TOTAL VERTEX IRREGULARITY STRENGTH OF LADDER RELATED GRAPHS

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ABSTRACT*: We investigate modifications of the well-known irregularity strength of graphs, namely the total vertex irregularity strength. In this paper, we determine the exact value of the total vertex irregularity strength of families of ladder related graphs, namely, triangular ladder, diagonal ladder, triangular snake and double triangular snake.*

Keywords: vertex irregular total labeling, total vertex irregularity strength, triangular ladder, diagonal ladder, triangular snake, double triangular snake.

Subject Classification: 05C78

1. INTRODUCTION AND DEFINITIONS

As a standard notation, assume that $G = G(V, E)$ is a finite, simple and undirected graph with p vertices and q edges. A labeling of a graph is any mapping that sends some set of graph elements to a set of numbers (usually positive integers). If the domain is the vertex-set or the edge-set, the labeling are called respectively vertex-labeling or edge-labeling. If the domain is $V \cup E$ then we call the labeling a total labeling. In many cases it is interesting to consider the sum of all labels associated with a graph element. This will be called the *weight* of element.

Motivated by total labeling mentioned in a book of Wallis [10], Bača et al. in [5] introduced a vertex irregular total labeling of graphs. For a simple graph $G = (V, E)$ with vertex set V and edge set E, a labeling $\phi: V \cup E \rightarrow \{1, 2, . . . , k\}$ is called *total k−labeling*. The associated vertex weight of a vertex $x \in V(G)$ under a total k−labeling ϕ is defined as

$$wt(x) = \phi(x) + \sum_{y \in N(x)} \phi(xy),$$

where $N(x)$ is the set of neighbors of x. A total k-labeling ϕ is defined to be a *vertex irregular total labeling* of a graph G if for every two different vertices x and y of G,

$wt(x) \neq wt(y)$

The minimum k for which a graph G has a vertex irregular total k-labeling is called the *total vertex irregularity strength* of G, $tvs(G)$.

In this paper, we study properties of the vertex irregular total labeling and determine a value of the total vertex irregularity strength for classes of ladder related graphs, such as triangular ladder, diagonal ladder, triangular snake and double triangular snake.

Triangular Ladder, denoted by TL_n, is the graph obtained from ladder by adding single diagonal to each rectangle. Thus the vertex set of TL_n is $\{v_{i,j} \mid 1 \leq i \leq 2, 1 \leq j \leq n\}$ and the edge set of L_n is

$E(G) = \{v_{i,j}v_{i,j+1} | 1 \leq i \leq 2, 1 \leq j \leq n - 1\} \cup \{v_{i,j}v_{i+1,j} | i = 1, 1 \leq j \leq n\} \cup \{v_{i,j}v_{i+1,j+1} | i = 1, 1 \leq j \leq n - 1\}$

Diagonal Ladder, denoted by DL_n, is the graph obtained from ladder by adding two diagonals to each rectangle. Thus the vertex set of DL_n is $\{v_{i,j} \mid 1 \leq i \leq 2, 1 \leq j \leq n\}$ and the edge set of DL_n is

$E(G) = \{v_{i,j}v_{i,j+1} | 1 \leq i \leq 2, 1 \leq j \leq n-1\} \cup \{v_{i,j}v_{i+1,j} | i = 1, 1 \leq j \leq n\} \cup \{v_{i,j}v_{i+1,j+1} | i = 1, 1 \leq j \leq n - 1\} \cup \{v_{i,j}v_{i+1,j-1} | i = 1, 2 \leq j \leq n\}$

Triangular Snake, denoted by TS_n, is the graph obtained from a non-trivial path P_n: $v_1, v_2, . . . , v_n$ by adding new vertices $u_1, u_2, . . . , u_{n-1}$ joining each u_i with v_i and v_{i+1} $(1 \leq i \leq n - 1)$. Thus the vertex set of TS_n is

$\{v_i, u_j | 1 \leq i \leq n, 1 \leq j \leq n - 1\}$

and the edge set of TS_n is

$\{v_iv_{i+1}, v_iu_i, u_iv_{i+1} | 1 \leq i \leq n - 1\}$

Double Triangular Snake, denoted by DTS_n, is the graph obtained from a triangular snake TS_n by adding new vertices $w_1, w_2, . . . , w_{n-1}$ joining each w_i with v_i and v_{i+1} $(1 \leq i \leq n - 1)$. Thus the vertex set of DTS_n is

$\{v_i, u_j, w_j | 1 \leq i \leq n, 1 \leq j \leq n - 1\}$

and the edge set of DTS_n is

$\{v_iv_{i+1}, v_iu_i, u_iv_{i+1}, v_iw_i, w_iv_{i+1} | 1 \leq i \leq n - 1\}$

2. KNOWN RESULTS

The following theorem proved in [5], establishes lower and upper bound for the total vertex irregularity strength of a (p, q)-graph.

