POLYNOMIOGRAPHY VIA AN ITERATIVE METHOD SUGGESTED BY S. K. KHATTRI AND I. K. ARGYROS

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ABSTRACT*: In this paper, we present polynomiography using an iterative method for solving nonlinear equations suggested by S. K. Khattri and I. K. Argyros, having convergence of fourth order. Polynomiography is the art and science of visualization in approximation of zeros of complex polynomials. The images thus obtained are called polynomiographs. The obtained polynomiographs are very interesting and quite new. We believe that the results of this paper enrich the functionality of the existing polynomiography software.*

Keywords: *Polynomials, Iterative method, Fractals, Polynomiographs.*

INTRODUCTION

Polynomials are one of the most significant objects in many fields of mathematics. Polynomial root-finding has played a key role in the history of mathematics. It is one of the oldest and most deeply studied mathematical problems. The last interesting contribution to the polynomials root finding history was made by Kalantari [16,17], who introduced the polynomiography. As a method which generates nice looking graphics, it was patented by Kalantari in USA in 2005 [17,18]. Polynomiography is defined to be "the art and science of visualization in approximation of the zeros of complex polynomials, via fractal and non fractal images created using the mathematical convergence properties of iteration functions" [16]. An individual image is called a "polynomiograph". Polynomiography combines both art and science aspects. Polynomiography gives a new way to solve the ancient problem by using new algorithms and computer technology. Polynomiography is based on the use of one or an infinite number of iterative methods formulated for the purpose of approximation of the root of polynomials e.g. Newton's method, Halley's method, Householder's method etc. The word "fractal", which partially appeared in the definition of polynomiography, was coined by the famous mathematician Benoit Mandelbrot [15]. Both fractal images and polynomiographs can be obtained via different iterative schemes. Fractals are self-similar has typical structure and independent of scale. On the other hand, polynomiographs are quite different. The "polynomiographer" can control the shape and designed in a more predictable way by using different iterative methods to the infinite variety of complex polynomials. Generally, fractals and polynomiographs belong to different classes of graphical objects.

Polynomiography has diverse applications in mathematics, science, education, art and design. According to Fundamental Theorem of Algebra, any complex polynomial with complex coefficients $\{a_n, a_{n-1}, ..., a_1, a_0\}$

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0 \quad (1)$$

of degree n has n roots (zeros) which may or may not be distinct. The degree of polynomial describes the number of basins of attraction and placing roots on the complex plane manually localization of basins can be controlled.

Usually, polynomiographs are colored based on the number of iterations needed to obtain the approximation of some polynomial root with a given accuracy and a chosen iteration method. The description of polynomiography, its theoretical background and artistic applications are described in [16,17,18].

ITERATION

During the last century, various numerical techniques for solving nonlinear equation $f(x) = 0$ have been successfully applied. For examples see [1-8, 12-14], and the reference therein. Now we define:

For a given x_0 , compute the approximate solution x_{n+1} by the following iterative schemes:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = y_n - \frac{f(y_n)}{\lambda f'^{(x_n)} + (1-\lambda)\left(\frac{f(y_n) - f(x_n)}{y_n - x_n}\right)},$$

where $\lambda = -1$

Which is a fouth order iterative method (KAM) for solving nonlinear equations, suggested by S. K. Khattri and I. K. Argyros in 2010 [20]. Let $p(z)$ be the complex polynomial, then

$$y_n = z_n - \frac{p(z_n)}{p'(z_n)}$$

$$z_{n+1} = y_n - \frac{f(y_n)}{\lambda f'(z_n) + (1-\lambda)\left(\frac{f(y_n)-f(z_n)}{y_n - z_n}\right)}, \quad \text{where } \lambda = -1$$

where $z_o \in \mathbb{C}$ is a starting point, is a fouth order iterative method (KAM) for solving nonlinear complex equations. The sequence $\{z_n\}_{n=0}^{\infty}$ is called the orbit of the point z_o converges to a root z^* of p then, we say that z_o is attracted to z^*. A set of all such starting points for which $\{z_n\}_{n=0}^{\infty}$ converges to root z^* is called the basin of attraction of z^*.

