FILTER BASES AND SUPRA PERFECT FUNCTIONS

Y. Y. Yousif
Department of Mathematics, College of Education for Pure Science Ibn-Al-Haitham, University of Baghdad, Iraq,
E-mail : yoyayousif@yahoo.com

ABSTRACT: *We introduce some new generalizations of some definitions which are, supra closure converge to a point, supra closure directed toward a set, almost supra converges to a set, almost supra cluster point, a set supra H-closed relative, supra closure continuous functions, supra weakly continuous functions, supra compact functions, supra rigid a set, almost supra closed functions and supra perfect functions. And we state and prove several results concerning it.*

Key Words: Filter base, Directed toward a set, Closure converges, Closure directed toward**,** Supra closure converge**,** Almost supra converges, Almost supra cluster, Supra compact function, Supra rigid subset, Supra perfect function**.**
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1. INTRODUCTION AND PRELIMINARIES

The notion "filter" first appeared in F. Riesz [13] and the setting of convergence in terms of filters was sketched by H. Cartan in [5] and [6] and was developed by N. Bourbaki in [4]. G. T. Whyburn in [15] introduces the notion directed toward a set and the generalization of this notion is studied in section 2. R. F. Dickman and J. R. Porter in [7] introduce the notion almost convergence, J. R. Porter and J. R. Thomas in [12] introduce the notion quasi-H-closed and the analogues of this notions are studied in section 3. N. Levine in [8] introduce the notion θ-continuous functions, D. R. Andrew and E. K. Whittlesy in [2] introduce the notion weakly θ-continuous functions, in [7] introduce the notions θ-compact functions, θ-rigid a set, almost closed functions and the analogues of this notions are studied in section 4. In [15] introduce the notion θ-perfect functions and the analogue of this notion is studied in section 5. The neighborhood denoted by nbd. The closure (resp. interior) of a subset A of a space X denoted by Cl(A) (resp. Int(A)).

Definition 1.1. [4] A nonempty family $\Im$ of nonempty subsets of X is said to be filter if it satisfies the following conditions:

(a) If $F_1, F_2 \in \Im$, then $F_1 \cap F_2 \in \Im$,

(b) If $F \in \Im$ and $F \subseteq F^* \subseteq X$, then $F^* \in \Im$.

Definition 1.2. [4] A nonempty family $\Im$ of nonempty subsets of X is said to be filter base if $F_1, F_2 \in \Im$ then $F_3 \subseteq F_1 \cap F_2$ for some $F_3 \in \Im$.

The filter generated by a filter base $\Im$ consists of all supersets of elements of $\Im$. An open filter base on a space X is a filter base with open members. The set $\aleph_x$ of all nbds of $x \in X$ is a filter on X, and any nbd base at x is a filter base for $\aleph_x$. This filter called the nbd filter at x.

Definition 1.3. [4] Let $\Im$ be a filter base on a space X. We say that $\Im$ converges to $x \in X$ (written as $\Im \to x$) iff each open set U about x contains some element $F \in \Im$. We say $\Im$ has x as a cluster point (or $\Im$ cluster at x) iff each open set U about x meets all element $F \in \Im$. Clear that if $\Im \to x$, then $\Im$ cluster at x.

Definition 1.4. [4] Let $\Im$ and G be filter bases on X. Then G is said to be finer than $\Im$ (written as $\Im <$ G) if for all $F \in \Im$, there is $G \in$ G such that $G \subseteq F$ and that $\Im$ meets G if $F \cap G \neq \phi$ for all $F \in \Im$ and $G \in$ G. Notice, $\Im \to x$ iff $\aleph_x < \Im$.

Definition 1.5. [15] Let $\Im$ be a filter base on a space X. We say that $\Im$ directed toward (shortly, *d-t*) a set $A \subseteq X$, provided each filter base finer than $\Im$ has a cluster point in A. (Note: Any filter base cann't be *d-t* the empty set).

Definition 1.6. [4] A filter $\Im$ is said to be an ultrafilter if there is no strictly finer filter G than $\Im$. Thus the ultrafilter are the maximal filters.

Definition 1.7. A subset A of a space X is said to be

(a) *r*-open [14] if A = Int(Cl(A));

(b) *pre*-open [10] if $A \subseteq$ Int(Cl(A)).

(c) *semi*-open [9] if $A \subseteq$ Cl(Int(A)).

(d) *b*-open [3] if $A \subseteq$ Cl(Int(A))$\cup$Int(Cl(A)).

(e) α-open [11] if $A \subseteq$ Int(Cl(Int(A))).

(f) β-open [1] if $A \subseteq$ Cl(Int(Cl(A))).

The complement of an *r*-open (resp. *pre*-open, *semi*-open, *b*-open, α-open, β-open) is said to be *r*-closed (resp. *pre*-closed, *semi*-closed, *b*-closed, α-closed, β-closed).

The supra closure (briefly j-closure) of $A \subseteq X$ is denoted by $\mathrm{Cl}^j(A)$ and defined by $\mathrm{Cl}^j(A) = \cap\{F \subseteq X;\ F$ is j-closed and $A \subseteq F\}$, where j$\in\{r, pre, semi, b, \alpha, \beta\}$.

2. Filter Bases and Closure Directed Toward a Set

Lemma 2.1. [15] Let $f : X \to Y$ be an injective function.

(a) If $\Im = \{F : F \subseteq X\}$ is a filter base in X, then $f(\Im) = \{f(F) : F \in \Im\}$ is a filter base in Y.

