AN IMPROVEMENT OF THE INTERPOINT DISTANCE SUM INEQUALITY WITH APPLICATION IN WIRELESS NETWORKS

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ABSTRACT: Let n points be arbitrarily placed in $B(D)$, a disk in R^2 having diameter D. Denote by l_{ij} the Euclidean distance between point i and j. The main result of this paper is to show that $\sum (\min_{j\neq i} l_{ij}^2) < \frac{D^2}{0.4018}$. We then raise a conjecture that the optimal denominator of the right hand side is 4/9 . The special case n = 3 is proved. Our results have a direct application for the best successful data transmission in wireless networks.

Keywords: combinatorial geometry, distance geometry, interpoint distance sum inequality, polyhedral concave program.

1. INTRODUCTION

Denote by arg $\min_{j\in J}\{Sj\}$ the index of the smallest point in the set {Sj} ($j\in J$). The following result was established in [1] to estimate upper bounds on the maximum number of simultaneously successful wireless transmissions and the maximum achievable per node end-to-end throughput under the general network scenario. For more results see [2-13].

Theorem 1.1 ([1])**.** *Let $B(D)$ be a disk in R^2 having diameter D. Let n points be arbitrarily placed in $B(D)$. Suppose each point is indexed by a distinct integer between* 1 *and n. Let l_{ij} be the Euclidean distance between point i and j. Define the mth closest point to point i, a_{im}, and the Euclidean distance between point i and the mth closest point to point i, u_{im}, as follows:*

$$a_{i1} := \arg\min_{j\in\{1,2,\dots,n\}}, \{l_{ij}\}, \quad 1\le i \le n, \quad j\neq i$$

$$a_{i1} := \arg\min_{j\in\{1,2,\dots,n\}}, \{l_{ij}\}, \quad 1\le i \le n, 2\le m \le n\text{-}1, \quad j\notin\{i\}\cup\{a_{ik}\}_{k=1}^{m-1}$$

$$u_{im} := l_{ia_{im}}, \quad 1\le i \le n, 1\le m \le n\text{-}1,$$

Then

$$\sum_{i=1}^{n} u_{im}^2 \le mD^2/c_1, \qquad 1\le m\le n-1 \qquad (1.1)$$

where

$$c_1 := \frac{2}{3} - \frac{\sqrt{3}}{2n} \approx 0.3910$$

In [2] observed that the interpoint distance sum inequality (1.1) can be simply but significantly strengthened with a proof following from (1.1) and the fact that $u_{im} \le D$.

Proposition 1.2 ([2])**.** *Define B(D), D,n,l_{ij},a_{im},u_{im},c_i as in Theorem 1.1. Then*

$$\sum_{i=1}^{n} u_{im}^2 \le \min(m/c_1, n)D, \quad 1\le m \le n\text{-}1 \qquad (1.2)$$

As a direct application, In [2] improved the upper bounds on the maximum number of simultaneously successful wireless transmissions and the maximum achievable per-node end-toend throughput under the same general network scenario as in Arpacioglu and Haas [1]. Later, we refined a special case of the interpoint distance sum inequality (1.1) when m = 1 as follows.

Theorem 1.3 ([3])**.** *Define B(D), D,n,l_{ij},a_{im},u_{im}, as in Theorem 1.1. Then*

$$\sum_{i=1}^{n} u_{i1}^2 \le \frac{D^2}{0.3972} \qquad (1.3)$$

In this article, we make a further improvement of Theorem 1.3, that is, the denominator of the right hand of the inequality (1.3), 0.3972, can be strengthened to be 0.4018. It is unknown what the optimal denominator is. We conjecture 4/9 is optimal and we show it is true for the special case n = 3. Finally, we use the similar approach to refine Proposition 1.2. This article is organized as follows. In section 2, Theorem 1.3 is improved and a conjecture is raised. In section 3, we use the similar approach to further improve Proposition 1.2. Conclusions are made in section 4.

2. MAIN RESULT

In this section, we improve Theorem 1.3 and then raise a conjecture. We need two lemmas.