APPLICATIONS

The applications of the fouth order iterative method (KAM) for solving nonlinear equations for solving nonlinear complex equations perturbs the shape of polynomial basins and makes the polynomiographs look more “fractal". The aim of using the fouth order iterative method for solving nonlinear equations for solving nonlinear complex equations is to create images that are quite new, different from images by the Newton's method and Householder's method free from second derivatives [2] and [9,10,11], and interesting from the aesthetic point of view.

In this section we present some examples of polynomiographs for different complex polynomials equation $p(z)=0$. The different colors of images depend upon number of iterations to reach a root with given accuracy $\varepsilon = 0.001$. One can obtain infinitely many nice looking polynomiographs by changing parameter k, where k is the upper bound of the number of iterations.

Polynomiograph for $z^2 - 1 = 0$

The polynomiograph of the complex polynomial $z^2 - 1 = 0$, via KAM is presented in the following figure :

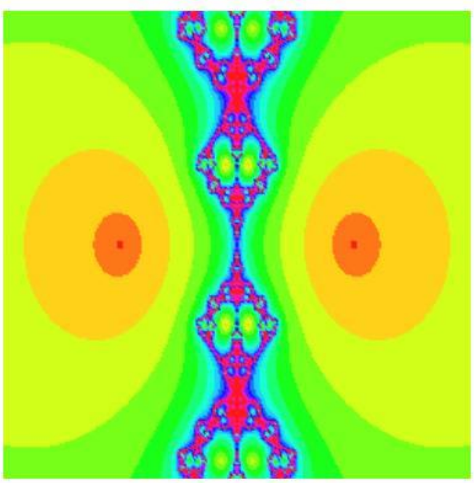

Fig. 1. Polynomiography for $z^2 - 1 = 0$.

Polynomiograph for $z^2 + z - 1 = 0$

The polynomiograph of the complex polynomial $z^2 + z - 1 = 0$, via KAM is presented in the following figure :

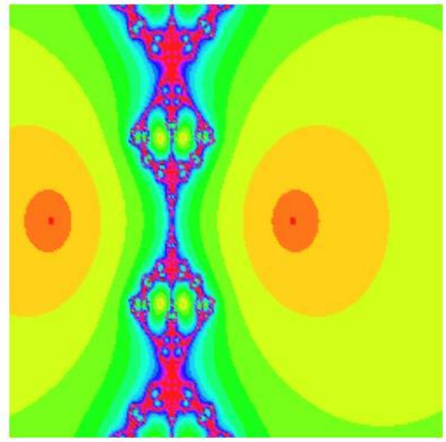

Fig. 2. Polynomiography for $z^2 + z - 1 = 0$.

Polynomiograph for $z^3 - 1 = 0$

The polynomiograph of the complex polynomial equation $z^3 - 1 = 0$, via KAM is presented in the following figure :

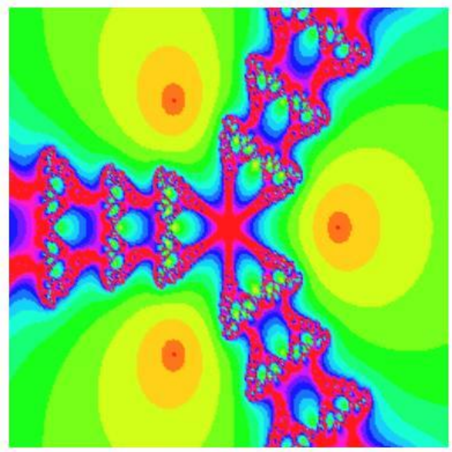

Fig. 3. Polynomiography for $z^3 - 1 = 0$.

Polynomiograph for $z^3 + z - 3 = 0$

The polynomiograph of the complex polynomial equation $z^3 + z - 3 = 0$, via KAM is presented in the following figure :

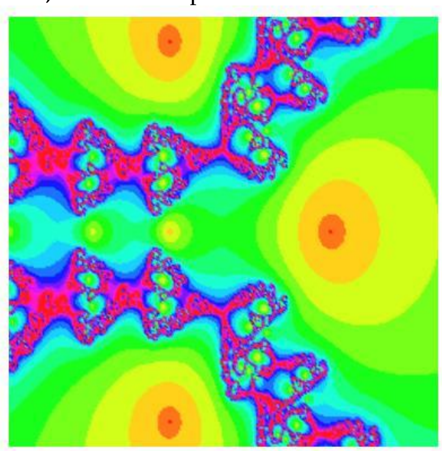

Fig. 4. Polynomiography for $z^3 + z - 3 = 0$.

