A NUMERICAL STUDY OF NON-LINEAR BOUNDARY VALUE PROBLEM WITH ROBIN BOUNDARY CONDITIONS

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ABSTRACT: *The Adomian decomposition method (ADM) is a creative and effective method for exact solution of functional equations of various kinds. Adomian decomposition method solves wide class of linear and non-linear, ordinary or partial differential equations. This paper presents the Adomian decomposition method for the solution of nonlinear boundary value problem using Robin boundary conditions. In this approach, the solution is found in the form of a convergent power series with easily computed components. To show the efficiency of the method, numerical results and graphical representation of results are presented and compared with exact solution.*

Keywords: *Adomian Decomposition Method, Robin Boundary Conditions, Nonlinear Boundary Value Problem*

1. INTRODUCTION

We apply ADM for solving nonlinear second-order differential equation that is boundary value problems (BVPs) using Robin boundary conditions [1]. For the solution of Robin BVPs, Some conditions include the set of Dirichlet boundary conditions as well as mixed sets of Robin and Neumann, Robin and Dirichlet, Neumann and Dirichlet, Robin boundary and Neumann conditions, Dirichlet and Robin, Dirichlet and Neumann [2-3]. There should be an estimation of split series in every subdomain with the help of famous latest recursion method for nonlinear BVPs by our innovative latest recursion method. For Initial Value Problems (IVPs), the Sub-solutions are united by application the form of connection at the inner boundary position in equivalence to the multistage ADM [6]. There is an introduction of Multistage ADM, which can simply behave nonlinear problem while the unique sequence diverges above the specific field [7-9]. A further mean of the multistage ADM for BVPs is to resolve nonlinear Neumann BVPs relying upon the solution theory of change the unique BVP into two sub-BVPs, where each is subject to a mixed set of Dirichlet and Neumann boundary conditions [5]. The ADM is considered organized system for useful solution of linear or nonlinear and deterministic or stochastic operator problems simultaneously with ordinary differential equations, partial differential equations, integro–differential equations, integral equations etc [10-12].

2. MATERIAL AND METHODS

2.1. Adomian Decomposition Method with robin boundary condition

Let the common nonlinear deterministic differential problem in Adomian's operator-theoretic type be

$$Lu + Ru + Nu = g \quad (1)$$

where g is the system input and u is the system output, and where L is the linear operator which frequently is at present the highest order differential operator, R is the linear rest operator, and N is the nonlinear operator

$$Lu = g - Ru - Nu. \quad (2)$$

Since L has been understood to be invertible, relate the converse linear operator L^{-1}equally side of eq. (2)

$$L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu$$

By the definition of integral operator,

$$L^{-1}Lu = u - \varphi$$

Where φ identically satisfies $L\varphi \equiv 0$

$I_j\varphi = c_j \ or \ B_j\varphi = b_j$

for $j = 1,2,\dots,p$, where the $I_j \ and \ B_j$ be the initial and boundary value operator, correspondingly for IVPs; the set of initial conditions

$I_1u = u(a)\,,\ I_2u = \frac{du}{dx}(a)$

Furthermore in favor of Dirichlet BVPs, the set of Dirichlet boundary conditions

$B_1u = u(a_1),\ B_2u = u(a_2),$

We have

$$u - \varphi = L^{-1}g - L^{-1}Ru - L^{-1}Nu \quad (3)$$

y the homogenous term φ, taking the operator L, to both sides of eq. (3), we get

$$u = \varphi + L^{-1}g - L^{-1}Ru - L^{-1}Nu$$

Or

$$u = \gamma - L^{-1}Ru - L^{-1}Nu$$

We describe the calculation of the a priori familiarconditions as $\gamma = \varphi + L^{-1}g$

That is the corresponding nonlinear Volterra integral equation designed used for the solution for either IVPs or BVPs depending on how we estimate

$$u = \sum_{n=0}^{\infty} u_n \quad and \quad Nu = \sum_{n=0}^{\infty} A_n$$

Correspondingly, wherever the Adomian polynomials are reliant leading the solution components from $u_0(x)$ through $u_n(x)$, inclusively, $A_n(x) = A_n\big(u_0(x),\cdots,u_n(x)\big)$.

f is understood to be analytic, as

$$A_n(x) = \frac{1}{n!}\frac{\partial^n}{\partial\lambda^n}[f\big(x,u(x;\lambda)\big)]_{\lambda=0},$$

Where λ is a constant.

