RELATION BETWEEN MEAN LABELING AND (*A*,*D*)-EDGE-ANTIMAGIC VERTEX LABELING *

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ABSTRACT*: An injective mapping* f, $f: V(G) \to \{0,1,2,\ldots,|E(G)|\}$ *is called mean labeling of* $G=(V,E)$ *if the induced edge function* g, $g: E(G) \to \{1,2,3,\cdots,|E(G)|\}$ *defined as*

$$g(uv) = \begin{cases} \dfrac{f(u)+f(v)}{2}, & \text{if } f(u) \text{ and } f(v) \text{ are of the same parity,} \\ \dfrac{f(u)+f(v)+1}{2}, & \text{otherwise} \end{cases}$$

is bijective.

A bijective mapping h, $h: V(G) \to \{0,1,2,\ldots,|V(G)|\}$ *is called an* (a,d) *-edge-antimagic vertex labeling, if the set of edge-weights* $\{h(u)+h(v):\ uv \in E(G)\}$ *forms an arithmetic sequence with the intial term* a *and the difference* d *, where* a *is a positive and* d *is a nonnegative integer.*

In this paper, we study the relation between mean labeling and (a,d) *-edge-antimagic vertex labeling. Moreover, we show that two classes of caterpillars admit mean labeling.*

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NOTATION AND PRELIMINARY RESULTS

As a standard notation, assume that $G=(V,E)$ is a finite simple and undirected graph. A labeling of a graph is any mapping that sends some set of graph elements to a set of numbers (usually positive integers). If the domain is the vertex-set or the edge-set, the labelings are called respectively vertex-labelings or edge-labelings. If the domain is $V \cup E$ then we call the labeling a total labeling. In many cases it is interesting to consider the sum of all labels associated with a graph element. This will be called the weight of the graph element.

An injective mapping f, $f: V(G) \to \{0,1,2,\ldots,|E(G)|\}$ is called *mean labeling* of $G=(V,E)$ if the induced edge function g, $g: E(G) \to \{1,2,3,\cdots,|E(G)|\}$ defined as

$$g(uv) = \begin{cases} \dfrac{f(u)+f(v)}{2}, & \text{if } f(u) \text{ and } f(v) \text{ are of the same parity,} \\ \dfrac{f(u)+f(v)+1}{2}, & \text{otherwise} \end{cases}$$

For every $uv \in E(G)$, is a bijection. A graph that admits a mean labeling is called a *mean graph*. On Figure 1 is illustrated a mean graph with the corresponding mean labeling.

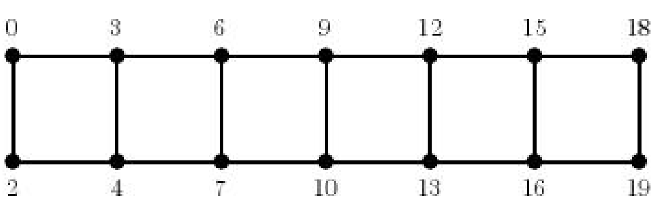

Figure 1: A mean graph with the corresponding mean labeling.

The concept of mean labeling was introduced by Somasundaram and Ponraj [5]. In [4, 5, 6, 7] and [8] they prove the following graphs are mean graphs: P_n, C_n, $K_{2,n}$, $K_2 + mK_1$, $K_n + 2K_2$, $C_m \cup P_n$, $P_m \times P_n$, $P_m \times C_n$, C_{nP_n}, P_{mK_1} triangular snake, triangular snakes, quadrilateral snakes, K_n if and only if $n < 3$, $K_{1,n}$ if and only if $n < 3$, bistars $B_{m,n} (m > n)$ if and only if $m < n+2$, the subdivision graph of the star $K_{1,n}$ if and only if $n < 4$, and the friendship graph $C^{(t)}3$ if and only if $t < 2$. They also prove that W_n is not a mean graph for $n > 3$ and enumerate all mean graphs of order less than 5. Finding the mean labeling seems to be hard even for simple graphs, see [2].