Polynomiograph for $z^4 - 1 = 0$

The polynomiograph of the complex polynomial equation $z^4 - 1 = 0$, via KAM is presented in the following figure :

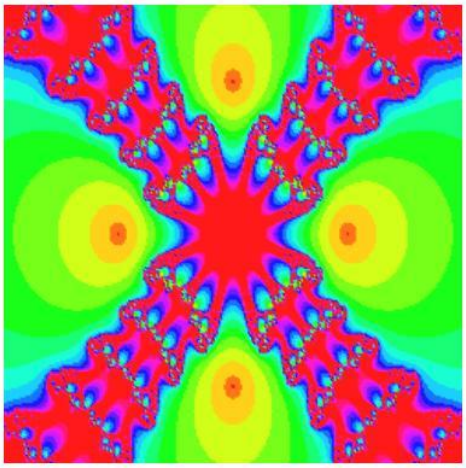

Fig. 5.Polynomiography for $z^4 - 1 = 0$.

Polynomiograph for $z^4 + z^3 + 3 = 0$

The polynomiograph of the complex polynomial equation $z^4 + z^3 + 3 = 0$, via KAM is presented in the following figure :

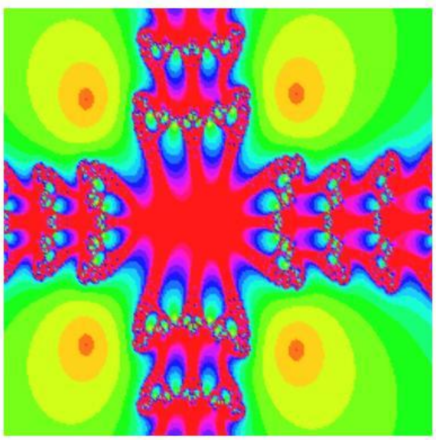

Fig. 6.Polynomiography for $z^4 + z^3 + 3 = 0$.

Polynomiograph for $z^5 - 1 = 0$

The polynomiograph of the complex polynomial equation $z^5 - 1 = 0$, via KAM is presented in the following figure :

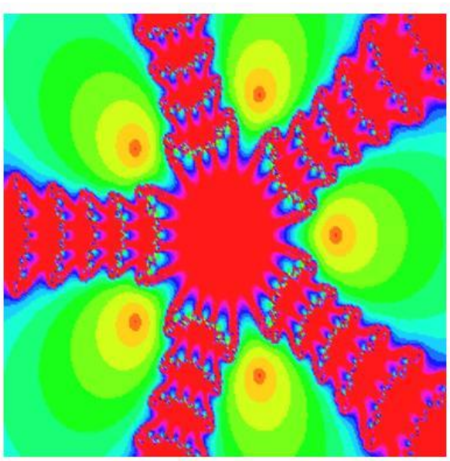

Fig. 7.Polynomiograph for $z^5 - 1 = 0$.

Polynomiograph for $z(z^2 + 1)(z^2 + 2) = 0$

The polynomiograph of the complex polynomial equation $z(z^2 + 1)(z^2 + 2) = 0$, via KAM is presented in the following figure :

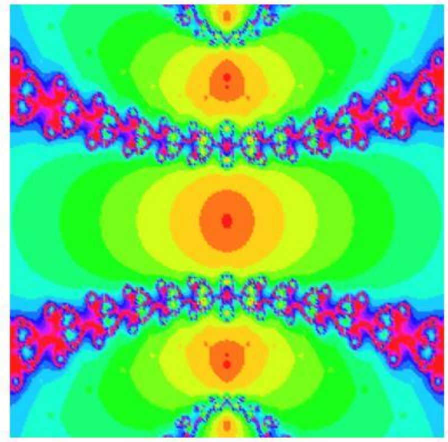

Fig. 8.Polynomiograph for $z(z^2 + 1)(z^2 + 2) = 0$.

