SOLUTION OF PAINLEVÉ EQUATION-I USING A RAMP INPUT OF AN EQUIVALENT STATE SPACE MODEL
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ABSTRACT: An analytical solution of Painlevé equation-I about a nominal point is performed using the state space model of an equivalent system. The Ramp response of the system is taken using Laplace Transformation. The equivalent state space model to Painlevé equation-I turned out a nonlinear system. The nonlinear system is transformed in an equivalent linear system about an operating point. The ramp input of the transformed linear system is obtained. Which is the solution of the Painlevé equation-I.
Keywords: Painlevé equation; state space model; ramp input; Laplace transformation
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INTRODUCTION
The transcendental functions defined by six second-order nonlinear equations are called as Painlevé transcendent. These functions described different physical processes. The use of Painlevé transcendent is shown in references [2,3,4]. Many researchers have applied transcendent in diverse field of engineering [5,6,7]. The initial value problem of Painlevé equation-I (PE-I) can be written as,
\[ y''(t) + 6y(t)^2 + t = 0, \quad y'(0) = 1 \]  \( (1) \)
In reference [9], the authors have linearized Painlevé equation-I by developing Fuchs-Garnier pairs which are linear in spectral parameter. The linear systems may be written as scalar or matrix equations and are typically referred to as Lax pairs for the Painlevé equations. The authors suggested calling these systems the Fuchs-Garnier pairs for the Painlevé equations to pay tribute to the two scientists who first introduced these systems in the beginning of the XX-th century.
A methodology for solution of Painlevé equation-I using computational intelligence technique is presented in [10] based on neural networks and particle swarm optimization hybridized with active set algorithm. The authors used the strength of feed forward artificial neural networks (ANNs) in their computations.
In the present studies an equivalent state space model of Painlevé equation-I is developed. The model appeared as a nonlinear. The model is then converted into linear state space model. The ramp input of the model is taken which is the solution of Painlevé equation-I.
The organization of the paper is as follows: an equivalent state space model is developed and explained in the section 2. The linearization procedure of the state space model is enlightened in third section. The fourth section is reserved for the ramp response of the model.

BLOCK DIAGRAM AND STATE SPACE MODEL
State space modeling is increasingly used in control mathematics. There are several advantages of state space modeling. It is a straightforward to implement and it provides a simple representation of relatively complex problem. The state space model of Painlevé equation-I provides a straightforward understanding of the equation. Before we switch to the state space model it is worth discussing the block diagram representation of the PE-1. The block diagram modeling may provide the reader the better understanding of the composition and interconnection of the components of PE-1. It can describe the cause and effect relationships of the system.
The PE-1 can be divided in two main components in order to simplify the solution of equation, input R(s) and input/output relationship called a transfer function G(s) as shown in figure. Figure 1, shows the block diagram of a system with transfer function G(s) and output Y(s). The input of the system is R(s) which is a ramp input in our case. The ramp function is signal that changes constantly with time. Mathematically, a ramp function is represented by [r(t) = Rtu_0(t)] Where R is real constant and u_0(t) is a step input.

![Block Diagram](image-url)

In order to define the equivalent block diagram and state space model of PE-I, it can be divided in to two parts (1) \[ y''(t) + 6y(t)^2 \] and (2) \[ r(t) = Rtu_0(t) \] Where R=1 and u_0(t)=1. Now Painlevé equation-I becomes,
\[ y''(t) + 6y(t)^2 + r(t) \]  \( (2) \)
Now we have a nonlinear dynamic system with output variable Y(s) and input variable R(s). In order to find the transfer function G(s) we need to linearize the system. This is discussed in next section. If we assume \( x_1=y \), then the state space model of PE-I can be written as
\[ \begin{align*}
  x_1 &= x_2 \\
  x_2 &= 6x_2^2 + r(t)
\end{align*} \]  \( (3) \)