$$u(x;\lambda) = \sum_{n=0}^{\infty} \lambda^n u_n(x),$$

$$f\big(x,u(x;\lambda)\big) = \sum_{n=0}^{\infty} \lambda^n A_n(x)$$

And

$$n!\,u_n(x) = \frac{\partial^n}{\partial\lambda^n}[u(x;\lambda)]_{\lambda=0},$$

Here the first several Adomian polynomials are

$$A_0(x) = f\big(x,u_0(x)\big)$$

$$A_1(x) = u_1(x)\frac{\partial}{\partial u_0}f\big(x,u_0(x)\big)$$

$$A_2(x) = u_2(x)\frac{\partial}{\partial u_0}f(x,u_0(x) + \frac{u_1^2(x)}{2!}\frac{\partial^2}{\partial u_0^2}f\big(x,u_0(x)\big)$$

$$A_3(x) = u_3(x)\frac{\partial}{\partial u_0}f\big(x,u_0(x)\big) + u_1(x)u_2(x)\frac{\partial^2}{\partial u_0^2}f\big(x,u_0(x)\big) + \frac{u_1^3(x)}{3!}\frac{\partial^3}{\partial u_0^3}f\big(x,u_0(x)\big)$$

$A_4(x) =$
$u_4(x)\frac{\partial}{\partial u_0}f\big(x,u_0(x)\big) +$
$\left(\frac{u_2^2(x)}{2!}+u_1(x)u_3(x)\right)\frac{\partial^2}{\partial u_0^2}f\big(x,u_0(x)\big) +$
$\frac{u_1^2(x)}{2!}u_2(x)\frac{\partial^3}{\partial u_0^3}f\big(x,u_0(x)\big) + \frac{u_1^4}{4!}\frac{\partial^4}{\partial u_0^4}f\big(x,u_0(x)\big)$
$A_5(x) = u_5(x)\frac{\partial}{\partial u_0}f\big(x,u_0(x)\big) + \Big(u_2(x)u_3(x) +$
$u_1(x)u_4(x)\frac{\partial^2}{\partial u_0^2}f(x,u_0)\Big) +$
$\left(u_1(x)\frac{u_2^2(x)}{2!}+\frac{u_1^2(x)}{2!}u_3(x)\right)\frac{\partial^3}{\partial u_0^3}f\big(x,u_0(x)\big) +$
$\frac{u_1^3(x)}{3!}u_2(x)\frac{\partial^4}{\partial u_0^4}f\big(x,u_0(x)\big) + \frac{u_1^5(x)}{5!}\frac{\partial^5}{\partial u_0^5}f(x,u_0))$

We observe that
$A_0 = f(x,u_0), and\ A_n = \sum_{k=1}^{n} c_n^k f^{(k)}(x,u_0), \quad n \geq 1$
so the Adomian polynomials rapidly and to high instructions as
$c_n^1 = u_n, \quad for\ \ n \geq 1 \quad and$
$c_n^k = \frac{1}{n}\sum_{j=0}^{n-k}(j+1)u_{j+1}c_{n-1-j}^{k-1}, \qquad for\ \ 2 \leq k \leq n$
The Adomian decomposition sequence used for the solution u(x) as well as the sequence of Adomian polynomials adapted in the direction of the nonlinearity Nu(x) as, we get
$\sum_{n=0}^{\infty} u_n = \gamma - L^{-1}R\sum_{n=0}^{\infty} u_n - L^{-1}\sum_{n=0}^{\infty} A_n$
The solution components $u_n(x)$can be calculated$u_0(x)$, using the standard Adomian recursion method
$u_0(x) = \gamma(x), u_{n+1}(x) = -L^{-1}Ru_n - L^{-1}A_n, \ \ n \geq 0$

2.2. Constitutive Equations

Suppose a nonlinear differential equation of second order of the type
$\frac{d^2}{dx^2}u(x) - f\big(u(x)\big) = 0\,, \ \ a \leq x \leq b$ (4)
Through Robin boundary form
$pu(a) + ru'(a) = \alpha$ (5)
$qu(b) + su'(b) = \beta$ (6)
where f (u(x)is systematic nonlinearity and p, q, r, s assure
$ps - qr + pq(b-a) \neq 0$ (7)
If$p,q,s \geq 0, r \leq 0, p, r$ are not all zeroes,q, s are not all zeroes, and p, q are not all zeroes, and then we have $ps - qr + pq(b-a) > 0$
We will take care of more common cases for values of p, q, r, s that is not inadequate by
Eq. (3.7), when $p = q = 0$, i.e. the Neumann boundary forms
We revise Eq. (3.4) in Adomian's operator-theoretic type
$Lu = Nu, \quad a \leq x \leq b$ (8)
Where $L(.) = \frac{d^2}{dx^2}(.)$is the linear differential operator to be on its head and Nu=f (u(x)).
We think the exact definite integral operators $L_{a,a}^{-1}$ which is definite as
$L_{a,a}^{-1} = \int_a^x\int_a^x(.)dxdx$
Applying the operator $L_{a,a}^{-1}$ of Eq. (3.8) yield
$u(x) - u(a) - (x-a)u'(a) = L_{a,a}^{-1}Nu$ (9)
by eq. (9), evaluate u(x) at x=b to get
$u(b) = u(a) + (b-a)u'(a) + [L_{a,a}^{-1}Nu]_{x=b}$ (10)
where
$[L_{a,a}^{-1}(.)]_{x=b} = \int_a^x\int_a^x(.)dxdx$