Theorem 1 [5] Let G be a (p, q)-graph with minimum degree $\delta = \delta(G)$ and maximum degree $\Delta = \Delta(G)$. Then

$$\left\lceil \frac{p+\delta}{\Delta+1} \right\rceil \leq tvs(G) \leq p + \Delta - 2\delta + 1 \qquad (1)$$

If G is an r-regular (p, q)-graph then from Theorem 1 it follows:

$$\left\lceil \frac{p+r}{r+1} \right\rceil \leq tvs(G) \leq p - r + 1$$

For a regular Hamiltonian (p, q) graph G, it was showed in [5] that $tvs(G) = \left\lceil \frac{p+2}{3} \right\rceil$. Thus for cycle C_p we have that $tvs(C_p) = \left\lceil \frac{p+2}{3} \right\rceil$

Recently, a much stronger upper bound on total vertex irregularity strength of graphs has been established in [4]. In [6, 7, 8], Nurdin et al. found the exact values of total vertex irregularity strength of trees, several types of trees and disjoint union of t copies of path. Whereas the total vertex irregularity strength of cubic graphs, wheel related graphs, Jahangir graph $J_{n,2}$ for $n \geq 4$ and circulant graph $C_n(1, 2)$ for $n \geq 5$ has been determined by Ahmad et al. [1, 2, 3]. K. Wijaya et al. [11, 12] found the exact value of the total vertex irregularity strength of wheels, fans, suns, friendship, and complete bipartite graphs. Slamin et al. [9] determined the total vertex irregularity strength of disjoint union of sun graphs.

3. MAIN RESULT

We start this section with the result on the total vertex irregularity strength of triangular ladder TL_n graph in the following theorem.

Theorem 2 *The total vertex irregular strength of triangular ladder TL_n, for $n \geq 8$, is*

$$tvs\ (TL_n) = \left\lceil \frac{2n+2}{5} \right\rceil$$

Proof.

Recall that the vertex set and edge set of triangular ladder are

$$V(G) = \{v_{i,j} \mid 1 \leq i \leq 2, 1 \leq j \leq n\}$$

$$E(G) = \{v_{i,j}v_{i,j+1} \mid 1 \leq i \leq 2, 1 \leq j \leq n-1\} \cup \{v_{1,j}v_{2,j} \mid 1 \leq j \leq n\}$$

$$[\{v_{1,j}v_{2,j+1} \mid 1 \leq j \leq n-1\}$$

The triangular ladder TL_n has 2 vertices of degree 2, 2 vertices of degree 3, and $2n - 2$ vertices of degree 4. The lower bound of the total vertex irregular strength of triangular ladder TL_n follows from (1). Thus

$$tvs\ (TL_n) \geq \left\lceil \frac{2n+2}{5} \right\rceil$$

We now prove the upper bound by providing labelling construction for TL_n.

Let $\left\lceil \frac{2n+2}{5} \right\rceil = k$

$$\phi(v_{1,j}) = \begin{cases} 2, & \text{for } j = 1 \\ 1, & \text{for } 2 \leq j \leq k,\ j = n-1,\ n \\ \min\{j-k+2,\ k\}, & \text{for } k+1 \leq j \leq n-3 \\ n+2-k-\left\lceil \frac{n}{2} \right\rceil, & \text{for } j = n-2 \end{cases}$$

$$\phi(v_{2,j}) = \begin{cases} 2, & \text{for } j = 1 \\ 1, & \text{for } 1 \leq j \leq k \\ n-2k+3, & \text{for } j = k+1 \\ \left\lceil \frac{n-2k+5}{2} \right\rceil + \left\lceil \frac{j-k-2}{2} \right\rceil & \text{for } k+2 \leq j \leq n-2 \\ 2n+2-4k, & \text{for } j = n-1 \text{ \& k and j have same parity} \\ 2n+2-3k-\left\lceil \frac{2n-3k+1}{2} \right\rceil & \text{for } j = n-1 \text{ \& k and j have different parity} \\ 3, & \text{for } j = n \end{cases}$$