Lemma 2.1. *Define B(D), D,n,l_{ij},a_{im},u_{im}, as in Theorem 1.1. Then for all $i\neq k\neq l\neq i$,*

$$u_{i1}^2 + u_{k1}^2 + u_{l1}^2 \le \frac{D^2}{4/9} \qquad (2.1)$$

Proof. Let n points in $B(D)$ whose center is denoted by O. For any three different points $i\neq k\neq l\neq i$, without loss of generality, we assume

$$u_{i1} \ge u_{k1} \ge u_{l1} \qquad (2.2)$$

The inequality (2.1) clearly holds if $u_{i1} < \frac{\sqrt{3}}{2}D$. Below we further assume $u_{i1} \ge \frac{\sqrt{3}}{2}D$. Let the point j be such that $l_{ij} = u_{i1}$, that is, we have that

$$l_{ij} \le l_{ij'}, \ \forall j' \neq j \qquad (2.3)$$

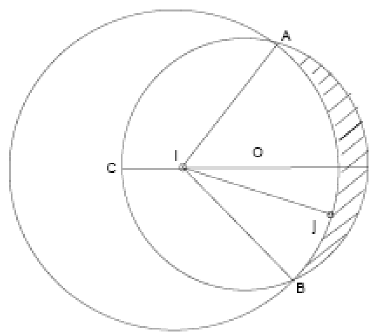

Figure 1: The left-hand side disk is $B_i(u_{i1})$ and the right-hand side is $B(D)$

Denote by $B_i(u_{i1})$ be the disk of diameter u_{i1} and center i. Then O is in the interior of $B_i(u_{i1})$ and j is on the boundary of $B_i(u_{i1})$. Denote by A and B the two intersecting points of disks $B(D)$ and $B_i(U_{i1})$, respectively. Let C be the point on the boundary of $B(D)$ such that the three points C, i and O are on a straight line in order. To make it clear, we mark them in Figure 1.

By the knowledge of plane geometry, we have

$$\cos(\angle ACO) = \frac{|AC|}{D}, \tag{2.4}$$

$$\frac{|Ai|}{\sin(\angle ACO)} = \frac{|AC|}{\sin(\angle AiO)}, \tag{2.5}$$

where $|AC|$ denotes the distance between A and C, $\angle ACO$ is the angle with C being the vertex and the other two, A and O on the legs, $|Ai|$ and $\angle AiO$ are defined similarly. Since $\angle ACO \le \angle AiO$ and $|Ai| = u_{i1}$, we have

$$\cos(\angle ACO) = \frac{AC}{D} = \frac{u_{i1}}{D}.\frac{\sin(\angle AiO)}{\sin(\angle ACO)} \ge \frac{u_{i1}}{D} \tag{2.6}$$

Therefore, we have

$$|AB|^2 = |Ai|^2 + |Bi|^2 - 2|Ai||Bi|\cos(\angle AiB) \tag{2.7}$$

$$= 2u_{i1}^2 - 2u_{i1}^2\cos(2\angle AiO) \tag{2.8}$$

$$= 2u_{i1}^2 - 2u_{i1}^2(2\cos^2(\angle AiO) - 1) \tag{2.9}$$

$$\le 2u_{i1}^2 - 2u_{i1}^2\left(2(u_{i1}/D)^2 - 1\right) \tag{2.10}$$

$$= 4D^2(u_{i1}/D)^2\left(1 - (u_{i1}/D)^2\right) \tag{2.11}$$

$$\le \frac{3}{4}D^2, \tag{2.12}$$

where the equality (2.7) follows from the cosine theorem, the equality (2.8) is because $|Ai| = |Bi| = u_{i1}$ and $\angle AiO = \angle BiO$, the inequality (2.10) holds due to Inequality (2.6) and the finally inequality is true since $D \ge u_{ui} \ge \frac{\sqrt{3}}{2}D$ as assumed and the function $f(x) = x^2(1 - x^2)$ is strict decreasing for $x = u_{i1}/D \in [\sqrt{3}/2, 1]$. Furthermore, Equality (2.8), Inequality (2.12) and the fact $D \ge u_{i1} \ge \frac{\sqrt{3}}{2}D$ imply

$$\cos(\angle AiB) = \frac{2u_{i1}^2 - |AB|^2}{2u_{i1}^2} \ge \frac{\frac{3}{2}D^2 - \frac{3}{4}D^2}{2D^2} = \frac{3}{8}$$