Differentiating (9) and then evaluating $u'(x) at\ x = b$give in
$u'(b) = u'(a) + \int_a^b Nudx$ (11)
Substitute eqn's (11) and (12) into Eq. (9), we acquire
$qu(b) + su'(b) = \beta$
$q[u(a) + (b-a)u'(a) + \big[L_{a,a}^{-1}Nu\big]_{x=b}\big] + s[u'(a) +$
$\int_a^b Nudx] = \beta$
$qu(a) + (q(b-a)+s)u'(a) = \beta - q[L_{a,a}^{-1}Nu]_{x=b} -$
$s\int_a^b Nudx$ (12)
Eqns. (5) and (12) represent a scheme of two linearly independent equations in the two remain undecided coefficients $u(a) and\ u'(a)$,
$\Delta = \begin{vmatrix} p & r \\ q & q(b-a)+s \end{vmatrix} = ps - qr + pq(b-a),$
This is nonzero by our statement (7)
We have resulting$u(a)\ and\ u'(a)$ in conditions of the particular significance of the method constraint$\alpha, \beta, a, b, p, q, r\ and\ s\ as$
$pu(a) + ru'(a) = \alpha$
$qu(a) + (q(b-a)+s)u'(a) = \beta - q[L_{a,a}^{-1}Nu]_{x=b} -$
$s\int_a^b Nudx$
$u(a) = \frac{1}{\Delta}\begin{vmatrix} \alpha & r \\ \beta - q[L_{a,a}^{-1}Nu]_{x=b} - s\int_a^b Nudx & (q(b-a)+s) \end{vmatrix}$
$u(a) = \frac{1}{\Delta}[q\alpha(b-a) + s\alpha - r\beta + rq[L_{a,a}^{-1}Nu]_{x=b} +$
$rs\int_a^b Nudx]$ (13)
$u'(a) = \frac{1}{\Delta}\begin{vmatrix} p & \alpha \\ q & \beta - q[L_{a,a}^{-1}Nu]_{x=b} - s\int_a^b Nudx \end{vmatrix}$
$u'(a) = \frac{1}{\Delta}[p\beta - pq[L_{a,a}^{-1}Nu]_{x=b} - ps\int_a^b Nudx - q\alpha]$(14)
Substituting Eqns (13) and (14) into Eq. (10) yield
$u(x) = \frac{1}{\Delta}[s\alpha - r\beta + p\beta(x-a) + q\alpha(b-x)] + L_{a,a}^{-1}Nu -$
$\frac{p(x-a)-r}{\Delta}[q[L_{a,a}^{-1}Nu]_{x=b} + s\int_a^b Nudx]$
This now is free of any uncertain coefficients after that relate the decomposition of the solution u(x) and the nonlinearity Nu(x),
$u(x) = \sum_{n=0}^{\infty} u_n(x), \qquad Nu(x) = \sum_{n=0}^{\infty} A_n(x)$
Respectively, wherever the$A_n(x)$ are the Adomian polynomials
$u_0 = \frac{1}{\Delta}[s\alpha - r\beta + p\beta(x-a) + q\alpha(b-x)]$
$u_n(x) = L_{a,a}^{-1}A_{n-1} - \frac{p(x-a)-r}{\Delta}[q[L_{a,a}^{-1}A_{n-1}]_{x=b} +$
$s\int_a^b A_{n-1}dx]\,, \ \ n \geq 1$
The mth-stage estimationfind by the ADM is
$\varphi_m(x) = \sum_{n=0}^{m-1} u_n(x)$
We have ensure that each one approximation $\varphi_m(x), m \geq 1$, accurately assure the boundary conditions (5) and (6) by our designed.
We insert $N-1$ points in the particular interval $[a,b]$,
$a = x_0 < x_1 < \cdots x_i < x_{N-1} < x_N = b.$
Let the principles of the solution at the innerposition are $u(x_i) = \eta_i, for\ i = 1,2,\cdots,N-1$, which represent $N-1$ undecided coefficients.
The left hand subinterval $[a, x_1]$, we solve the nonlinear BVP through a Dirichlet boundary and a mixed set of Robin conditions
$Lu = Nu\,, \quad a \leq x \leq x_1$

$pu(a) + ru'(a) = \alpha$, $u(x_1) = \eta_1$

And denote the mth-stage approximation as

$\varphi_m^{(1)}(x) = \varphi_m^{(1)}(x; \eta_1) = \sum_{k=0}^{\infty} u_k^{(1)}(x)$

On the interior subintervals $[x_{i-1}, x_i], i = 2,3,\cdots,N-1$, we solve the nonlinear BVP with a mixed set of Robin and Dirichlet boundary conditions

$Lu = Nu$, $x_{i-1} \leq x \leq x_i$

$u(x_{i-1}) = \eta_{i-1}$, $u(x_i) = \eta_i$

and denote the mth-stage approximation as

$\varphi_m^{(i)}(x) = \varphi_m^{(i)}(x; \eta_{i-1}, \eta_i) = \sum_{k=0}^{m-1} u_k^{(i)}(x)$, $i = 2,3,\cdots,N-1$