(b) If G = $\{G : G \subseteq f(X)\}$ is a filter base in $f(X)$, $\Im = \{f^{-1}(G) : G \in$ G$\}$ is a filter base in X. For each $\phi \neq A \subseteq X$ and any filter base G in $f(A)$, then $\{A \cap f^{-1}(G) : G \in$ G$\}$ is a filter base in A.

(c) If $\Im = \{F : F \subseteq X\}$ is a filter base in X, G = $\{f(F) : F \in \Im\}$, G* is finer than G, and $\Im^* = \{f^{-1}(G^*) : G^* \in$ G*$\}$, then the collection of sets $\Im^{**} = \{F \cap F^*$ for all $F \in \Im$ and $F^* \in \Im^*\}$ is finer than both of $\Im$ and $\Im^*$.

Now, we will generalizations Definitions (1.3) and (1.5) as follows.

Definition 2.2. Let $\Im$ be a filter base on a space X. We say that $\Im$ closure converges to $x \in X$ (written as $\Im \rightsquigarrow x$) iff all open set U about x, the Cl(U) contains some element $F \in \Im$. We say $\Im$ has x as a closure cluster point (or $\Im$ closure cluster at x) iff all open set U about x the Cl(U) meets all element $F \in \Im$.

Clear that if $\Im \rightsquigarrow x$, then $\Im$ closure cluster at x. Cl($\aleph_x$) is used to denote the filter base {Cl(U) : $U \in \aleph_x$}. Notice, $\Im \rightsquigarrow x$ iff Cl($\aleph_x$) < $\Im$.

Definition 2.3. Let $\Im$ be a filter base on a space X. We say that $\Im$ closure directed toward (shortly, *cl-d-t*) a set $A \subseteq X$, provided each filter base finer than $\Im$ has a closure cluster point in A.

Theorem 2.4. Let $\Im$ be a filter base on a space X. $\Im \rightsquigarrow x \in X$ iff $\Im$ is *cl-d-t* x.

Proof. ($\Rightarrow$) If $\Im \rightsquigarrow x$, all open set U about x, Cl(U) contains an element of $\Im$ and thus contains an element of each filter base $\Im^* < \Im$, so that $\Im^*$ actually closure converges to x.

($\Leftarrow$) If $\Im$ is *cl-d-t* x, it must $\Im \rightsquigarrow x$. For if not, there is an open set U in X about x such that Cl(U) don't contains an element of $\Im$. Denote by $\Im^*$ the collection of sets $F^* = F \cap (X - Cl(U))$ for $F \in \Im$, then the sets F^* are nonempty. Also $\Im^*$ is a filter base and indeed $\Im^* < \Im$, because result in $F_1^* = F_1 \cap (X - Cl(U))$ and $F_2^* = F_2 \cap (X - Cl(U))$, so there is an $F_3 \subseteq F_1 \cap F_2$ and this lead to

$F_3^* = F_3 \cap (X - Cl(U)) \subseteq F_1 \cap F_2 \cap (X - Cl(U))$

$= F_1 \cap (X - Cl(U)) \cap F_2 \cap (X - Cl(U))$.

By construction x is not a closure cluster point of $\Im^*$. This contradiction yields that, $\Im \rightsquigarrow x$.

Theorem 2.5. Let $f : X \to Y$ be an injective function and given $B \subset Y$. If for each filter base G in $f(X)$ *cl-d-t* a point $y \in B$, the inverse filter M = $\{f^{-1}(G) : G \in$ G$\}$ is *cl-d-t* $f^{-1}(y)$, then for any filter base $\Im$ in $f(X)$ *cl-d-t* a set B, E = $\{f^{-1}(F) : F \in \Im\}$ is *cl-d-t* $A = f^{-1}(B)$.

Proof. Suppose that the hypothesis is true and any $y \in B$ which is a closure cluster point of a filter base finer than $\Im$ must be in $f(X)$. Thus $B \cap f(X) \neq \phi$, also $\Im$ is *cl-d-t* $B \cap f(X)$. Thus we may assume $B \subseteq f(X)$. Let M be a filter base finer than E. Then G = $\{f(M) : M \in$ M $\}$ finer than $\Im$ by Lemma (3.1, a). Thus G has a closure cluster point z in B and a filter base G* finer than G closure converges to z and thus is *cl-d-t* z. By Assumption M* = $\{f^{-1}(G^*) : G^* \in$ G*$\}$ is *cl-d-t* $f^{-1}(z)$. Also by Lemma (3.1, c), M and M* have a common filter base M** finer than of them. Thus M** has a closure cluster point x in $f^{-1}(z)$. Because x is a closure cluster point of M and $x \in f^{-1}(z) \subset A$, our result follows.

Theorem 2.6. A function $f : X \to Y$ is closed and $f^{-1}(y)$ compact for each $y \in Y$ iff for each filter base $\Im$ in $f(X)$ *cl-d-t* a set $B \subseteq$ Y, the collection E = $\{f^{-1}(F) : F \in \Im\}$ is *cl-d-t* $f^{-1}(B)$.

