\,4\xi(-\mathrm{sech}^3(\mathrm{x}) + \tanh^2(\mathrm{x})\,\mathrm{sech(x)})\right)\\ +\frac{\partial}{\partial \xi}L^{-1}\left(\frac{1}{s^2}L\{\sin(4\xi sech(x))\}\right)\end{pmatrix}d\xi \quad (7)\ (24)$$

$$u_1(x,t) = 4tsech(x) + \int_0^t (s-t)\begin{pmatrix}\frac{\partial}{\partial \xi}\left(-4\xi^2(-\mathrm{sech}^3(\mathrm{x}) + \tanh^2(\mathrm{x})\,\mathrm{sech(x)})\right)\\ +L^{-1}\left(\frac{1}{s^2}L\left\{\sin\frac{4sech(x)}{s^2+(4sech(x))^2}\right\}\right)\end{pmatrix}d\xi \quad (25)$$

$$u_1(x,t) = 4tsech(x) + \int_0^t (s-t)\left(\frac{\partial}{\partial \xi}\begin{pmatrix}-4\xi^2(-\mathrm{sech}^3(\mathrm{x})\\ +\tanh^2(\mathrm{x})\mathrm{sech(x)})\end{pmatrix} + \xi(\sin(4\xi\mathrm{sech}(x))\right)d\xi \quad (26)$$

$$u_1(x,t) = 4tsech(x) + \int_0^t (s-t)\begin{pmatrix}\left(-\,8\xi(-\mathrm{sech}^3(\mathrm{x}) + \tanh^2(\mathrm{x})\,\mathrm{sech(x)})\right)\\ +\xi\cos(4\xi\,\mathrm{sech}(x))\,4\,\mathrm{sech}(x) + (\sin(4\xi\,\mathrm{sech}(x)))d\xi\end{pmatrix} \quad (27)$$

$$u_1 = 4tsech(x) - \frac{4}{3}t^3(sech^3(x) + tanh^2(x)\,\mathrm{sech}(x)) - \frac{t}{4\mathrm{sech}(x)} + \frac{\sin(4tsech(x))}{(4\,\mathrm{sech(x)})^2} \quad (28)$$

$$\vdots$$

$$u(x,t) \cong u_1(x,t) \quad (29)$$

$$u(x,t) = 4\,tan^{-1}\,(t\,sech(x))$$

Table (1) shows a comparison between the results of exact solution and the solution obtained by the LT-VIM of example 1 at *x* = 5 and its illustrate in figure (1). While table (2) shows the absolute error between the exact solution and the solution obtained by the LT-VIM of example 1 at x = 5.

Table (1): Comparison between LT-VIM and exact solutions when x = 5

t	Exact Solution	LT-VIM
0.1	0.0053	0.0053
0.2	0.0107	0.0105
0.3	0.0161	0.0154
0.4	0.0215	0.0198
0.5	0.0269	0.0235

Table (2): The absolute error between exact solution and LT-VIM of example (1) when x = 5

t	Exact Solution	LT-VIM	Absolute Error between exact solution and LT-VIM
0.1	0.0053	0.0053	0
0.2	0.0107	0.0105	2×10^{-4}
0.3	0.0161	0.0154	7×10^{-4}
0.4	0.0215	0.0198	1.7×10^{-3}
0.5	0.0269	0.0235	3.4×10^{-3}

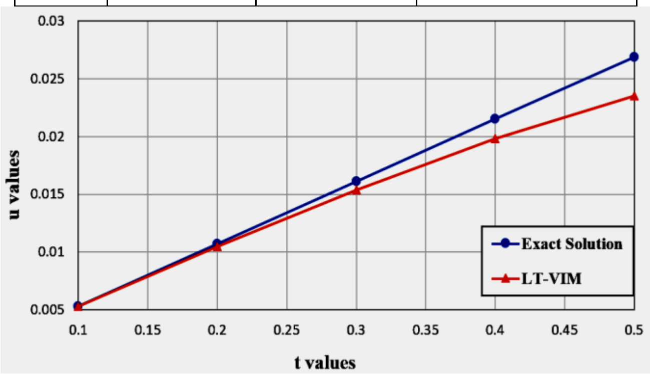

Figure (1): The results of Table 1

It is observed that the exact solution can be reached for any x and t values from the first iteration by using the LT-VIM method, while the Modified Decomposition Method and the Homotopy Perturbation Method have reach the exact solutions from the 4-order approximate solution [5,13]. In addition, the Modified Homotopy Perturbation Method has reach the exact solutions from the 3-order approximate solution [14]. Also, the results show that the suggested method provides the most accurate solution (error less than the error margin).