On the right hand subinterval $[x_{N-1}, b]$

$Lu = Nu$, $x_{N-1} \leq x \leq b$

$u(x_{N-1}) = \eta_{N-1}$, $qu(b) + su'(b) = \beta$

and denote the mth-stage approximation as

$\varphi_m^{(N)}(x) = \varphi_m^{(N)}(x; \eta_{N-1}) = \sum_{k=0}^{m-1} u_k^{(N)}(x)$

Matching the N approximations $\varphi_m^{(i)}(x)$, $i = 1,2,\cdots,N$

$$\frac{d\varphi_m^{(i)}}{dx}(x_i) = \frac{d\varphi_m^{(i+1)}}{dx}(x_i), \quad i = 1,2,\cdots,N-1 \qquad (15)$$

solve the $N-1$ coefficients $\eta_1, \eta_2, \cdots, \eta_{N-1}$. If, as m increases, there exist solutions for the coefficients $\eta_1, \eta_2, \cdots, \eta_{N-1}$

We indicate by $\eta_k[\varphi_m]$the approximate value of η_k obtained by matching $\varphi_m^{(i)}(x), i = 1,2,\ldots,N$, according to Eqs (3.15). Combining

$\varphi_m^{(1)}(x; \eta_1[\varphi_m]), \varphi_m^{(2)}(x; \eta_1[\varphi_m], \eta_2[\varphi_m]), \cdots, \varphi_m^{(N)}(x; \eta_{N-1}[\varphi_m])$

We can employ the boxcar function to state the mth-stage solution approximant as

$\varphi_m(x) = \sum_{i=0}^{N} \varphi_m^{(i)}(x) \prod(x; x_{i-1}, x_i)$

so the boxcar function can be describe as

$\prod(x; c, d) = H(x; c) - H(x; d)$

And where

$$H(x; h) = \begin{cases} 0 & x < h, \\ 1 & x \geq h. \end{cases}$$

Case-1

Boundary conditions of the Dirichlet

$u(a) = \alpha$, $u(b) = \beta$

Correspond to the case of $p = q = 1, r = s = 0$ in Eqs. (3.5) and (3.6). Hence we have $\Delta = b - a$, we get

$$u(x) = \frac{\beta(x-a) + \alpha(b-x)}{b-a} + L_{a,a}^{-1} Nu(x) - \frac{x-a}{b-a}[L_{a,a}^{-1} Nu(x)]_{x=b}$$

Case-2

Dirichlet boundary and the mixed set of Robin conditions

$pu(a) + ru'(a) = \alpha$, $u(b) = \beta$

corresponds to the case of $q = 1, s = 0$. Hence we have $\Delta = p(b-a) - r$,we have

$$u(x) = \frac{\alpha(b-x) + p\beta(x-a) - r\beta}{p(b-a) - r} + L_{a,a}^{-1} Nu(x) - \frac{p(x-a) - r}{p(b-a) - r}[L_{a,a}^{-1} Nu(x)]_{x=b}$$

Case-3

Neumann boundary and the mixed set of Robin conditions

$pu(a) + ru'(a) = \alpha$, $u'(b) = \beta$

correspond to the case of $q = 0, s = 1$. Hence we have $\Delta = p$,we have

$$u(x) = \beta(x-a) + \frac{\alpha - r\beta}{p} + L_{a,a}^{-1} Nu - \frac{p(x-a) - r}{p}\int_a^b Nudx$$

Case-4

Robin boundary and the mixed set of Dirichlet conditions

$u(a) = \alpha$, $qu(b) + su'(b) = \beta$

Corresponds to the case of$p = 1, r = 0$, Hence we have

$\Delta = q(b-a) + s$

$$u(x) = \frac{\beta(a-x) + q\alpha(x-b) - s\alpha}{q(a-b) - s} + L_{b,b}^{-1} Nu(x) - \frac{q(x-b) - s}{q(a-b) - s}[L_{b,b}^{-1} Nu(x)]_{x=a}$$

Case-5

Robin boundary and the mixed set of Neumann conditions

$u'(a) = \alpha$, $qu(b) + su'(b) = \beta$

Corresponds to the case of $p = 0, r = 1$. Hence we have

$\Delta = -q$

$$u(x) = \alpha(x-b) + \frac{\beta - s\alpha}{q} + L_{b,b}^{-1} Nu - \frac{q(x-b) - s}{q}\int_b^a Nudx$$

Case-6

Neumann boundary and the mixed set of Dirichlet conditions

$u(a) = \alpha$, $u'(b) = \beta$

Corresponds to the case of$p = s = 1, q = r = 0$, corresponding Volterra integral equation is