Proof. ($\Rightarrow$) Assume that f is closed and $f^{-1}(y)$ compact for each $y \in Y$. Then by Theorem (2.4) and (2.5) it suffices to prove that if G is a filter base in $f(X)$ closure converging to $y \in B$, then M = $\{f^{-1}(G) : G \in$ G$\}$ is *cl-d-t* $f^{-1}(y)$. For if not, there is a filter base M* finer than M, no point of $f^{-1}(y)$ is a closure cluster point of M*. For all $x \in f^{-1}(y)$, by assumption there is an open set U_x about x and $M_x^* \in$ M* with $M_x^* \cap U_x = \phi$. Since $f^{-1}(y)$ is compact, there are a finite numbers of open sets U_{x_i} such that $f^{-1}(y) \subseteq U = \cup U_{x_i}$. Let $M^* \in$ M* such that $M^* \subseteq \cap M_{x_i}^*$ and let $V = Y - f(X - U)$ be the open set. Then $f(M^*) \cap V = \phi$ since $M^* \subset X - Cl(U)$. Thus since $f(M^*) \in$ G*, G* cannot have y as a closure cluster point.

($\Leftarrow$) Suppose that the hypothesis is true and f is not closed. Let $A \subseteq X$ be a closed set and for some $y \in Y - f(A)$ is a closure cluster point of $f(A)$. Let G be a filter base of sets $f(A) \cap V$ for each open sets $V \subseteq Y$ such that $y \in V$, then G is a filter base in $f(X)$ and G $\rightsquigarrow y$. Let M = $\{f^{-1}(G) : G \in$ G$\}$ and M* = $\{A \cap M : M \in$ M$\}$. It clear that M* < M. But $X - A$ is open and $f^{-1}(y) \subseteq X - A$, M* has no closure cluster point in $f^{-1}(y)$. This contradiction yields that f be a closed function. Finally, to prove $f^{-1}(y)$ is compact. This is easy for $y \in Y - f(X)$. Also for $y \in f(X)$, $\{y\}$ is a filter base in $f(X)$ *cl-d-t* y. By assumption, $\{f^{-1}(y)\}$ *cl-d-t* $f^{-1}(y)$. This means that all filter base in $f^{-1}(y)$ has a closure cluster point in $f^{-1}(y)$, so that $f^{-1}(y)$ is compact.

Corollary 2.7. A function $f : X \to Y$ is closed and $f^{-1}(y)$ compact for each $y \in Y$ iff each filter base in $f(X) \rightsquigarrow y \in Y$ has pre-image filter base *cl-d-t* $f^{-1}(y)$.

Corollary 2.8. If f : X $\to$ Y is closed and $f^{-1}(y)$ compact for each $y \in Y$, for each compact set $K \subseteq Y$, $f^{-1}(K)$ is compact.

Proof. Let $K \subseteq Y$ be a compact set and $\Im$ is a filter base in $f^{-1}(K)$, G = $\{f(F) : F \in \Im\}$, is a filter base in K and in $f(X)$ and is *cl-d-t* K. Thus $\Im^* = \{f^{-1}(G) : G \in$ G$\}$ is *cl-d-t* $f^{-1}(K)$ so that $\Im^* < \Im$ and $\Im^*$ has a closure cluster point in $f^{-1}(K)$.

3. Filter Bases and Almost Supra Convergence

By analogue of definition almost convergence in [7] we define.

Definition 3.1. Let $\Im$ be a filter base on a space X. We say $\Im$ almost supra converges (briefly almost j-converges) to a subset $A \subseteq X$ (written as $\Im_j \rightsquigarrow A$) if for each cover A of A by subsets open in X, there is a finite subfamily B $\subseteq$ A and $F \in \Im$ such that $F \subseteq \cup\{Cl^j(B) : B \in$ B$\}$. We say $\Im$ almost j-converges to $x \in X$ (written as $\Im_j \rightsquigarrow x$) if $\Im_j \rightsquigarrow \{x\}$. Now, $Cl(\aleph_x) \rightsquigarrow x$, whereas, $Cl^j(\aleph_x)_j \rightsquigarrow x$, where j$\in\{r, pre, semi, b, \alpha, \beta\}$.

Also, we introduce the following definitions:

Definition 3.2. A point $x \in X$ is called an almost supra cluster (briefly almost j-cluster) point of a filter base $\Im$ (written as $x \in al_jc_X\Im$) if $\Im$ meets $Cl^j(\aleph_x)$, where j$\in\{r, pre, semi, b, \alpha, \beta\}$.

For a set $A \subseteq X$, the almost j-closure of A, denoted as $al_jCl(A)$ is $al_jc_X\{A\}$ if $A \neq \phi$ i.e. $\{x \in X$: every j-closed nbd of x meets $A\}$ and is ϕ if $A = \phi$; A is almost j-closed if $A = al_jCl(A)$. Correspondingly, the almost j-interior of A, denoted as al_jIntA, is $\{x \in X$; $Cl^j(U) \subseteq A$ for some open set U containing $x\}$; A is almost j-interior if $A = al_jInt(A)$, where j$\in\{r, pre, semi, b, \alpha, \beta\}$.

Theorem 3.3. Let $\Im$ and G be filter bases on a space X, $A \subseteq X$ and $x \in X$.

(a) If $\Im_j \rightsquigarrow A$, then $Cl^j(\aleph_A) < \Im$.
(b) If $\Im_j \rightsquigarrow x$, iff $Cl^j(\aleph_x) < \Im$.
(c) If $\Im <$ G, then al_jc_XG $\subseteq al_jc_X\Im$.
(d) If $\Im <$ G and $\Im_j \rightsquigarrow A$, then $G_j \rightsquigarrow A$.
(e) $al_jc_X\Im = \cap\{Cl^j(F) : F \in \Im\}$.
(f) If $\Im_j \rightsquigarrow x$ and $x \in A$, then $\Im_j \rightsquigarrow A$.
(g) If $\Im_j \rightsquigarrow A$ iff $\Im_j \rightsquigarrow A \cap al_jc_X\Im$.
(h) If $\Im_j \rightsquigarrow A$, then $A \cap al_jc_X\Im \neq \phi$.
(i) If $U \subseteq X$ is open, then $al_jCl(U) = Cl(U)$.
(j) If $\Im$ is a open filter base, then $al_jCl\Im = al_jc_X\Im$.
(k) If U is an open ultrafilter on X, then U$\rightsquigarrow x$ iff $U_j \rightsquigarrow x$. Where j$\in\{r, pre, semi, b, \alpha, \beta\}$.