It follows that $\angle AiB$ is an acute angle. According to Equation (2.3), any other points j' ($i \ne j' \ne j$) is in $B(D) \backslash B_i(u_{i1})$, i.e., the shadow of Figure 1. We notice that the maximum Euclidean distance between any two points in $B(D) \backslash B_i(u_{i1})$ is $|AB|$ since $\angle AiB$ is an acute angle. Therefore,

$$u_{i1}^2 + u_{k1}^2 + u_{l1}^2 \le u_{i1}^2 + 2|AB|^2 \tag{2.13}$$

$$\le u_{i1}^2 + 4u_{i1}^2 - 4u_{i1}^2\left(2(u_{il} \backslash D)^2 - 1\right) \tag{2.14}$$

$$= D^2(u_{il}/D)^2\left(9 - 8(u_{il}/D)^2\right) \tag{2.15}$$

$$\le \frac{9}{4}D^2 \tag{2.16}$$

where Inequality (2.14) follows from Inequality (2.10) and the final inequality (2.16) holds since $D \ge u_{i1} \ge \frac{\sqrt{3}}{2}D$ as assumed and the function $f(x) = x^2(9 - 8x^2)$ is strict decreasing for $x = \frac{u_{i1}}{D} \in \left[\frac{\sqrt{3}}{2}, 1\right]$.

Lemma 2.2. Define $B(D)$,D, n, l_{ij} , a_{im}, u_{im} *as in Theorem 1.1. Then for any increasing concave function* $f(\sqrt{x})$ *and all indices* $i \ne k \ne l \ne i$,

$$f\left(\frac{u_{i1}}{D}\right) + f\left(\frac{u_{k1}}{D}\right) + f\left(\frac{ul1}{D}\right) \le 3f\left(\frac{\sqrt{3}}{2}\right) \tag{2.17}$$

Proof. According to Lemma 2.1, it is sufficient to consider the following optimization problem

$$\max\ f(u_{i1}/D) + f(u_{k1}/D) + f(u_{l1}/D) \tag{2.18}$$

$$s.t. \quad x_i + x_k + x_l \le \frac{9}{4} \tag{2.19}$$

By introducing $x_i = \frac{u_{i1}^2}{D^2}, x_k = \frac{u_{k1}^2}{D^2}$ *and* $x_1 = \frac{u_{l1}^2}{D^2}$, the above problem is equivalent to

$$\max\ f(\sqrt{x_i}) + f(\sqrt{x_k}) + f(\sqrt{x_l}) \tag{2.20}$$

$$s.t. \quad x_i + x_k + x_l \le 9/4 \tag{2.21}$$

Since $f(\sqrt{x})$ is concave, by Jensen's inequality, we have

$$f(\sqrt{x_i}) + f(\sqrt{x_k}) + f(\sqrt{x_l}) \le 3f\left(\sqrt{\frac{x_i + x_k + x_l}{3}}\right) \le 3f\left(\frac{\sqrt{3}}{2}\right) \tag{2.22}$$

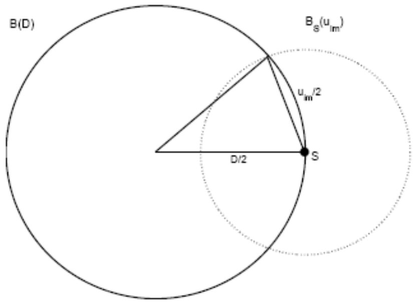

Figure 2: Computation of the overlap ratio between $B(D)$ and $B_s(u_{im})$.

Theorem 2.1. *Define* $B(D)$,D, n, l_{ij} , a_{im}, u_{im} *as in Theorem 1.1. Then*

$$\sum_{i=1}^{n} u_{i1}^2 < \frac{D^2}{0.4018} \tag{2.23}$$

Proof. The case $n = 2$ is trivial to verify since $u_{i1} \le D$. So we assume $n \ge 3$. The first part of this proof is based on that of Theorem 1.1 [1]. Denote the disk of diameter x and center i by $B_i(x)$. Define the following sets of disks

$$R1 := \{B_i(u_{i1}) : 1 \le i \le n\}.$$

As shown in [1], all disks in R_1 are non-overlapping, i.e., the distance between the centers of any two disks is greater than or equal to the sum of the radii of the two disks. Denote by $A(X)$ the area of a region X. We try to find a lower bound on

$$f_{i1} := A(B(D) \cap B_i(u_{i1})) / A(B_i(u_{i1}))$$

for every $1 \le i \le n$. Pick any point S from the boundary of $B(D)$ and consider the overlap ratio

$$f_{i1}^s := \frac{A(B(D) \cap B_S(u_{i1}))}{A(B_S(u_{i1}))}, \ 1 \le i \le n.$$