Let a > 0 and $d \geq 2$ are two fixed integers. A labeling h, $h: V(G) \to \{0,1,2,\ldots,|V(G)|\}$ is called an (a,d)-edge-antimagic vertex labeling, for short (a,d)-EAV labeling, if the set of the edge-weights forms an arithmetic sequence with the initial term *a* and the difference *d*, i.e.

$\{h(u)+h(v):\ uv \in E(G)\} = \{a, a+d, \ldots, (|E(G)|-1|)d\}$.

A graph that admits an (a,d)-EAV labeling is called an (a,d)-EAV graph. For more detail see [1, 3, 9].

A *caterpillar* is a tree that can be converted into a path or a single vertex by deleting all vertices of degree one. We will deal with two types of caterpillars.

A *comb graph*, denoted by Cb_n, is a caterpillar with the vertex set

$V(Cb_n) = \{x_i : 1 \leq i \leq n+2\} \cup \{y_i : 1 \leq i \leq n\}$

and the edge set

$E(Cb_n) = \{x_i x_{i+1} : 1 \leq i \leq n+1\} \cup \{x_{i+1} y_i : 1 \leq i \leq n\}$.

For illustration see Figure 2.

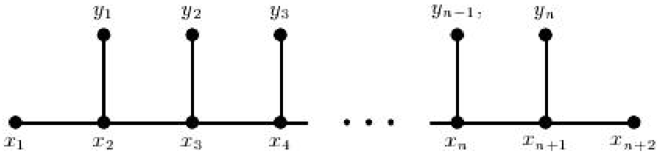

Figure 2: A comb graph Cb_n .

A *special caterpillar*, denoted by Sc_n , is a caterpillar with the vertex set

$V\left(Sc_n\right) = \{x_i : 1 \le i \le n\} \cup \{y_i : 1 \le i \le 2n\}$

and the edge set

$E\left(Sc_n\right) = \{x_i x_{i+1} : 1 \le i \le n-1\} \cup \{x_i y_{2i-1}, x_i y_{2i} : 1 \le i \le n\}$.

For illustration see Figure 3.

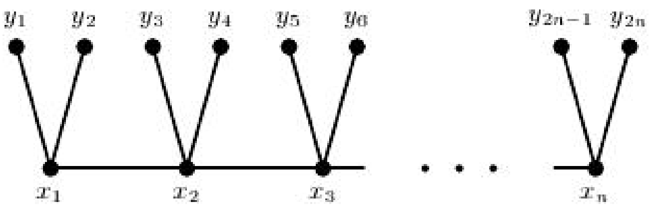

Figure 3: A special caterpillar graph Sc_n .

In this paper, we study properties (a,d) -EAV labeling and mean labeling and we prove that the comb graph Cb_n and the special caterpillar Sc_n are the mean graphs.

RELATION BETWEEN MEAN LABELING AND (*a,d*) - EAV LABELING

The main result in this section is given in the following theorem.

Theorem 1. *If G is an* (a,d) *-EAV graph,* $d \ge 2$ *, then G is also a mean graph.*

Proof. Let *G* be an *(a, d)*-EAV graph. Thus, there exists an *(a, d)*-EAV labeling *f* of *G*,

$$f : V(G) \to \{1,2,\ldots,|V(G)|\} \subset \{0,1,2,\ldots,|E(G)|\},$$

such that

$$\{f(u)+f(v): uv \in E(G)\} = \{a, a+d, \ldots, (|E(G)-1|)d\}.$$

We consider an edge function *g* of the graph *G* defined in the following way

$$g(uv) = \begin{cases} \dfrac{f(u)+f(v)}{2}, & \text{if } f(u) \text{ and } f(v) \text{ are of the same parity,} \\ \dfrac{f(u)+f(v)+1}{2}, & \text{otherwise} \end{cases}$$

Now we will consider two cases according to the parity of the difference *d*.