Proof. The proof is easy, so it is omitted.

By analogue of definition quasi-H-closed relative in [12] we define.

Definition 3.4. The subset A of a space X is said to be quasi-supra H-closed (briefly quasi-$_j$H-closed) relative to X if every cover A of A by open subsets of X contains a finite subfamily B $\subseteq$ A such that $A \subseteq \cup\{Cl^j(B) : B \in$ B $\}$. If X is Hausdorff, we say that A is $_j$H-closed relative to X. If X is quasi-$_j$H-closed relative to itself, then X is said to be quasi-$_j$H-closed (resp. $_j$H-closed), where j$\in\{r, pre, semi, b, \alpha, \beta\}$.

Theorem 3.5. The following are equivalent for a subset $A \subseteq X$:

(a) A is quasi-$_j$H-closed relative to X.
(b) For all filter base $\Im$ on A, $\Im_j \rightsquigarrow A$.
(c) For all filter base $\Im$ on A, $al_jc_X\Im \cap A \neq \phi$. Where $j \in \{r, pre, semi, b, \alpha, \beta\}$

Proof. Clearly (a) $\Rightarrow$ (b), and by Theorem (3.3, h), (b) $\Rightarrow$ (c).To show (c) $\Rightarrow$ (a), let A be a cover of A by open subsets of X such that the j-closed of the union of any finite subfamily of A is not cover A. Then $\Im = \{A - Cl^j_X(\cup_S U_s) : S$ is finite subfamily of A$\}$ is a filter base on A and $al_jc_X\Im \cap A = \phi$. This contradiction yields that A is quasi-$_j$H-closed relative to X, where $j \in \{r, pre, semi, b, \alpha, \beta\}$.

By concepts of closure directed toward a set and almost j-convergence are characterized and related in the next result.

Theorem 3.6. Let $\Im$ be a filter base on a space X and $A \subseteq X$.
(a) Then $\Im$ is *cl-d-t A* iff for all cover A of A by open subsets of X, there is a finite subfamily $B \subseteq A$ and an $F \in \Im$ such that $F \subseteq \cup\{Cl^j(B) : B \in B\}$, where $j \in \{r, pre, semi, b, \alpha, \beta\}$.
(b) Then for every filter base G, $\Im <$ G implies $al_jc_X G \cap A \neq \phi$ iff $\Im_j \rightsquigarrow A$, where $j \in \{r, pre, semi, b, \alpha, \beta\}$.

Proof. The proof of the two facts are similar; so, we will only prove the fact (b):

($\Rightarrow$) Suppose for every filter base G, $\Im <$ G implies $al_jc_X G \cap A \neq \phi$. If $\Im_j \rightsquigarrow x$ for some $x \in A$, then by Theorem (3.3, f), $\Im_j \rightsquigarrow A$. So, suppose that for every $x \in A$, $\Im$ does not $_j\rightsquigarrow x$. Let A be a cover of A by subsets open in X. For each $x \in A$, there is an open set U_x containing x and $V_x \in$ A such that $U_x \subseteq V_x$ and $F - Cl^j_X(U_x) \neq \phi$ for every $F \in \Im$. Thus, $G_x = \{F - Cl^j_X(U_x) : F \in \Im\}$ is a filter base on X and $\Im < G_x$. Now, $x \notin al_jc_X G_x$. Assume that $\cup\{G_x : x \in A\}$ forms a filter subbase with G denoting the generated filter. Then $\Im <$ G and $al_jc_X G \cap A = \phi$. This contradiction implies there is a finite subset $B \subseteq A$ and $F_x \in \Im$ for $x \in B$ such that. $\phi = \cap\{F_x - Cl^j_X(U_x) : x \in B\}$.There is $F \in \Im$ such that $F \subset \cap\{F_x : x \in B\}$. It easily follows that $\phi = \cap\{F - Cl^j_X(U_x) : x \in B\}$ and $F \subseteq \cup\{Cl^j_X(V_x) : x \in B\}$. Thus $\Im_j \rightsquigarrow A$.

($\Leftarrow$) Suppose $\Im_j \rightsquigarrow A$ and G is a filter base such that $\Im <$ G. By Theorem (3.3, d), $G_j \rightsquigarrow A$, and Theorem (3.3, h), $al_jc_X G \cap A \neq \phi$.

4. Filter Bases and Supra Rigidity

By analogues of definitions θ-continuous functions in [12] and weakly θ-continuous functions in [8] we define.

Definition 4.1. A function $f : X \to Y$ is said to be j-closure continuous (resp. j-weakly continuous) if for every $x \in X$ and every nbd V of $f(x)$, there exists a nbd U of x in X such that $f(Cl^j(U)) \subseteq Cl^j(V)$ (resp. $f(U) \subseteq Cl^j(V)$). Clearly, every continuous function is j-closure continuous, where $j \in \{pre, semi, b, \alpha, \beta\}$.

The notions of almost j-convergence and almost j-cluster can be used to characterize j-closure continuous.