Using Figure 2, one can obtain the geometrical computation formula: $f_{i1}^S = f(y)|_{y = \frac{u_{i1}}{D}}$, where

$$f(y) := \frac{1}{\pi}\left(1 - \frac{2}{y^2}\right)\arccos\left(\frac{y}{2}\right) + \frac{1}{y^2} - \frac{1}{\pi}\sqrt{\frac{1}{y^2} - \frac{1}{4}}, \tag{2.24}$$

and $f(0) = \lim_{y \to 0} f(y) = 0.5$.

Since $f_{i1} \ge f_{i1}^S$, we have

$$A(B_i(u_{i1}) \cap B(D)) \ge f\left(\frac{u_{i1}}{D}\right) A(B_i(u_{i1})). \tag{2.25}$$

Adding all the n inequalities in (2.25), we obtain

$$\sum_{i=1}^{n} A(B_i(u_{i1}) \cap B(D)) \geq \sum_{i=1}^{n} \left(f\left(\frac{u_{i1}}{D}\right) A(B_i(u_{i1})) \right). \qquad (2.26)$$

Since all disks in R_1 are non-overlapping, we have

$$\sum A(B_i(u_{i1}) \cap B(D)) \leq A(B(D)). \qquad (2.27)$$

Inequalities (2.26) and (2.27) imply

$$A(B(D)) \geq \sum_{i=1}^{n} \left(f\left(\frac{u_{i1}}{D}\right) A(B_i(u_{i1})) \right).$$

Notice that $A(B(D)) = \pi D^2/4$ and $A(B_i(u_{i1})) = \pi u_{i1}^2/4$. Therefore,

$$D^2 \geq \sum_{i=1}^{n} \left(f\left(\frac{u_{i1}}{D}\right) u_{i1}^2 \right). \qquad (2.28)$$

To upper bound $\sum_{i=1}^{n} u_{i1}^2$, it is sufficient to consider the following optimization problem ($n \geq 3$):

$$\max \sum_{i=1}^{n} u_{i1}^2 \qquad (2.29)$$

$$s.t.. \sum \left(f\left(\frac{u_{i1}}{D}\right) u_{i1}^2 \right) \leq D^2, \qquad (2.30)$$

$$0 \leq u_{i1} \leq D, \quad i = 1, \ldots, n. \qquad (2.31)$$

Introducing $y_i := \frac{u_{i1}}{D}$ for $i = 1, \ldots, n$, the above optimization problem (2.29)-(2.31) is equivalent to

$$\max \ D^2 \sum_{i=1}^{n} y_i^2 \qquad (2.32)$$

$$s.t. \ \sum_{i=1}^{n} F(y_i) \leq 1, \qquad (2.33)$$

$$0 \leq y_i \leq 1, \ i = 1, \ldots, n, \qquad (2.34)$$

where

$$F(y) = f(y) y^2, \qquad \forall y \in [0,1]. \qquad (2.35)$$

It is not difficult to verify that the above function $F(y)$ is strictly increasing and strictly convex for y in its domain [0, 1]. The variations of $F(y)$ and its first derivative $F'(y)$ are shown in Figure 4. We can also verify that $F(\sqrt{x})$ is an increasing concave function for $x \in [0, 1]$, as shown in Figure 3. According to Lemma 2.2, we can further add inequalities in Problem (2.32)-(2.34):

$$F(y_i) + F(y_k) + F(y_l) \leq 3F\left(\frac{\sqrt{3}}{2}\right), \quad \forall i \neq k \neq l \neq i. \qquad (2.36)$$

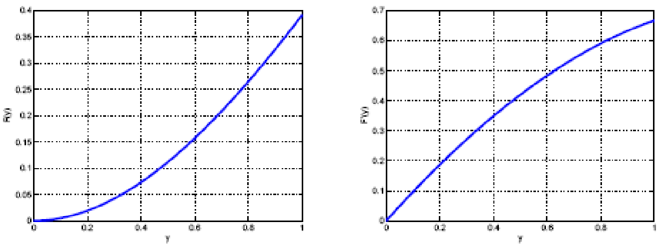

Figure 3: Variation of $F(y)$ and $F'(y)$.