2.2 Example 2

consider the following sine-Gordon equation

$$u_{tt} - u_{xx} = \sin u \tag{30}$$

subject to the initial conditions:

$$u(x,0) = \pi + \eta \cos\beta x \,, u_t(x,0) = 0 \tag{31}$$

Taking Laplace transform for equation (30):

$$L[u_{tt}] - L[u_{xx}] = L[\sin u] \tag{32}$$

$$s^2 Lu(x,t) - su(x,o) - u_t(x,0) - L[u_{xx}] = L(\sin u) \tag{33}$$

Taking the inverse Laplace transform for the Equation (32):

$$u(x,t) - (\pi + \eta\cos(\beta x)) - L^{-1}\left(\frac{1}{s^2}L[u_{xx}]\right) = L^{-1}\left(\frac{1}{s^2}L[\sin u]\right) \tag{34}$$

Derivate by $\frac{\partial}{\partial t}$ both sides of equation (34):

$$u_t(x,t) - \frac{\partial}{\partial t}\left[L^{-1}\frac{1}{s^2}L[u_{xx}] + L^{-1}\frac{1}{s^2}L[\sin u]\right] = 0 \tag{35}$$

The correction function:

$$u_{n+1} = u_n + \int_0^t \lambda\left((u_n)_\xi(x,\xi) - \frac{\partial}{\partial \xi}\left[L^{-1}\frac{1}{s^2}L[u_n]_{xx} - L^{-1}\frac{1}{s^2}L[\sin u_n]\right]\right) d\xi \tag{36}$$

$$u_0 = \pi + \eta\cos(\beta x) \qquad ; \qquad \lambda = s - t \tag{37}$$

$$u_1 = u_0 + \int_0^t (s-t)\left((u_0)_\xi(x,\xi) - \frac{\partial}{\partial \xi}\left[L^{-1}\frac{1}{s^2}L[u_0]_{xx} - L^{-1}\frac{1}{s^2}L[\sin u_0]\right]\right) d\xi \tag{38}$$

$$u_1 = \pi + \eta\cos(\beta x) + \int_0^t (s-t)\left(0 + \frac{\partial}{\partial \xi}\begin{pmatrix} L^{-1}\frac{1}{s^2}L[\pi + \eta\cos(\beta x)]_{xx} \\ -L^{-1}\frac{1}{s^2}L[\sin(\pi + \eta\cos(\beta x))] \end{pmatrix}\right) d\xi \tag{39}$$

$$u_1 = \pi + \eta\cos(\beta x) + \int_0^t (s-t)\left(\frac{\partial}{\partial \xi}\begin{pmatrix} L^{-1}\frac{1}{s^2}L[-\eta\cos(\beta x)].\beta^2 \\ -L^{-1}\frac{1}{s^2}L[\sin(\eta\cos(\beta x))] \end{pmatrix}\right) d\xi \tag{40}$$

$$u_1 = \pi + \eta\cos(\beta x) - \int_0^t \left((t[-\eta\cos(\beta x)].\beta^2 - t[\sin(\eta\cos(\beta x))]]\right) d\xi \tag{41}$$

$$u_1 = \pi + \eta\cos(\beta x) - \frac{1}{3}\left[[\eta\cos(\beta x)].\beta^2 - \sin(\eta\cos(\beta x))\right]t^3 \tag{42}$$

$$u_1 = \pi + \eta\cos(\beta x) - \frac{t^3}{3}\left[[\eta\cos(\beta x)].\beta^2 - \sin(\eta\cos(\beta x))\right] \tag{43}$$

$$\therefore u(x,t) = \lim_{n\to\infty} \tilde{u}_n \tag{44}$$

$$u(x,t) \cong u_1(x,t) \tag{45}$$

In example 2, it is shown that the LT-VIM gives the same approximation solution of sine-Gordon equation from the first iteration, while the Modified Decomposition Method has reaches the same approximation solution but from the 3-order [5]. In addition, the Reduced Differential Transform Method has reached the approximation solution from the 5th term [2].

3. CONCLUSION

In this work, the combined Laplace transform- Variation Iteration Method (LT-VIM) has been successfully applied for solving models of sin-Gordon equation with initial conditions to achieve exact and approximate solutions. The LT-VIM has worked effectively to handle these models by giving the solutions from the 1st iteration comparing with other methods and give a wider applicability. The results illustrate that the suggested method is a strong mathematical tool to solve sine-Gordon equation and can be a promising method to generate a solution for other nonlinear equations.

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