NEW FIXED POINT ITERATIVE METHOD FOR SOLVING NONLINEAR FUNCTIONAL EQUATIONS

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ABSTRACT: *In this paper, we present a new fixed point iterative method for solving nonlinear functional equations and analyzed. The new fixed point iterative method has convergence of order two. The new fixed point iterative method converges faster than the fixed point method. The comparison table demonstrates the faster convergence of new fixed point method.*

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1 INTRODUCTION

The problem, to recall, is solving equations in one variable. We are given a function f and would like to find atleast one solution of the equation $f(x)=0$. Note that, we do not put any restrictions on the function f; we need to be able to evaluate the function; otherwise, we cannot even check that a given $x=$□ is true, that is $f(r)=0$. In reality, the mere ability to be able to evaluate the function does not suffice. We need to assume some kind of "good behavior". The more we assume, the more potential we have, on the one hand, to develop fast iteration scheme for finding the root. At the same time, the more we assume, the fewer the functions are going to satisfy our assumptions! This is a fundamental paradigm in numerical analysis.

We know the fundamental algorithm for solving nonlinear equations is so-called fixed point iteration method [1].

In the fixed-point iteration method for solving nonlinear equation $f(x)=0$, the equation is usually rewritten as

$$x=g(x), \quad (1)$$

where

(i) there exists $[a,b]$ such that $g(x)\in[a,b]$ for all $x\in[a,b]$,

(ii) there exists $[a,b]$ such that $|g'(x)|\leq L<1$ for all $x\in[a,b]$.

Considering the following iteration scheme

$$x_{n+1}=g(x_n), n=0,1,2\ldots, \quad (2)$$

and starting with a suitable initial approximation x_0, we built up a sequence of approximations, say $\{x_n\}$, for the solution of nonlinear equation, say ξ. the scheme will be converge to ξ, provided tha

(i) the initial approximation x_0 is chosen in the interval $[a,b]$,

(ii) $|g'(x)|<1$ for all $x\in[a,b]$,

(iii) $a\leq g(x)\leq b$ for all $x\in[a,b]$.

It is well known that the fixed point method has first order convergence.

Shin *et al*. described a new second order iterative method for solving nonlinear equations [18] extracted from fixed point method by following the approach of [8] as follows:

If $g'(x)\neq 1$, we can modify (1) by adding $\theta\neq -1$ to both sides as:

$$\theta x+x=\theta x+g(x),$$

$$(1+\theta)x=\theta x+g(x),$$

which implies that

$$x=\frac{\theta x+g(x)}{1+\theta}=g_\theta(x) \quad (3)$$

In order for $g_\theta(x)$ to be efficient, we can choose θ such that $g'_\theta(x)=0$, we yields

$$\theta=-g'(x),$$

so that (3) takes the form

$$x=\frac{-xg'(x)+g(x)}{1-g'(x)}.$$

For a given x_0, we calculate the approximation solution x_{n+1}, by the iteration scheme

$$x_{n+1}=\frac{-x_n g'(x_n)+g(x_n)}{1-g'(x_n)}, g'(x_n)\neq 1.$$

This is so-called a new second order iterative method for solving nonlinear equations, which converges quadratically.

Table: Comparison of FPM and NFPIM

Method	N	N_f	$\vert f(x_{n+1})\vert$	x_{n+1}
$f(x)=x+\ln(x-2), g(x)=2+e^{-x}$				
$x_0=2.2$				
FPM	32	32	5.410786e-30	2□12002823898764122948468797527 2
NFPIM	4	12	1.497263e-51	2□12002823898764122948468797527 2
$f(x)=x^3+4x^2-10, g(x)\sqrt{\frac{10}{4+x}}$				
$x_0=1.5$				
FPM	34	34	6.189210e-30	1□36523001341409684576080682898 2
NFPIM	4	12	6.592144e-46	1□36523001341409684576080682898 2
$f(x)=x^2-e^x-3x+2, g(x)=\ln(x^2-3x+2)$				
$x_0=0.8$				
FPM			Diverged	
NFPIM	6	18	4.911909e-46	0□25753028543986076045536730493 7
$f(x)=x^3+x^2-3x-3, g(x)=(3+3x-x^2)^{\frac{1}{3}}$				
$x_0=1$				
FPM	24	24	9.368499e- 30	1□73205080756887729352744634150 6
NFPIM	5	15	6.239681e-42	1□73205080756887729352744634150 6
$f(x)=x^3+4x^2+8x+8, g(x)=-(1+\frac{1}{2}x^2+\frac{1}{8}x^3)$				
$x_0=-1.9$				
FPM	97	97	5.570340e- 30	-2.000000000000000000000000000000
NFPIM	5	15	1.039840e-50	-2.000000000000000000000000000000