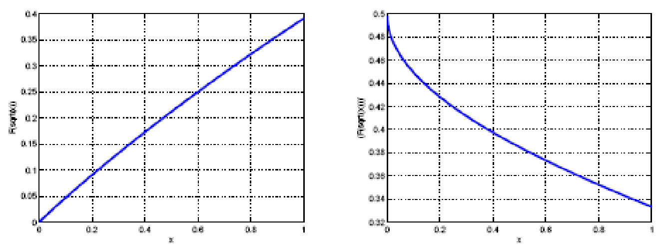

Figure 4: Variation of $F(\sqrt{x})$ and $(F(\sqrt{x}))'$.

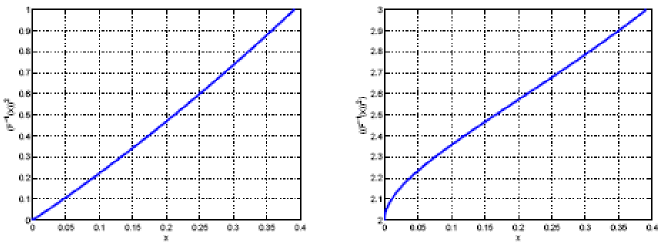

Figure 5: Variation of $(F^{-1}(x))^2$ and $((F^{-1}(x))^2)'$.

Let $x_i = F(y_i)$ for $i = 1, \ldots, n$, or equivalently, $y_i = F^{-1}(x_i)$ where $F^{-1}(.)$ is the inverse function for $F(.)$ and it exists since $F(\ldots)$ is strictly increasing in its domain. Therefore, based on this one-to-one mapping, Problem (2.32)-(2.34) is equivalent to

$$\max \quad D^2 \sum_{i=1}^{n} (F^{-1}(x_i))^2 \qquad (2.37)$$

$$s.t. \quad \sum_{i=1}^{n} x_i \leq 1, \qquad (2.38)$$

$$x_i + x_k + x_l \leq 3F\left(\frac{\sqrt{3}}{2}\right), \ \forall i \neq k \neq l \neq i, \qquad (2.39)$$

$$0 \leq x_i \leq F(1), \quad i = 1, \ldots, n, \qquad (2.40)$$

The objective function (2.37) is separable and convex for $(x_i) \in [0, F(1)]^n$. The variations of $(F^{-1}(x))^2$ and its first derivative function are shown in Figure 5. Besides, the feasible region defined by (2.38)-(2.40) is a convex polyhedron. The strict convexity of $(F^{-1}(x))^2$ implies that Problem (2.37)-(2.40) is a polyhedral concave programming problem. By making use of [4. Corollary 32.3.4], every optimal solution of Problem (2.37)-(2.40) can be attained at one of the vertices of the solution region (2.38)-(2.40). Below we discuss the vertices. Notice that $F(1)=f(1) = \frac{2}{3} - \frac{\sqrt{3}}{2\pi} \in (0.391, 0.392)$ and $F(\sqrt{3}/2) \in (0.304, 0.305)$. If there exists three indices $I \neq k \neq l \neq i$ such that the corresponding constraint (2.39) holds as an equality, that is, $x_i + x_k + x_l = 3F(\sqrt{3}/2) \in (0.914, 0.915)$. It implies that $x_i > 0.914 - x_k - x_l > 0.914 - 0.392 - 0.392 = 0.13$. Similarly, $x_k > 0.13$ and $x_l > 0.13$. We conclude that at most one of the inequalities (2.39) can hold as equality. If it is not true, there are at least two inequalities become equalities, for example, one contains indices $I \neq k \neq l \neq i$ and the other contains a different index j ($j \neq i,k,l$). Then $x_i + x_k + x_l = 3F(\sqrt{3}/2) \in (0.914, 0.915)$ and $x_j > 0.13$. It follows that $\sum_{i=1}^{n} x_i \geq x_i + x_k + x_j + x_l > 0.914 + 0.13 = 1.044$ which is a contradiction due to the constraint (2.38). Let (x_i) be a vertex of the feasible region (2.38)-(2.40). If all inequalities (2.39) are strict, (x_i) is a vertex of the region (2.38) and (2.40). Since $nF(1) \geq 3F(1) > 1 > 2F(1)$, the vertex (x_i) has two components with the value $F(1)$, one component with the value $x* = F(y*)$ satisfying $x* = F(y*) = 1 - 2F(1) \in (0.21, 0.22)$ and the others are zeros. But the sum of the three positive elements is equal to $F(1) + F(1) + (1 - 2F(1)) = 1$, which violates one of the constraint (2.39). As a conclusion, at any vertex (X_i), only one of the inequalities (2.39) holds as an equality. Due to the exchangeability of the elements of (x_i), without loss of generality, we can find the optimal solution from the vertices of the following feasible region:

$$\left\{(x_i)\in[0,F(1)]^n : x_1+x_2+x_3=3F\left(\frac{\sqrt{3}}{2}\right), \sum_{i=4}^{n} x_i \le 1-3F\left(\frac{\sqrt{3}}{2}\right)\right\} \quad (2.41)$$

There can be only two kinds of vertices. Each of the first kind has only four positive elements: $F(1)$, $F(1)$, $3F(\sqrt{3}/2) - 2F(1)$ and $1 - 3F(\sqrt{3}/2)$. Each of the second kind has only three positive elements: $F(1)$, $F(1)$ and $3F(\sqrt{3}/2) - 2F(1)$. Clearly, the optimal vertex solution belongs to the first kind. The corresponding four positive elements of (y_i) are hence 1, 1, y^*_1 and y^*_2, where y^*_1 and y^*_2 uniquely solve the following equations, respectively,

$$F(y) = 3F(\sqrt{3}/2) - 2F(1), \quad (2.42)$$

$$F(y) = 1 - 3F(\sqrt{3}/2), \quad (2.43)$$

Therefore, the optimal objective values of Problem (2.32)-(2.34) is $D^2(1^2+1^2+(y^*_1)^2+(y^*_2)^2) < 2.48856D^2$. In conclusion, we have proved for all valid u_{i1} that

$$\sum_{i=1}^{n} u_{i1}^2 < 2.48856D^2 < \frac{D^2}{0.4018}.$$

Notice that the best known denominator on the right hand side of the inequality (2.23) before this article is 0.3972. Naturally, one can ask what the optimal (i.e., largest) denominator is. Interestingly, from the proof of Theorem 2.1, especially from the solution structure of Problem (2.32)-(2.34), we conjecture that there are only three nonzero elements u_{i1} in the maximum distance distribution. Hence, we turn to consider the special case $n = 3$. As a direct corollary of Lemma 2.1, we immediately have

Proposition 2.2. *Define $B(D)$, D, l_{ij}, a_{im}, u_{im} as in Theorem 1.1. Let $n = 3$. Then*

$$\sum_{i=1}^{n} u_{i1}^2 \le \frac{D^2}{4/9}.$$

The equality holds if and only if $u_{11} = u_{21} = u_{31} = \sqrt{3}D/2$.

Based on the above observations, we raise the following conjecture for the general case.

Conjecture 2.1. *Define $B(D)$, D, n, l_{ij} as in Theorem 1.1. Then*

$$\sum_{i=1}^{n}\left(\min_{j\neq i} l_{ij}^2\right) \le \frac{D^2}{4/9}.$$

3 AN APPLICATION

In this section, we use the similar approach to improve Proposition 1.2 where we have a direct application of our results for successful data transmission in wireless networks.

Theorem 3.1. *Define $B(D)$, D, n, l_{ij}, a_{im}, u_{im}, c_1 as in Theorem 1.1. Then*

$$\sum_{i=1}^{n} u_{im}^2 \le \min\left(\left\lfloor\frac{m}{c_1}\right\rfloor + c_m^2, n\right)D^2, \quad 1 \le m \le n-1, \quad (3.1)$$

where $c_m \in (0, 1)$ is the unique solution of the equation

$$F(y) = m - \left\lfloor\frac{m}{c_1}\right\rfloor c_1, \quad (3.2)$$

and the function $F(y)$ is defined in (2.35).