$$\phi(v_{1,j}v_{1,j+1}) = \begin{cases} 1, & \text{for } j = n-1 \\ k, & \text{for } j = n-2 \\ 1, & \text{for } 1 \leq j \leq 2k-3 \\ \left\lceil \frac{j-2k+5}{2} \right\rceil, & \text{for } 2k\text{-}2 \leq j \leq n-3 \end{cases}$$

$$\phi(v_{2,j}v_{2,j+1}) = \begin{cases} 1, & \text{for } 1 \leq j \leq k \\ k, & \text{for } k+1 \leq j \leq n-2,\ k \text{ and } j \text{ have different parity} \\ \left\lceil \frac{n-3k+j+3}{2} \right\rceil, & \text{for } k+1 \leq j \leq n-2,\ k \text{ and } j \text{ have same parity} \\ k, & \text{for } j = n-1 \end{cases}$$

$$\phi(v_{1,j}v_{2,j}) = \begin{cases} \min\{j,k\}, & \text{for } 1 \leq j \leq n-1 \\ 1, & \text{for } j = n \end{cases}$$

$$\phi(v_{1,j}v_{2,j+1}) = \min\{j,k\}, \text{ for } 1 \leq j \leq n-1$$

This labeling gives weight of the vertices as follows:

$$wt(v_{1,j}) = \begin{cases} 3, & \text{for } j = n \\ 2j+3, & \text{for } 1 \leq j \leq k+1 \\ n+k+3, & \text{for } j = n-2 \\ 3k+2, & \text{for } j = n-1 \\ j+k+4, & \text{for } k+2 \leq j \leq 2k-3 \\ j+k+5, & \text{for } 2k-2 \leq j \leq n-3 \end{cases}$$

$$wt(v_{2,j}) = \begin{cases} 2j+2, & \text{for } 1 \leq j \leq k \\ 2n+4, & \text{for } j = n \\ n+j+3, & \text{for } k+1 \leq j \leq n-1 \end{cases}$$

It is easy to check in both these cases that the weights of the vertices are different, that is *{3, 4, . . . , 2n + 2}*. This labelling construction shows that

$$tvs\ (TL_n) \leq \left\lceil \frac{2n+2}{5} \right\rceil$$

Combining with the lower bound, we conclude that

$$tvs\ (TL_n) = \left\lceil \frac{2n+2}{5} \right\rceil$$

The proof is now complete.

The illustration of the Theorem 2 is shown in Figure 1.

Let DL_n be a diagonal ladder graph, we find the total vertex irregularity strength of such a ladder in the following theorem.

Theorem 3 *The total vertex irregular strength of diagonal ladder DL_n, for $n \geq 3$ is*

$$tvs\ (DL_n) = \left\lceil \frac{2n+3}{6} \right\rceil$$

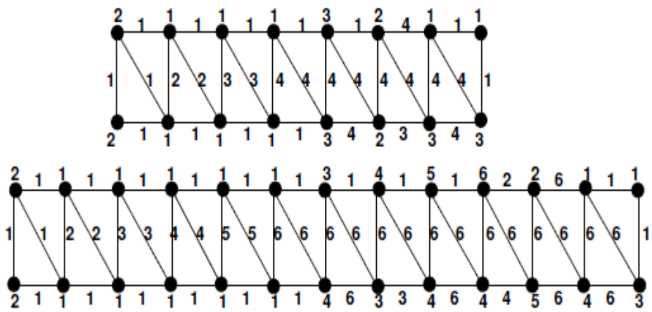

Figure 1: The vertex irregular total labeling of triangular ladder TL_8 and TL_{13}

Proof. Recall that the vertex set of DL_n is $\{v_{i,j} \mid 1 \leq i \leq 2,\ 1 \leq j \leq n\}$ and the edge set of DL_n is

$E(G) = \{v_{i,j}v_{i,j+1} \mid 1 \leq i \leq 2,\ 1 \leq j \leq n-1\} \cup \{v_{1,j}v_{2,j} \mid 1 \leq j \leq n\} \cup \{v_{1,j}v_{2,j+1} \mid 1 \leq j \leq n-1\} \cup \{v_{1,j}v_{2,j-1} \mid 2 \leq j \leq n\}$

The diagonal ladder DL_n has 4 vertices of degree 3, and 2n − 2 vertices of degree 5. The lower bound of the total vertex irregular strength of triangular ladder TL_n follows from (1). Thus

$$tvs\ (DL_n) \geq \left\lceil \frac{2n+3}{6} \right\rceil$$

We now prove the upper bound by providing labelling construction for DL_n.