Theorem 4.2. Let $f : X \to Y$ be a function. The following are equivalent:
(a) f is j-closure continuous.
(b) For all filter base $\Im$ on X, $\Im_j \rightsquigarrow x$ implies $f(\Im) \to f(x)$.
(c) For all filter base $\Im$ on X, $f(al_jc\Im) \subseteq al_jc\, f(\Im)$.
(d) For all open $U \subseteq Y$, $f^{-1}(U) \subseteq al_jInt f^{-1}(al_jCl(U))$. Where $j \in \{pre, semi, b, \alpha, \beta\}$

Proof. The proof of the equivalence of (a), (b) and (d) is straightforward.

(a) $\Rightarrow$ (c) Suppose $\Im$ is a filter base on X, $x \in al_jc\Im$, $F \in \Im$ and V is a nbd of $f(x)$. There is a nbd U of x such that $f(Cl^j(U)) \subseteq Cl^j(V)$. Since $Cl^j(U) \cap F \neq \phi$, then $Cl^j(V) \cap f(F) \neq \phi$. So, $f(x) \in al_jc\, f(\Im)$. This shows that $f(al_jc\Im) \subseteq al_jc\, f(\Im)$.

(c) $\Rightarrow$ (a) Let U be an ultrafilter containing $f(Cl^j(\aleph_x))$. Now, $f^{-1}(U)$ is a filter base since $f(X) \in$ U and $f^{-1}(U)$ meets $Cl^j(\aleph_x)$. So, $f^{-1}(U) \cup Cl^j(\aleph_x)$ is contained in some ultrafilter V. Now $f f^{-1}(U)$ is an ultrafilter base that generates U. Since $f f^{-1}(U) < f(V)$, then $f(V)$ also generates U; hence $al_jcf(V) = al_jcU$. Since $x \in al_jc(V)$, then $f(x) \in f(al_jcV) \subseteq al_jc\, f(V) = al_jcU$. So, U meets $Cl^j(\aleph_{f(x)})$ and $Cl^j(\aleph_{f(x)}) \subseteq \cap\{U : U$ ultrafilter, $U \supseteq f(Cl^j(\aleph_x))\}$, (denote this intersection by G). But G is the filter generated by $(Cl^j(\aleph_x))$ (see [4] Proposition I.6.6); so $Cl^j(\aleph_{f(x)}) < f(Cl^j(\aleph_x))$. Hence f is j-closure continuous, where $j \in \{pre, semi, b, \alpha, \beta\}$.

Corollary 4.3. If $f : X \to Y$ is j-closure continuous and $A \subseteq X$, then $f(al_jCl(A)) \subseteq al_jCl(f(A))$, where $j \in \{pre, semi, b, \alpha, \beta\}$.

Here are some similarly proven facts about j-weakly continuous functions.

Theorem 4.4. Let $f : X \to Y$ be a function. The following are equivalent:
(a) f is j-weakly continuous.
(b) For all filter base $\Im$ on X, $\Im \to x$ implies $f(\Im)_j \rightsquigarrow f(x)$.
(c) For all filter base $\Im$ on X, $f(al_jc\Im) \subseteq al_jc\, f(\Im)$.
(d) For all open $U \subseteq Y$, $f^{-1}(U) \subseteq Int\, f^{-1}(Cl^j(U))$. Where $j \in \{pre, semi, b, \alpha, \beta\}$.

Theorem 4.5. If $f : X \to Y$ is j-weakly continuous, then
(a) For all $A \subseteq X$, $f(Cl^j(A)) \subseteq al_jCl\, f(A)$.
(b) For all $B \subseteq Y$, $f(Cl^j(Int(Cl^j f^{-1}(B)))) \subseteq Cl^j(B)$.
(c) For all open $U \subseteq Y$, $f(Cl^j(U)) \subseteq Cl^j f(U)$. Where $j \in \{pre, semi, b, \alpha, \beta\}$.

By analogues of definitions θ-compact functions, θ-rigid a set and almost closed in [7] we define.

Definition 4.6. A function $f : X \to Y$ is said to be supra compact (briefly j-compact) if for every subset K quasi-$_j$H-closed relative to Y, $f^{-1}(K)$ is quasi-$_j$H-closed relative to X, where $j \in \{pre, semi, b, \alpha, \beta\}$.

Definition 4.7. A subset A of a space X is said to be supra rigid (briefly j-rigid) provided whenever $\Im$ is a filter base on X and $A \cap al_jc_X\Im = \phi$, there is an open U containing A and $F \in \Im$ such that $Cl^j(U) \cap F = \phi$, where $j \in \{pre, semi, b, \alpha, \beta\}$.

Definition 4.8. A function $f : X \to Y$ is said to be almost supra closed (briefly almost j-closed) if for any set $A \subseteq X$, $f(al_jCl(A)) = al_jCl\, f(A)$, where $j \in \{pre, semi, b, \alpha, \beta\}$.

Definition 4.9. A space X is said to be supra Urysohn (briefly j-Urysohn) if every pair of distinct points are contained in disjoint j-closed nbds, where $j \in \{pre, semi, b, \alpha, \beta\}$.

Before characterizing j-rigidity, we show that a j-closure continuous, j-compact function into a j-Urysohn space with a certain property (the "j-closure" and "quasi-$_j$H-closed relative" analogue of property α in [15]) is almost j-closed.

Theorem 4.10. Suppose $f : X \to Y$ is a j-closure continuous and j-compact and Y is j-Urysohn with this property: For each $B \subseteq Y$ and $y \in al_jCl(B)$, there is a subset K quasi-$_j$H-closed relative to Y such that $y \in al_jCl(K \cap B)$. Then f is almost j-closed, where $j \in \{pre, semi, b, \alpha, \beta\}$.