Proof. The inequality $\sum_{i=1}^{n} u_{im}^2 \le nD^2$ directly follows from the fact $u_{im} \le D$. It is sufficient to prove $\sum_{i=1}^{n} u_{im}^2 \le \left(\left\lfloor\frac{m}{c_1}\right\rfloor + c_m^2\right)D^2$. We assume $n \ge 3$ because in the case $n = 2$ one has $\left(\left\lfloor\frac{m}{c_1}\right\rfloor + c_m^2\right) > \left\lfloor\frac{1}{c_1}\right\rfloor = 2$. Besides, the special case $m = 1$ exactly corresponds to Theorem 2.1. Below we always assume $2 \le m \le n-1$.

Denote the disk of diameter x by $B_i(x)$, whose center is at point i. Define the following sets of disks

$$R_m := \{B_i(u_{im}) : 1 \le i \le n\},\ 2 \le m \le n-1. \quad (3.3)$$

There can be overlaps between some pairs of disks in R_m. But as shown in [1], any arbitrarily chosen point within $B(D)$ can belong to at most m overlapping disks from R_m. Then for every $2 \le m \le n-1$, we have

$$\sum_{i=1}^{n} A(B_i(u_{im}) \cap B(D)) \le mA(B(D)), \quad (3.4)$$

where $A(X)$ is the area of a region X. Then similarly to the deduction of the inequality (2.28), we have

$$mD^2 \ge \sum_{i=1}^{n}\left(f\left(\frac{u_{im}}{D}\right)u_{im}^2\right). \quad (3.5)$$

Also, we solve the following optimization problem ($n \ge 3$) to obtain the upper bound of $\sum_{i=1}^{n} u_{im}^2$.

$$\max \quad D^2\sum_{i=1}^{n} y_i^2 \quad (3.6)$$

$$s.t. \quad \sum_{i=1}^{n} F(y_i) \le m, \quad (3.7)$$

$$0 \le y_i \le 1, \quad i = 1,\dots,n, \quad (3.8)$$

where $y_i = \frac{u_{i1}}{D}$ for $i = 1,\dots,n$ and the function $F(y)$ is defined in (2.35). Similarly to the analysis of Problem (2.32)-(2.34), we obtain an optimal solution to Problem (3.6)-(3.8), which is $y^*_i = 1$ for $i = 1,\dots,\left\lfloor\frac{m}{F(1)}\right\rfloor$, $y^*_i = \left\lfloor\frac{m}{F(1)}\right\rfloor+1,\dots,n-1$ and $y^*_n = c_m \in (0,1)$ such that $F(c_m) = m - \left\lfloor\frac{m}{F(1)}\right\rfloor F(1)$. We notice that $0 < m - \left\lfloor\frac{m}{F(1)}\right\rfloor F(1) < F(1)$, $F(y) \in [0, F(1)]$ for $y \in [0,1]$ and $F(1) = c_1 = \frac{2}{3} - \frac{\sqrt{3}}{2\pi}$. Hance, the equation $F(y) = m - \left\lfloor\frac{m}{F(1)}\right\rfloor F(1)$ always has a unique solution $c_m \in (0,1)$. And then the optimal objective function value of problem (3.6)-(3.8) is $\left(\left\lfloor\frac{m}{F(1)}\right\rfloor + c_m^2\right)D^2$, which upper bounds $\sum_{i=1}^{n} u_{im}^2$.

Remark 3.1. *Notice that $F(y) = f(y)y^2 > c_1y^2$ for $y \in (0, 1)$. We have*

$$c_1c_m^2 < F(c_m) = m - \left\lfloor\frac{m}{c_1}\right\rfloor c_1.$$

Therefore, Theorem 3.1 strictly improves Proposition 1.2 since

$$\left\lfloor\frac{m}{c_1}\right\rfloor + c_m^2 < \frac{m}{c_1}.$$

4 .CONCLUSION

In this article, we improve the interpoint distance sum inequality for general m. The special case $m = 1$ of this inequality can be restated as follows. Let n points be arbitrarily placed in $B(D)$, a disk in $\mathbb{R}^2$ having diameter D. Denote by l_{ij} the Euclidean distance between point i and j. In this article, we show

$$\sum_{i=1}^{n}\left(\min_{j\neq i} l_{ij}^2\right) < \frac{D^2}{0.4018}$$

where the best known denominator on the right hand side of the inequality before this article is 0.3972. Naturally, one can ask what the optimal (or largest) denominator is. Based on some key observations, we raise a conjecture which states that the optimal denominator is 4/9, see Conjecture 2.1. The special case n=3 is proved.

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