Polynomiograph for $z^6 - 1 = 0$

The polynomiograph of the complex polynomial equation $z^6 - 1 = 0$, via KAM is presented in the following figure :

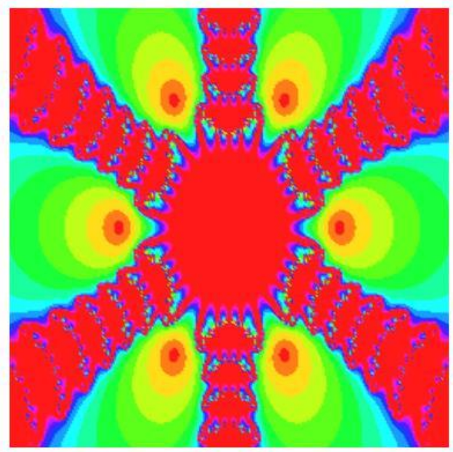

Fig. 9.Polynomiograph for $z^6 - 1 = 0$.

Polynomiograph for $z^6 + z^3 + 4 = 0$

The polynomiograph of the complex polynomial equation $z^6 + z^3 + 4 = 0$, via KAM is presented in the following figure

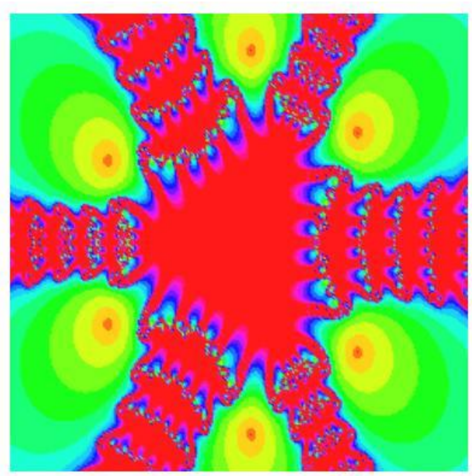

Fig. 10.Polynomiograph for $z^6 + z^3 + 4 = 0$.

Polynomiograph for $z^7 - 1 = 0$

The polynomiograph of the complex polynomial equation $z^7 - 1 = 0$, via KAM is presented in the following figure :

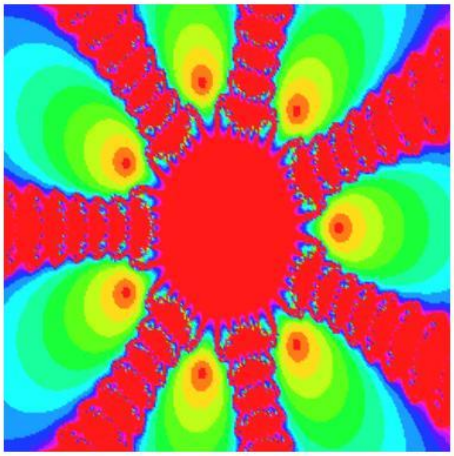

Fig. 11.Polynomiograph for $z^7 - 1 = 0$.

Polynomiograph for $z^7 - z^2 + 1 = 0$

The polynomiograph of the complex polynomial equation $z^7 - z^2 + 1 = 0$, via KAM is presented in the following figure

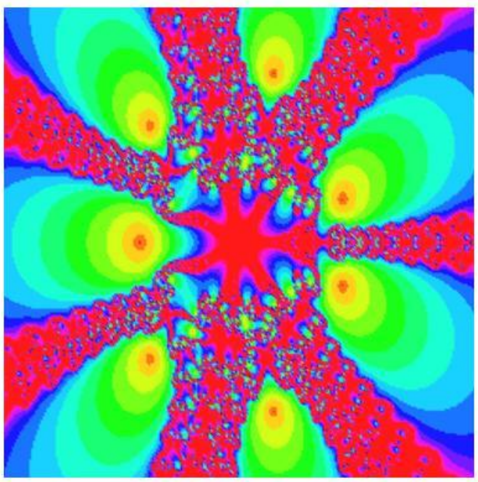

Fig. 12.Polynomiograph for $z^7 - z^2 + 1 = 0$.