Proof. Let $A \subseteq X$. By corollary (4.3), $f(al_jCl(A)) \subseteq al_jClf(A)$. Suppose $y \in al_jClf(A)$. There is a subset K quasi-$_j$H-closed relative to Y such that $y \in al_jCl(K \cap f(A))$. Then $\Im = \{Cl^j(U) \cap K \cap f(A) : U \in \aleph_y\}$, is a filter base on Y such that $\Im_j \rightsquigarrow y$. Now, $G = \{A \cap f^{-1}(F) : F \in \Im\}$ is a filter base on $A \cap f^{-1}(K)$. Since $f^{-1}(K)$ is quasi-$_j$H-closed relative to X, then there is $x \in al_jc_X G \cap f^{-1}(K)$. By theorem (4.2), $f(x) \in al_jc_Y f(G) \subseteq al_jc_Y\Im$. Since $\Im_j \rightsquigarrow y$ and Y is j-Urysohn, $al_jc_Y\Im = \{y\}$. Thus, $y \in f(al_jCl(A))$, where $j \in \{pre, semi, b, \alpha, \beta\}$.

Theorem 4.11. Let A be a subset of a space X. The following are equivalent:

(a) A is j-rigid in X.

(b) For all filter base $\Im$ on X, if $A \cap al_jc_X\Im = \phi$, then for some $F \in \Im$, $A \cap al_jCl(F) = \phi$.

(c) For all cover A of A by open subsets of X, there is a finite subfamily $B \subseteq A$ such that $A \subseteq Int\, Cl^j(\cup B)$. Where $j \in \{pre, semi, b, \alpha, \beta\}$.

Proof. The proof that (a) $\Rightarrow$ (b) is straightforward. (b) $\Rightarrow$ (c) Let A be a cover of A by open subsets of X and $\Im = \{\cap_{U \in B} (X - Cl^j(U)) : B$ is a finite subset of A$\}$. If $\Im$ is not a filter base, then for some finite subfamily $B \subseteq A$, $X \subseteq \cup\{Cl^j(U) : U \in B\}$; thus, $A \subseteq X \subseteq Int\, Cl^j(\cup B)$ which completes the proof in the case that $\Im$ is not a filter base. So, suppose $\Im$ is a filter base. Then $A \cap al_jc\Im = \phi$ and there is an $F \in \Im$ such that $A \cap al_jCl(F) = \phi$. For each $x \in A$, there is open V_x of x such that $Cl^j(V_x) \cap F = \phi$. Let $V = \cup\{V_x : x \in A\}$. Now, $V \cap F = \phi$. Since $F \in \Im$, then for some finite subfamily $B \subseteq A$, $F = \cap\{X - Cl^j(U) : U \in B\}$. It follows that $V \subseteq Cl^j(\cup B)$ and hence, $A \subseteq Int\, Cl^j(\cup B)$, where $j \in \{pre, semi, b, \alpha, \beta\}$.

(c) $\Rightarrow$ (a) Let $\Im$ be a filter base on X such that $A \cap al_jc\Im = \phi$. For all $x \in A$ there is open V_x of x and $F_x \in \Im$ such that $Cl^j(V_x) \cap F_x = \phi$. Now $\{V_x : x \in A\}$ is a cover of A by open subsets of X; so, there is finite subset $B \subseteq A$ such that $A \subseteq Int\, Cl^j(\cup\{V_x : x \in B\})$. Let $U = Int\, Cl^j(\cup\{V_x : x \in B\})$. There is $F \in \Im$ such that $F \subseteq \cap\{F_x : x \in B\}$. Since $Cl^j(U) = \cup\{Cl^j(V_x) : x \in B\}$, then $Cl^j(U) \cap F = \phi$. Thus A is j-rigid in X, where $j \in \{pre, semi, b, \alpha, \beta\}$.

5. Filter Bases and Supra Perfect Functions

In Corollary (2.7) prove that a function $f : X \to Y$ is perfect (i.e. closed and $f^{-1}(y)$ compact for each $y \in Y$) iff for all filter base $\Im$ on $f(X)$, $\Im \rightsquigarrow y \in Y$, implies $f^{-1}(\Im)$ is *cl-d-t* $f^{-1}(y)$ and in Corollary (2.8) proved that a perfect function is compact (i.e. inverse image of compact sets are compact). In view Theorem (3.6), we say that a function $f : X \to Y$ is supra perfect (briefly j-perfect) if for every filter base $\Im$ on $f(X)$, $\Im_j \rightsquigarrow y \in Y$ implies $f^{-1}(\Im)_j \rightsquigarrow f^{-1}(y)$, where $j \in \{pre, semi, b, \alpha, \beta\}$.

Theorem 5.1. Let $f : X \to Y$ be a function. The following are equivalent:

(a) f is j-perfect.

(b) For all filter base $\Im$ on X, $al_jc\, f(\Im) \subseteq f(al_jc\Im)$.

(c) For all filter base $\Im$ on $f(X)$, $\Im_j \rightsquigarrow B \subseteq Y$, implies $f^{-1}(\Im)_j \rightsquigarrow f^{-1}(B)$. Where $j \in \{pre, semi, b, \alpha, \beta\}$.