During the last century, the numerical techniques for solving nonlinear equations have been successfully applied (see, e. g., [2-19] and the references therein).

***Theorem 1*.[8]**

Suppose $g(x)\in C^p[a,b]$. If $g^{(k)}(x)=0$, for $k=1,2,...,p-1$ and $g^{(p)}(x)\neq 0,$ then the sequence $\{x_n\}$ is of order p.

In this paper, we presented a new fixed point iterative method for solving nonlinear functional equations having (NFPIM) convergence of order 2 extracted from fixed point iterative method for solving nonlinear equations motivated by the technique of Fernando et al. [11]. The proposed new fixed point iterative method applied to solve some problems in order to assess its validity and accuracy.

2 Main results

Let $f{:}X\subset R\rightarrow R$ for an open interval X is a scalar function and consider that the nonlinear equation $f(x)=0$ (or $x=g(x)$) ,

where $g(x){:}X\subset R\rightarrow R$,

then, we have a new second order iterative method [18]

$$x_{n+1}=\frac{-x_n g'(x_n)+g(x_n)}{1-g'(x_n)}, g'(x_n)\neq 1. \qquad (4)$$

By following the approach of Fernando et al. [11], we develop a new fixed point iterative method by replacing $g'(x_n)$ by harmonic mean $\frac{2g'(x_n)g'(v_n)}{g'(x_n)+g'(v_n)}$ as follows:

$$x_{n+1}=g(x_n)+\frac{2g'(x_n)g'(v_n)(g(x_n)-x_n)}{g'(x_n)+g'(v_n)-2g'(x_n)g'(v_n)}, \qquad (5)$$

$$v_n=g(x_n).$$

3 Convergence Analysis

Theorem 3.1 *Let $f{:}X\subset R\rightarrow R$ for an open interval X and consider that the nonlinear equation $f(x)=0$ (or $x=g(x)$) has a simple root $\alpha\in X$, where $g(x){:}X\subset R\rightarrow R$ be sufficiently smooth in the neighborhood of α; then the convergence order of new fixed point iterative method given in (4) is at least two.*

Proof.

To analysis the convergence of new fixed point iterative method (5), let

$$H(x) = g(x) + \frac{2g'(x)g'(v)(g(x)-x)}{g'(x)+g'(v)-2g'(x)g'(v)}; g'(x) \neq 1$$

$$v = g(x).$$

Let α be a simple zero of *f* and *f*(α)=0 (or *g*(α)=α), then we can easily deduce by using the software Maple that

$$H(a)=a,$$

$$H'(a)=0,$$

$$H''(a)= \frac{g'(a)g''(a)}{-1+g'(a)}. \qquad (6)$$

Now, from (6) it can be easily seen that $H''(a)¹0$, then according to theorem 1, new fixed point iterative method (5) has second order convergence.

4 Applications

In this section we included some nonlinear functions to illustrate the efficiency of our developed new fixed point iterative method (NFPIM). We compare the NFPIM with Fixed point method (FPM) as shown in Table given at the end.

Table: (at the end) Shows the numerical comparisons of new fixed point iterative method with Fixed point method. The columns represent the number of iterations *N* and the number of functions or derivatives evaluations N_f required to meet the stopping criteria, and the magnitude |*f*(*x*)| of *f*(*x*) at the final estimate x_n.

5 CONCLUSIONS

A new FPIM for solving nonlinear functions has been established. We can concluded from table that

1. The new FPIM has convergence of order two.
2. By using some examples the performance of NFPIM is also discussed. The NFPIM is performing very well in comparison to FPM as discussed in the table given above.

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