$u(x) = \alpha + \beta(x-a) + L_{a,b}^{-1} Nu(x)$,

Case-7

Dirichlet boundary and the mixed set of Neumann conditions

$u'(a) = \alpha$, $u(b) = \beta$

Corresponds to the case of$p = s = 0, q = r = 1$, equivalent nonlinear Volterra equation is

$u(x) = \beta + \alpha(x-b) + L_{b,a}^{-1} Nu(x)$

2.3. Inverse Linear Operators

Suppose the inverse linear operators

$L_{b,b}^{-1}(.) = \int_b^x \int_b^x (.) dxdx$

$L_{a,b}^{-1}(.) = \int_a^x \int_b^x (.) dxdx$

And

$L_{b,a}^{-1}(.) = \int_b^x \int_a^x (.) dxdx$

Relate the operator$L_{b,b}^{-1}(.)$,we include

$u - u(b) - (x-b)u'(b) = L_{b,b}^{-1} Nu$

We obtain

$$u(x) = \frac{1}{\Delta}[s\alpha - r\beta + p\beta(x-a) + q\alpha(b-x) + L_{b,b}^{-1} Nu + \frac{q(x-b) - s}{\Delta}[p[L_{b,b}^{-1} Nu]_{x=a} + r\int_b^a Nudx]$$

which we obtain the customized recursion system

$$u_0(x) = \frac{1}{\Delta}[s\alpha - r\beta + p\beta(x-a) + q\alpha(b-x)]$$

$$u_n(x) = L_{b,b}^{-1} A_{n-1} + \frac{q(x-b) - s}{\Delta}[p[L_{b,b}^{-1} A_{n-1}]_{x=a} + r\int_b^a A_{n-1} dx], n \geq 1$$

Applying the operator $L_{a,b}^{-1}(.)$ to Eq. (8) we include

$u - u(a) - (x-a)u'(b) = L_{a,b}^{-1} Nu$

Letting x=b, we have

$u(b) = u(a) + (b-a)u'(b) + [L_{a,b}^{-1} Nu]_{x=b}$

Differentiating,and letting x=a, we get

$u'(a) = u'(b) + \int_b^a Nudx$

Substituting the equations into the boundary conditions (5), (6) we get

$pu(a) + ru'(b) = \alpha - r\int_b^a Nudx$

$qu(a) + (s + q(b-a))u'(b) = \beta - q[L_{a,b}^{-1} Nu]_{x=b}$

Solving for u (a) and $u'(b)$

After that introduce their results

$$u(x) = \frac{1}{\Delta}[s\alpha - r\beta + p\beta(x-a) + q\alpha(b-x)] + L_{a,b}^{-1} Nu - \frac{pq(x-a) - qr}{\Delta}[L_{a,b}^{-1} Nu]_{x=b} - \frac{rs + qr(b-x)}{\Delta}\int_b^a Nudx$$

we get the customized recursion system as

$$u_0(x) = \frac{1}{\Delta}[s\alpha - r\beta + p\beta(x-a) + q\alpha(b-x)]$$

$u_n(x) = L_{a,b}^{-1}A_{n-1} - \frac{1}{\Delta}[q(p(x-a)-r)[L_{a,b}^{-1}A_{n-1}]_{x=b} + r(q(b-x)+s)\int_b^a A_{n-1}dx]\ , \quad n \geq 1$

Applying the operator $L_{b,a}^{-1}(.)$ to Eq. (8), we contain

$u - u(b) - (x-b)u'(a) = L_{b,a}^{-1}Nu$

By a related process as for operator $L_{a,b}^{-1}(.)$, we get this

$u(x) = \frac{1}{\Delta}[s\alpha - r\beta + p\beta(x-a) + q\alpha(b-x)] + L_{b,a}^{-1}Nu + \frac{pq(x-b)-ps}{\Delta}[L_{b,a}^{-1}Nu]_{x=a} + \frac{ps(a-x)+rs}{\Delta}\int_a^b Nudx$

the customized recursion system equivalent to

$u_0(x) = \frac{1}{\Delta}[s\alpha - r\beta + p\beta(x-a) + q\alpha(b-x)]$

$u_n(x) = L_{b,a}^{-1}A_{n-1} + \frac{1}{\Delta}[p(q(x-b)-s)[L_{b,a}^{-1}A_{n-1}]_{x=a} + s(p(a-x)+r)\int_a^b A_{n-1}dx], \quad n \geq 1$

We get to the first components in the over recursion system are the equal, and with the exchanging $a \leftrightarrow b, p \leftrightarrow q, r \leftrightarrow s\ and\ \Delta \leftrightarrow \Delta,$