Polynomiograph for $z^8 - 1 = 0$

The polynomiograph of the complex polynomial equation $z^8 - 1 = 0$, via KAM is presented in the following figure :

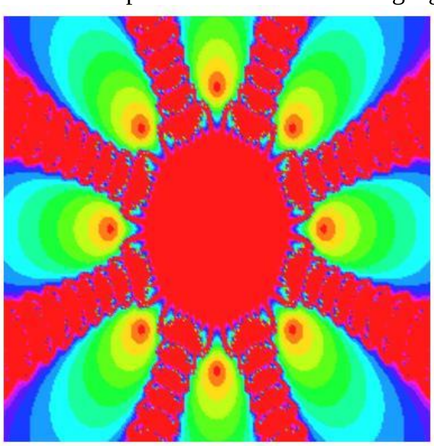

Fig. 13.Polynomiograph for $z^8 - 1 = 0$.

Polynomiograph for $z^8 - z^5 + 5 = 0$

The polynomiograph of the complex polynomial equation $z^8 - z^5 + 5 = 0$, via KAM is presented in the following figure

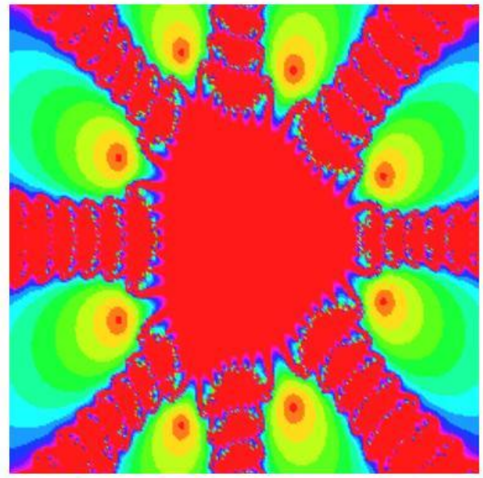

Fig. 14.Polynomiograph for $z^8 - z^5 + 5 = 0$.

Polynomiograph for $z^9 - 1 = 0$

The polynomiograph of the complex polynomial equation $z^9 - 1 = 0$, via KAM is presented in the following figure :

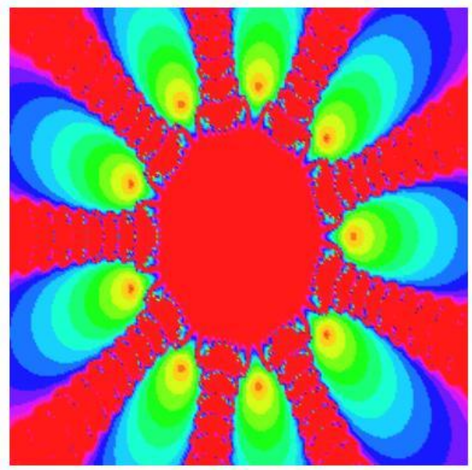

Fig. 15.Polynomiograph for $z^9 - 1 = 0$.

Polynomiograph for $z^9 - z^4 + 5 = 0$

The polynomiograph of the complex polynomial equation $z^9 - z^4 + 5 = 0$, via KAM is presented in the following figure

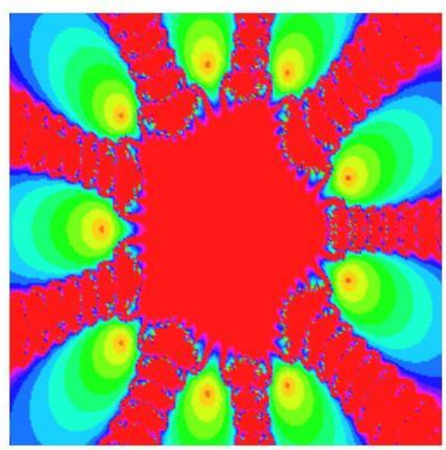

Fig. 16.Polynomiograph for $z^9 - z^4 + 5 = 0$.