LINEARIZATION PROCEDURE
Taylor series may be used to expand a nonlinear function about a reference or operating value [11]. Let us represent a nonlinear system by the following vector-matrix state equations:
\[ \frac{dx(t)}{dt} = f(x(t), r(t)) \]  \( (4) \)
Where \( \mathbf{x}(t) \) represents the \( n \times 1 \) state vector; \( \mathbf{r}(t) \), \( q \times 1 \) input vector; and \( f(\mathbf{x}(t), \mathbf{r}(t)) \), an \( n \times 1 \) function of state vector and input vector. Let the nominal operating trajectory be denoted by \( \mathbf{x}_0(t) \) and corresponding nominal input \( \mathbf{r}_0(t) \) with some fixed initial state. Expanding nonlinear state equation using Taylor series and neglecting higher order terms,

\[
\mathbf{x}_i = f_i(\mathbf{x}_0, \mathbf{r}_0) + \sum_{j=1}^{n} \left. \frac{\partial f_j}{\partial x_j} \right|_{\mathbf{x}_0, \mathbf{r}_0} (\mathbf{x}_j - \mathbf{x}_0) + \sum_{j=1}^{n} \left. \frac{\partial f_j}{\partial r_j} \right|_{\mathbf{x}_0, \mathbf{r}_0} (\mathbf{r}_j - \mathbf{r}_0)
\]

(5)

for \( i = 1, 2 \ldots n \).

Let

\[
\Delta \mathbf{x}_i = (\mathbf{x}_i - \mathbf{x}_0)
\]

\[
\Delta \mathbf{r}_i = (\mathbf{r}_i - \mathbf{r}_0)
\]

Then

\[
\mathbf{A} \Delta \mathbf{x} = \sum_{i=1}^{n} \left. \frac{\partial f_i}{\partial x_i} \right|_{\mathbf{x}_0, \mathbf{r}_0} \Delta \mathbf{x}_i + \sum_{j=1}^{n} \left. \frac{\partial f_j}{\partial r_j} \right|_{\mathbf{x}_0, \mathbf{r}_0} \Delta \mathbf{r}_j
\]

(6)

Using equation (5) in (3) and writing in matrix form

\[
\Delta \mathbf{x} = \mathbf{A} \Delta \mathbf{x} + \mathbf{B} \Delta \mathbf{r}
\]

\[
\mathbf{y} = \mathbf{C} \Delta \mathbf{x}
\]

(7)

Where

\[
\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 2x_{01} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}
\]

The nominal point is \( (x_{01}, x_{02}) \). The ramp response of the system is solution of PE-I which is discussed in the next section.

**RAMP RESPONSE OF THE MODEL**

In order to obtain the ramp response of the system, the state transition matrix needs to be calculated. The state transition matrix satisfies the homogeneous state equation, it represents the free response of the system. In other words, it governs the response which excited by the initial conditions only. The state-transition equation is defined as the solution of linear homogeneous state equation. This can be obtained by Laplace transform of the state equation as follows.

\[
\Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{x}(0) + (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} \mathbf{u}(s)
\]

(8)

The state-transition equation in time domain can be obtained by taking the inverse Laplace Transform of above equation.

\[
\Phi(t) = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}] \mathbf{x}(0) + \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} \mathbf{u}(s)]
\]

(9)

\[
\mathbf{x}(t) = \Phi(t) \mathbf{x}(0) + \int_0^t \phi(t-r) \mathbf{B} \mathbf{u}(r) dr
\]

\[
\mathbf{y}(t) = \mathbf{C} \Phi(t) \mathbf{x}(0)
\]

The transfer function can be obtained as

\[
\mathbf{y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} = \frac{1}{s^2 - 2x_{01} s + x_{02}}
\]

Non linear systems are difficult to analyze, it is desirable to perform a linearization whenever, situation justifies it. The nonlinear state equations are linearized around a nominal point \( (x_{01}, x_{02}) \). The Figure 2, shows the ramp response of the system which is equal to the solution of PE-I with zero initial condition. The nominal point is chosen to be \( x_{02} = 0.5 \).

**CONCLUSION**

An equivalent state space model of of Painlevé equation-I is developed whose ramp response about a nominal point is

![State space solution of PE-1](image)

Figure State space solution of PE-1 computed using Laplace transformation. The nonlinear term is linearized using Taylor series expansion. The equivalent linear model is then used to solve PE-I.

**REFERENCES**