We denote

$h^{[1]}(x) = \int A_{n-1}dx$

$h^{[2]}(x) = \int h^{[1]}(x)dx$

we have

$u_n(x) = h^{[2]}(x) - h^{[2]}(a) - h^{[1]}(b)(x-a) - \frac{1}{\Delta}[(pq(x-a)-qr)\left(h^{[2]}(b) - h^{[2]}(a) - h^{[1]}(b)(b-a) + (qr(b-x)+rs) \times \left(h^{[1]}(a) - h^{[1]}(b)\right)\right]$

we enclose

$\bar{u}_n(x) = h^{[2]}(x) - h^{[2]}(b) - h^{[1]}(a)(x-b) + \frac{1}{\Delta}[pq(x-b)-ps)\left(h^{[2]}(a) - h^{[2]}(b) - h^{[1]}(a)(a-b)\right) + (ps(a-x)+rs) \times (h^{[1]}(b) - h^{[1]}(a))]$

We have the identity $u_n(x) - \bar{u}_n(x) = 0$

3. Numerical Illustrations

Example-1: Let the variable coefficients linear BVP with Robin boundary form be

$u'' = \frac{1}{1+x}u + \frac{x}{1+x}u'\ , \quad 0 \leq x \leq 1$

$u(0) - 2u'(0) = -1\ , \quad u(1) + 2u'(1) = 3e$

Solution:

The exact solution of the BVP is $u^*(x) = e^x$ and here we have $a = 0\ , b = 1, \alpha = -1, \beta = 3e\ , p = 1, r = -2, q = 1, s = 2\ and\ \Delta = 5$. Thus

$u(x) = \frac{1}{5}[6e - 3 + (1+3e)x - (2+x)[L_{0,0}^{-1}Nu]_{x=1} - (4+2x)\int_0^1 Nudx] + L_{0,0}^{-1}Nu$

Nu degenerates to the sum of linear terms $Nu = \frac{1}{1+x}u + \frac{x}{1+x}u'$. the Adomian polynomials are

$A_n = \frac{1}{1+x}u_n + \frac{x}{1+x}u_n'\ ,\ n = 0,1,2, ...$

From the recursion scheme, we have

$u_0(x) = \frac{1}{5}[6e - 3 + (1+3e)x]$

$u_n(x) = \frac{1}{5}[-(2+x)[L_{0,0}^{-1}A_{n-1}]_{x=1} - (4+2x)\int_0^1 A_{n-1}dx] + L_{0,0}^{-1}A_{n-1}, n \geq 1$

The solution components are computed as

$u_1(x) = \frac{1}{5}[-(2+x)[L_{0,0}^{-1}A_0]_{x=1} - (4+2x)\int_0^1 A_0 dx] + L_{0,0}^{-1}A_0\ , \quad n \geq 1$

$A_0 = \frac{1}{1+x}u_0 + \frac{x}{1+x}u_0'$

$= \frac{1}{1+x}\left(\frac{1}{5}[6e - 3 + (1+3e)x]\right) + \frac{x}{1+x}\left[\frac{(1+3e)}{5}\right]$

$= \frac{1}{5(1+x)}[6e - 3 + x + 3ex + x + 3ex]$

$A_0 = \frac{1}{5(1+x)}[6e - 3 + 2x + 6ex]$

$A_0 = \frac{1}{5}\left[\frac{(6e-3)}{(1+x)} + \frac{x(2+6e)}{(1+x)}\right]$

$A_0 = \frac{1}{5}[\frac{(6e-3)}{(1+x)} + (2+6e)\left(1 - \frac{1}{1+x}\right)]$

$A_0 = \frac{1}{5}[\frac{6e-3-2-6e}{(1+x)} + (2+6e)]$

$A_0 = \frac{1}{5}\left[\frac{-5}{(1+x)} + (2+6e)\right]$

$A_0 = \frac{-1}{1+x} + \frac{(2+6e)}{5}$

$u_1(x) = \frac{1}{5}[-(2+x)[L_{0,0}^{-1}(\frac{-1}{1+x} + \frac{2+6e}{5})]_{x=1} - (4+2x)\int_0^1(\frac{-1}{1+x} + \frac{(2+6e)}{5})dx] + L_{0,0}^{-1}(\frac{-1}{1+x} + \frac{2+6e}{5})$

$= \frac{1}{5}[-(2+x)[\int_0^x\int_0^x\left(\frac{-1}{1+x} + \frac{2+6e)}{5}\right)dxdx]_{x=1} - (4+2x)\int_0^1\left(\frac{-1}{1+x} + \frac{2+6e}{5}\right)dx] + \int_0^x\int_0^x\left(\frac{-1}{1+x} + \frac{2+6e}{5}\right)dxdx$

$u_1 = \left(\frac{1}{5} + \frac{3e}{5}\right)x^2 + x(-\log(x+1) - \frac{3e}{5} + \frac{3}{5} - \frac{\log(4)}{5} + \frac{\log(64)}{5}) - \log(x+1) - \frac{6e}{5} - \frac{4}{5} + \frac{\log(256)}{5}$