Proof. (a) $\Rightarrow$ (b) Suppose $\Im$ is a filter base on X and $y \in al_jc\, f(\Im)$. For if not. Assume that $f^{-1}(y) \cap al_jc\Im = \phi$. For each $x \in f^{-1}(y)$, there is open U_x of x and $F_x \in \Im$ such that $Cl^j(U_x) \cap F_x = \phi$. Since $f^{-1}(Cl^j(\aleph_y))_j \rightsquigarrow f^{-1}(y)$ and $\{U_x : x \in f^{-1}(y)\}$ is an open cover of $f^{-1}(y)$, there is a $V \in \aleph_y$ and a finite subset $B \subseteq f^{-1}(y)$ such that $f^{-1}(Cl^j(V)) \subseteq \cup\{Cl^j(U_x) : x \in B\}$. There is an $F \in \Im$ such that $F \subseteq \cap\{F_x : x \in B\}$. Thus, $F \cap f^{-1}(Cl^j(V)) = \phi$ implying $Cl^j(V) \cap f(F) = \phi$, a contradiction as $y \in al_jc\, f(\Im)$. This shows that $y \in f(al_jc\, \Im)$, Where $j \in \{pre, semi, b, \alpha, \beta\}$.

(b) $\Rightarrow$ (c) Suppose $\Im$ is a filter base on $f(X)$ and $\Im_j \rightsquigarrow B \subseteq Y$. Let G be a filter base on X such that $f^{-1}(\Im) < G$. Then $\Im < f(G)$ and $al_jc\, f(G) \cap B \neq \phi$. Hence $f(al_jc\, G) \cap B \neq \phi$ and $al_jc\, G \cap f^{-1}(B) \neq \phi$. By Theorem (3.6, b), $f^{-1}(\Im)_j \rightsquigarrow f^{-1}(B)$, Where $j \in \{pre, semi, b, \alpha, \beta\}$.

(c) $\Rightarrow$ (a) Clearly.

Corollary 5.2. If $f : X \to Y$ is j-perfect, then:

(a) For all $A \subseteq X$, $al_jClf(A) \subseteq f(al_jClA)$.

(b) For all almost j-closed $A \subseteq X$, $f(A)$ is almost j-closed.

(c) f is j-compact. Where $j \in \{pre, semi, b, \alpha, \beta\}$.

Proof. (a) Is an immediate consequence of Theorem (5.1), and (b) follows easily from (a). To prove (c) Let K be quasi-$_j$H-closed relative to Y, and G be a filter base on $f^{-1}(K)$, then $f(G)$ is a filter base on K. By Theorem (3.5), $al_jcf(G) \cap K \neq \phi$ and by Theorem (5.1, b), $al_jcG \cap f^{-1}(K) \neq \phi$. By Theorem (3.5), $f^{-1}(K)$ is quasi-$_j$H-closed relative to X, where $j \in \{pre, semi, b, \alpha, \beta\}$.

Theorem 5.3. An j-closure continuous function $f : X \to Y$ is j-perfect iff

(a) f is almost j-closed, and

(b) $f^{-1}(y)$ j-rigid for each $y \in Y$, where $j \in \{pre, semi, b, \alpha, \beta\}$.

Proof. ($\Rightarrow$) If f is j-closure continuous and j-perfect, then by Corollaries (5.2) and (4.3), f is almost j-closed. To show $f^{-1}(y)$, for $y \in Y$, is j-rigid, Let $\Im$ be a filter base on X such that $f^{-1}(y) \cap al_jc\Im = \phi$. So, $y \notin f(al_jc\Im)$ and by Theorem (5.1, b), $y \notin al_jc\, f(\Im)$. There is open U of y and $F \in \Im$ such that $Cl^j(U) \cap f(F) = \phi$. Therefore, $f^{-1}(Cl^j(U)) \cap F = \phi$. Since f is j-closure continuous, then for any $x \in f^{-1}(y)$, there is open V of x such that $Cl^j(V) \subseteq f^{-1}(Cl^j(U))$. So, $f^{-1}(y) \cap Cl_j(F) = \phi$, where $j \in \{pre, semi, b, \alpha, \beta\}$.

($\Leftarrow$) Suppose a j-closure continuous function f satisfies (a) and (b). Let $\Im$ be a filter base on $f(X)$ such that $\Im_j \rightsquigarrow y$. Let G be a filter base on X such that $f^{-1}(\Im) < G$. So, $\Im < f(G)$ implying that $y \in al_jc\, f(G)$. So, for every $G \in G$, $y \in al_jClf(G) \subseteq f(al_jClG)$. Hence, $f^{-1}(y) \cap al_jClG \neq \phi$ for every $G \in G$. By (b), $f^{-1}(y) \cap al_jcG \neq \phi$. By Theorem (5.1), f is j-perfect, where $j \in \{pre, semi, b, \alpha, \beta\}$.

Actually, in the proof of the converse of Theorem (5.3), we have shown that property (a) of Theorem (5.3) can be reduced to this statement: For each $A \subseteq X$, $al_jClf(A) \subseteq f(al_jClA)$; in fact, we have shown the next corollary (the function is not necessarily j-closure continuous).

Corollary 5.4. Let $f : X \to Y$. If (a) for all $A \subseteq X$, $al_jClf(A) \subseteq f(al_jClA)$ and (b) $f^{-1}(y)$ j-rigid for each $y \in Y$, then f is j-perfect, where $j \in \{pre, semi, b, \alpha, \beta\}$.

Corollary 5.5. Let $f : X \to Y$. (a) f is almost j-closed, and (b) $f^{-1}(y)$ j-rigid for each $y \in Y$, then f^{-1} preserves j-rigidity, where $j \in \{pre, semi, b, \alpha, \beta\}$.