Polynomiograph for $z^{10} - 1 = 0$

The polynomiograph of the complex polynomial equation $z^{10}-1=0$, via KAM is presented in the following figure :

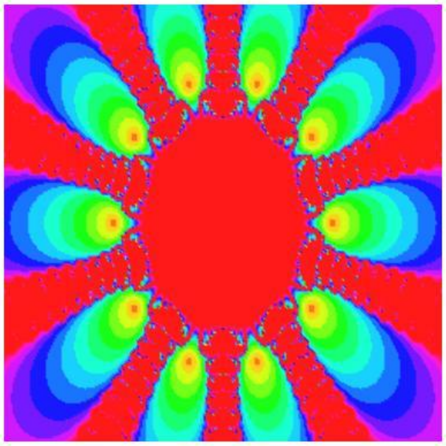

Fig. 17.Polynomiograph for $z^{10}-1=0$.

Polynomiograph for $z^{10}-z^{5}+10=0$

The polynomiograph of the complex polynomial equation $z^{10}-z^{5}+10=0$, via KAM is presented in the following figure :

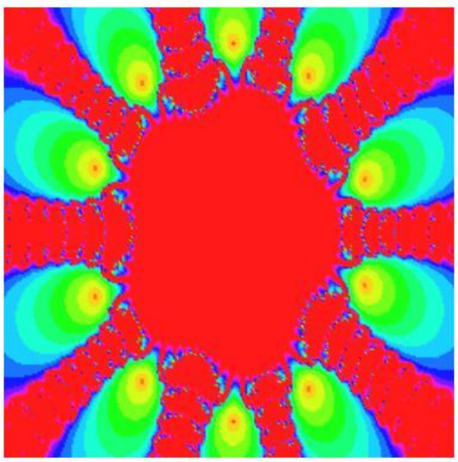

Fig. 18.Polynomiograph for $z^{10}-z^{5}+10=0$.

Polynomiograph for $z^{15}-1=0$

The polynomiograph of the complex polynomial equation $z^{15}-1=0$, via KAM is presented in the following figure :

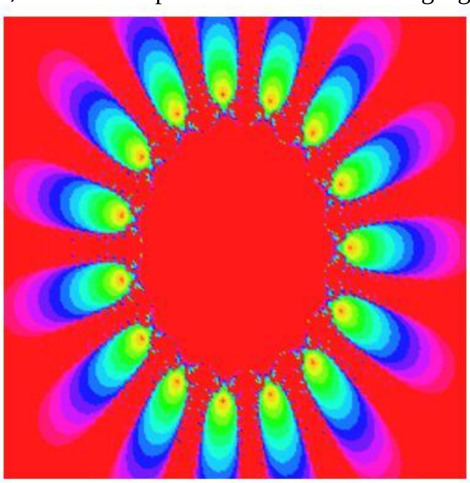

Fig. 19.Polynomiograph for $z^{15}-1=0$.

Polynomiograph for $z^{20}-1=0$

The polynomiograph of the complex polynomial equation $z^{20}-1=0$, via KAM is presented in the following figure :

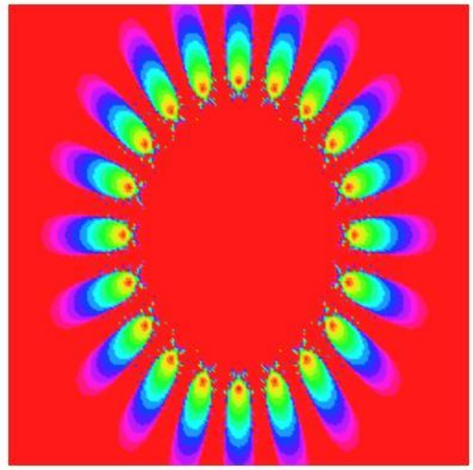

Fig. 20.Polynomiograph for $z^{20}-1=0$.

C0NCLUSIONS

We present some examples of polynomiographs for different complex polynomial equations $p(z)=0$. We used fouth order iterative method (KAM) , for solving nonlinear complex polynomial equations to create images that are quite new, different from images by the Newton's method and Householder's method free from second derivatives [2], and [9,10,11], and interesting from the aesthetic point of view.

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