Case 1. $2 \le d \equiv 0 \pmod 2$

In this case the edge-weights under the labeling *f* have the same parity. For $a \equiv 0 \pmod 2$ the edge-weights are even and we have

$$\{g(uv) = (f(u)+f(v))/2 : uv \in E(G)\} = \{a/2, (a+d)/2, (a+d)/2$$

EMBED Equation.DSMT4 For $a \equiv 1 \pmod 2$ the edge-weights are odd thus we obtain

$$\{g(uv) = \frac{f(u)+f(v)+1}{2} : uv \in E(G)\} = \{\frac{a+1}{2}, \frac{a+d+1}{2}, \ldots, \frac{a+(|E(G)-1|)d}{2}\}.$$

As $d \ne 0$, in both subcases the induced labeling *g* is a bijective mapping and thus *f* is a mean labeling of *G*.

Case 1. $3 \le d \equiv 1 \pmod 2$

In this case the edge-weights under the labeling *f* have the alternating parity. For $a \equiv 0 \pmod 2$ we get

$$\{g(uv): uv \in E(G)\} = \{\frac{a}{2}, \frac{a+d+1}{2}, \frac{a+2d}{2}, \frac{a+3d+1}{2}, \ldots\}.$$

For $a \equiv 1 \pmod 2$ we have

$$\{g(uv): uv \in E(G)\} = \{\frac{a+1}{2}, \frac{a+d}{2}, \frac{a+2d+1}{2}, \frac{a+3d}{2}, \ldots\}.$$

As $d \ne 1$, in both subcases the induced labeling *g* is a bijective mapping and thus *f* is a mean labeling of *G*.

MEAN LABELING OF SOME CLASSES OF CATERPILLARS

For the comb graph we proved.

Theorem 2. *For every positive integer n the comb* Cb_n *is a mean graph.*

Proof. Let *n* be a positive integer. We define the labeling f, $f : V(Cb_n) \to \{0,1,2,\ldots,2n+1\}$ in the following way

$$f(x_i) = \begin{cases} 2(i-1), & \text{if } 1 \le i \le n+1, \\ 2n+1, & \text{if } i = n+2 \end{cases}$$

$$f(y_i) = 2i-1, \quad \text{for } 1 \le i \le n.$$

For the induced edge labeling *g* of labeling *f* we have

$$g(x_i x_{i+1}) = 2i-1, \quad \text{for } 1 \le i \le n+1,$$

$$g(x_{i+1} y_i) = 2i, \quad \text{for } 1 \le i \le n.$$

It is easy to see that *g* is a bijective function that assigns the consecutive integers $1,2,\ldots,2n+1$ to the edges of Cb_n. Therefore Cb_n is a mean graph.

Theorem 3. *For every positive integere r n caterrpillar* Sc_n *is a mean graph.*

Proof. Let *n* be a positive integer. We define the vertex labeling f for Sc_n as follows:

$$f(x_i) = 3i-2, \quad \text{if } 1 \le i \le n,$$

$$f(y_i) = \begin{cases} \dfrac{3(i-1)}{2}, & \text{if } i \equiv 1 \pmod 2 \\ \dfrac{3i-2}{2},, & \text{if } i \equiv 1 \pmod 2 \end{cases}$$

For the induced edge labeling *g* of f we get.

$$g(x_i x_{i+1}) = 3i, \quad \text{if } 1 \le i \le n-1,$$

$$g(x_i y_{2i}) = 3i-1, \quad \text{if } 1 \le i \le n,$$

$$g(x_i y_{2i-1}) = 3i-2, \quad \text{if } 1 \le i \le n.$$

It is easy to see that *g* assigns the consecutive integers $1,2,\ldots,3n$ to the edges of Sc_n. Therefore Sc_n is a mean graph. W

CONCLUSION

In this paper we mention the relationship between the mean labeling and the *(a, d)*-edge-antimagic vertex labeling. For further investigation we recall to find relationship between the mean labeling and a graceful labeling or ~ -labeling or other kind of labeling. Moreover, we prove that the comb graph Cb_n and special caterpillar Sc_n are mean graphs.

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