Proof. Let $K \subseteq Y$ be j-rigid and $\Im$ be a filter base on X such that $al_jc_X\Im \cap f^{-1}(K) = \phi$. By Corollary (5.4) and Theorem (5.1), $al_jcf(\Im) \cap K = \phi$. So, there is $F \in \Im$ such that $al_jClf(F) \cap K = \phi$. But $al_jClf(F) = f(al_jClF)$. So, $al_jCl(F) \cap f^{-1}(K) = \phi$. So, by Theorem (4.11), $f^{-1}(K)$ is j-rigid, where $j \in \{pre, semi, b, \alpha, \beta\}$.

Theorem 5.6. Suppose $f : X \to Y$ has j-rigid point-inverses. Then:

(a) f is j-closure continuous iff for each $y \in Y$ and open set V containing y, there is an open set U containing $f^{-1}(y)$ such that $f(\mathrm{Cl}^j(U)) \subseteq \mathrm{Cl}^j(V)$, where j$\in$ {*pre*, *semi*, *b*, α, β}.

(b) If for each $y \in Y$ and open set U containing $f^{-1}(y)$, there is an open set V of y such that $f^{-1}(\mathrm{Cl}^j(V)) \subseteq \mathrm{Cl}^j(U)$, then for each $A \subseteq X$, $al_j\mathrm{Cl}(f(A) \subseteq f(al_j\mathrm{Cl}(A))$, where j$\in$ {*pre*, *semi*, *b*, α, β}.

Proof. (a) (⇒) Is obvious.

(⇐) Is straightforward using Theorem (4.11, c)

(b) Let $\phi \neq A \subseteq X$ and $y \notin f(al_j\mathrm{Cl}(A))$. Then $f^{-1}(y) \cap al_j\mathrm{Cl}(A) = \phi$. Now, $\Im = \{A\}$ is a filter base and $al_jc\Im \cap f^{-1}(y) = \phi$. So, there is open set U continuing $f^{-1}(y)$ such that $\mathrm{Cl}^j(U) \cap A = \phi$. There is open V of y such that $f^{-1}(\mathrm{Cl}^j(V)) \subseteq \mathrm{Cl}^j(U)$. So, $\mathrm{Cl}^j(V) \cap f(A) = \phi$. Hence $y \notin al_j\mathrm{Cl}f(A)$, where j$\in$ {*pre*, *semi*, *b*, α, β}.

The next result is closely related to Theorem (5.6, b); the proof is straightforward.

Theorem 5.7. Let $f : X \to Y$. The following are equivalent:

(a) For all j-closed $A \subseteq X$, $f(A)$ is j-closed, where j$\in$ {*pre*, *semi*, *b*, α, β}.

(b) For all $B \subseteq Y$ and j-open U containing $f^{-1}(B)$, there is j-open V containing B such that $f^{-1}(V) \subseteq U$, where j$\in$ {*pre*, *semi*, *b*, α, β}.

Theorem 5.8. If $f : X \to Y$ is j-closure continuous and Y is j-Urysohn, then f is j-perfect iff for all filter base $\Im$ on X, if $f(\Im)_j \rightsquigarrow y \in Y$, then $al_jc_X\Im \neq \phi$, where j$\in$ {*pre*, *semi*, *b*, α, β}.

Proof. (⇒) Suppose f is j-perfect and $f(\Im)_j \rightsquigarrow y$. So, $f^{-1}f(\Im)_j \rightsquigarrow f^{-1}(y)$. Since $f^{-1}f(\Im) < \Im$, then by Theorem (3.3, d), $\Im_j \rightsquigarrow f^{-1}(y)$, by Theorem (3.3, h), $al_jc\ \Im \neq \phi$.

(⇐) Suppose for every filter base $\Im$ on X, if $f(\Im)_j \rightsquigarrow y \in Y$, then $al_jc_X\Im \neq \phi$. Suppose G is a filter base on $f(X)$ such that $G_j \rightsquigarrow y \in Y$, and assume H is a filter base on X such that $f^{-1}(G) < H$. Then $G = ff^{-1}(G) < f(H)$. So, $f(H)_j \rightsquigarrow y$. Hence, $al_jc_XH \neq \phi$. Let $z \in Y - \{y\}$. Since Y is j-Urysohn, there are open sets U_z of z and U_y of y such that $\mathrm{Cl}^j(U_z) \cap \mathrm{Cl}^j(U_y) = \phi$. There is $H \in$ H such that $f(H) \subseteq \mathrm{Cl}^j(U_y)$. For each $x \in f^{-1}(z)$, there is open V_x of x such that $f(\mathrm{Cl}^j(V_x)) \subseteq \mathrm{Cl}^j(U_z)$. So, $\mathrm{Cl}^j(V_x) \cap H = \phi$. It follows that $f^{-1}(z) \cap al_jc_XH = \phi$ for each $z \in Y - \{y\}$. So, $al_jc_XH \cap f^{-1}(y) \neq \phi$ and f is j-perfect, where j$\in$ {*pre*, *semi*, *b*, α, β}.

Corollary 5.9. If $f : X \to Y$ is j-closure continuous, X is quasi-$_j$H-closed, and Y is j-Urysohn, then f is j-perfect, where j$\in${*pre*, *semi*, *b*, α, β}.

Proof. Since X is quasi-$_j$H-closed, then all filter base on X has nonvoid almost j-cluster; now, the corollary follows directly from Theorem (5.3), Where j$\in${*pre*, *semi*, *b*, α, β}.

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