$u_2 = \left(\frac{1}{10} + \frac{3e}{10}\right)x^3 + x^2(-\log(x+1) - \frac{3e}{2} + \frac{13}{10} + \frac{\log(256)}{10}) + x(\frac{1}{2}\log(x+1)^2 - \frac{17}{5}\log(x+1) + \frac{9}{5}e\log(x+1) - \frac{9e}{25} + \frac{22}{25} - \frac{1}{5}3\log(2)^2 + \frac{1}{25}\log(32)\log(2) - \frac{32\log(2)}{25} + \frac{3\log(64)}{5} - \frac{3}{25}e\log(4096)) + \frac{1}{2}\log(x+1)^2 - \frac{12}{5}\log(x+1) + \frac{9}{5}e\log(x+1) - \frac{76}{25} + \frac{72}{25}e - \frac{6\log(2)^2}{5} + \frac{2}{25}\log(2)\log(32) + \frac{6\log(64)}{5} - \frac{6}{25}e\log(4096) - \frac{64\log(2)}{25}$ ...

The maximal error constraint

$ME_n = \max_{0 \leq x \leq 1}|\varphi_n(x) - u^*(x)|$

Table-1: The maximal error constraint $ME_n\ for\ n = 1,2,\cdots,12$.

N	1	2	3	4	5	6
ME_n	1.93842	1.41912	0.985009	0.677592	0.465683	0.320021
N	7	8	9	10	11	12
ME_n	0.21992	0.15113	0.103857	0.0713709	0.0490464	0.0337049

Example-2: Solve the following BVP with Robin boundary condition

$u'' = \frac{-1}{8}(e^{-2u} + 4(u')^2)\ , \qquad 0 \leq x \leq 1$

$u(0) - 2u'(0) = -1\ ,\ u(1) + 2u'(1) = \frac{2}{3} + \log\frac{2}{3}$

Solution:

The exact solution is $u^*(x) = \log\frac{2+x}{2}$ **.** for this BVP we have $a = 0\ , b = 1, \alpha = -1, \beta = \frac{2}{3} + \log\frac{3}{2}\ , p = 1, r = -2, q = 1, s = 1, and\ \Delta = 5$.

$u(x) = \frac{-1}{3} + \frac{2}{5}\log\frac{3}{2} + x(\frac{1}{3} + \frac{1}{5}\log\frac{3}{2}) - \frac{1}{8}L_{0,0}^{-1}Nu + \frac{2+x}{40}[L_{0,0}^{-1}Nu]_{x=1} + \frac{2+x}{20}\int_0^1 Nudx,$

Where $Nu = e^{-2u} + 4(u')^2$

The nonlinearity Nu as $u(x) = \sum_{n=0}^{\infty}u_n(x)\ and\ Nu(x) = \sum_{n=0}^{\infty}A_n(x)$

Where the Adomian polynomials are

$A_0 = e^{-2u_0} + 4(u_0')^2$

$A_1 = -2e^{-2u_0}u_1 + 8u_0'u_1'$

$A_2 = 2e^{-2u_0}u_1^2 - 2e^{-2u_0}u_2 + 4(u_1')^2 + 8u_0'u_2'$

$A_3 = \frac{-4}{3}e^{-2u_0}u_1^3 + 4e^{-2u_0}u_1u_2 - 2e^{-2u_0}u_3 + 8u_1'u_2' + 8u_0'u_3' \ldots$

By the parameterized recursion scheme

$u_0 = c$

$u_1 = -c - \frac{1}{3} + \frac{2}{5}\log\frac{3}{2} + x(\frac{1}{3} + \frac{1}{5}\log\frac{3}{2}) - \frac{1}{8}L_{0,0}^{-1}A_0 + \frac{2+x}{40}[L_{0,0}^{-1}A_0]_{x=1} + \frac{2+x}{20}\int_0^1 A_0 dx$

$u_n = \frac{-1}{8}L_{0,0}^{-1}A_{n-1} + \frac{2+x}{40}[L_{0,0}^{-1}A_{n-1}]_{x=1} + \frac{2+x}{20}\int_0^1 A_{n-1}dx\,, \quad n \geq 2$

Table-2: The maximal error parameter ME_n $for\ n = 2\ through\ 19$

N	2	3	4	5	6	7
ME_n	0.0640962	0.086546	0.0222792	0.00933165	0.00405274	0.00342178
N	8	9	10	11	12	13
ME_n	0.00277756	0.00190359	0.00142094	0.00108823	0.00083530	0.00064324
N	14	15	16	17	18	19
ME_n	0.000502041	0.000397089	0.000315939	0.000252689	0.0002032294	0.000164427

Example-3: Solve the following BVP with Robin boundary condition

$u'' = -e^{-2u}\,,\ \ 0 \leq x \leq 1$

$u'(0) = 1\,,\ \ u'(1) = \frac{1}{2}$

Solution:

The exact solution of BVP is $u^*(x) = \log(1+x)$

We have $Lu = Nu$

$Lu = Nu\,,\ \ 0 \leq x \leq 0.5$

$u'(0) = 1\,, u(0.5) = \eta$

since

$u = \eta + x - 0.5 + L_{0.5,0}^{-1}Nu$

From which we develop the modified recursion system

$u_0 = \eta - 0.5 + x$

$u_n = L_{0.5,0}^{-1}A_{n-1}\,, \quad n \geq 1$

Where the Adomian polynomials $Nu = -e^{-2u}$ are

$A_0 = -e^{-2u_0}$

$A_1 = 2e^{-2u_0}u_1$

$A_2 = -2e^{-2u_0}u_1^2 + 2e^{-2u_0}u_2 \ldots$

$u_1 = \frac{1}{4}e^{1-2\eta} - \frac{1}{4}e^{1-2x-2\eta} + \frac{e^{-2\eta}}{4} - \frac{1}{2}e^{1-2\eta}x$

$u_2 = \frac{1}{8}e^{1-4\eta} + \frac{1}{16}e^{2-4\eta} - \frac{1}{32}e^{2-4x-4\eta} + \frac{1}{8}e^{1-2x-4\eta} - \frac{1}{8}e^{2-2x-4\eta} - \frac{3e^{-4\eta}}{32} + \frac{1}{4}e^{1-4\eta}x - \frac{1}{8}e^{2-4\eta}x - \frac{1}{4}e^{2-2x-4\eta}x\ \cdots$

where we indicate the nth-stage solution estimate

$\varphi_n^{(1)}(x;\eta) = \sum_{k=0}^{n-1}u_k$

$Lu = Nu\ \ ,\ \ 0.5 \leq x \leq 1$

$u(0.5) = \eta\,, u'(1) = \frac{1}{2}$

from

$u = \eta + \frac{1}{2}(x - 0.5) + L_{0.5,1}^{-1}Nu$

And the modified recursion scheme

$u_0 = \eta + \frac{1}{2}(x - 0.5)$

$u_n = L_{0.5,1}^{-1}A_{n-1}\,, \quad n \geq 1$

We get the calculated solution are

$u_1 = \frac{1}{2}e^{-\frac{1}{2}-2\eta} - e^{\frac{1}{2}-x-2\eta} + e^{-2\eta} - e^{-\frac{1}{2}-2\eta}x$

$u_2 = 2e^{-1-4\eta} + 3e^{\frac{-1}{2}-4\eta} - \frac{1}{2}e^{1-2x-4\eta} - 3e^{-x-4\eta} + 2e^{\frac{1}{2}-x-4\eta} - \frac{3e^{-4\eta}}{2} - 4e^{-1-4\eta}x + 2e^{-\frac{1}{2}-4\eta}x - 2e^{-x-4\eta}x\ \ldots$

where we indicate the nth-stage solution estimation $\varphi_n^{(2)}(x;\eta) = \sum_{k=0}^{n-1}u_k$. Solve the corresponding equation for the instability

$[\frac{d\varphi_n^{(1)}(x;\eta)}{dx}]_{x=0.5} = [\frac{d\varphi_n^{(2)}(x;\eta)}{dx}]_{x=0.5}$

The corresponding nth-stage approximation as

$\varphi_n(x) = \varphi_n^{(1)}(x;\eta[\varphi_n])\prod(x;0{,}0.5) + \varphi_n^{(2)}(x;\eta[\varphi_n])\prod(x;0.5{,}1)$

The maximal error parameters $ME_n = \max_{0\leq x\leq 1}|\varphi_n(x) - u^*(x)|$ are solved and given in table

Table-3: The maximal error parameter ME_n

n	1	2	3	4
ME_n	1.07741	0.129275	0.0526345	0.0161081
n	5	6	7	8
ME_n	0.00573991	0.00199712	0.000709397	0.000250784

Table-4: The values of $\eta[\varphi_n]$ $for\ n = 2,3,\ldots,10$.

n	2	3	4	5	6
$\eta[\varphi_n]$	0.459188	0.389312	0.410025	0.403754	0.406114
n	7	8	9	10	
$\eta[\varphi_n]$	0.405198	0.405577	0.405417	0.405486	

Table-5: The maximal error parameter ME_n .

n	2	3	4	5	6
ME_n	0.05898	0.0184545	0.00527023	0.00199138	0.00075682
n	7	8	9	10	
ME_n	0.000311709	0.000130807	0.0000567361	0.0000250119	

4. CONCLUSION

Adomian decomposition method has been known to be a powerful device for solving many functional equations as algebraic equations, ordinary and partial differential equations, integral equations and so on. Here we used this method for solving nonlinear BVP. It is demonstrated that this method has the ability of solving systems of both linear and non-linear differential equations. In above problems, there was a nonlinear system and we derived the exact solutions. For non-linear systems, we usually derive a very good approximation to the solutions with the Robin boundary conditions. It is also important that the Adomian decomposition method does not require discretization of the variables. It is not affected by computation round errors and one is not faced with necessity of large computer memory and time. Comparing the results with exact solutions, the Adomian decomposition method was clearly reliable techniques. It is important that this method unlike the most numerical techniques provides a closed form of the solution.

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