Let $\left\lceil \frac{2n+3}{6} \right\rceil = k$

We label the vertices and the edges of DL_n in the following way.

$$\phi(v_{i,j}) = \begin{cases} 1, & \text{for } i=1,2 \text{ and } j=1,2 \\ \left\lceil \frac{2(j-i+2)}{3} \right\rceil - (2-i), & \text{for } i=1,2 \text{ and } 3 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor \\ k-i+1, & \text{for } i=1,2 \text{ and } j=\left\lfloor \frac{n}{2} \right\rfloor + 1, \text{ when } n \equiv 0 \pmod 2 \\ k, & \text{for } i=1,2 \text{ and } j=\left\lfloor \frac{n}{2} \right\rfloor + 1, \text{ when } n \equiv 1 \pmod 2 \\ \left\lceil \frac{4n+3+2i-4j}{6} \right\rceil, & \text{for } i=1,2 \text{ and } \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq j \leq n \end{cases}$$

$$\phi(v_{i,j}\, v_{i,j+1}) = \begin{cases} 1, & \text{for } i=1 \text{ and } j=1 \\ 2, & \text{for } i=2 \text{ and } j=1 \\ \left\lceil \frac{2j+i-1}{3} \right\rceil, & \text{for } i=1,2 \text{ and } 2 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \\ k, & \text{for } i=1 \text{ and } j=\left\lfloor \frac{n}{2} \right\rfloor, \text{ when } n \equiv 1 \pmod 2 \\ \left\lceil \frac{2j+i-1}{3} \right\rceil, & \text{for } i=2 \text{ and } j=\left\lfloor \frac{n}{2} \right\rfloor, \text{ when } n \equiv 1 \pmod 2 \\ \left\lceil \frac{2j+i-1}{3} \right\rceil & \text{for } i=1,2 \text{ and } j=\left\lfloor \frac{n}{2} \right\rfloor, \text{ when } n \equiv 0 \pmod 2 \\ \left\lceil \frac{4n+1+2i-4j}{6} \right\rceil & \text{for } i=1,2 \text{ and } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq j \leq n-1 \end{cases}$$

$$\phi(v_{1,j}\, v_{2,j-1}) = \begin{cases} \left\lceil \frac{2(j-1)}{3} \right\rceil, & \text{for } 2 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor \\ k, & \text{for } j=\left\lfloor \frac{n}{2} \right\rfloor + 1, \left\lfloor \frac{n}{2} \right\rfloor + 2; \text{ when } n \equiv 1 \pmod 2 \\ \left\lceil \frac{2(j-1)}{3} \right\rceil, & \text{for } j=\left\lfloor \frac{n}{2} \right\rfloor + 1; \text{ when } n \equiv 0 \pmod 2 \\ \left\lceil \frac{4n+1+2i-4j}{6} \right\rceil & \text{for } j=\left\lfloor \frac{n}{2} \right\rfloor + 2; \text{ when } n \equiv 0 \pmod 2 \\ \left\lceil \frac{4n+1+2i-4j}{6} \right\rceil & \text{for } \left\lfloor \frac{n}{2} \right\rfloor + 3 \leq j \leq n \end{cases}$$

$$\phi(v_{1,j}\, v_{2,j}) = \begin{cases} \left\lceil \frac{2(j+1)}{3} \right\rceil - 1, & \text{for } 1 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor \\ k, & \text{for } j = \left\lfloor \frac{n}{2} \right\rfloor + 1 \\ \left\lceil \frac{4n-1+2i-4j}{6} \right\rceil, & \text{for } \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq j \leq n-1 \\ 2, & \text{for } j = n \end{cases}$$

$$\phi(v_{1,j}\, v_{2,j+1}) = \begin{cases} \left\lceil \dfrac{2j+i-1}{3} \right\rceil, & \text{for } 1 \le j \le \left\lfloor \dfrac{n}{2} \right\rfloor \\ k, & \text{for } j = \left\lfloor \dfrac{n}{2} \right\rfloor + 1, \\ & \textit{when } n \equiv 1 \pmod 2 \\ \left\lceil \dfrac{4n-3+2i-4j}{6} \right\rceil, & \text{for } j = \left\lfloor \dfrac{n}{2} \right\rfloor + 1, \\ & \textit{when } n \equiv 0 \pmod 2 \\ \left\lceil \dfrac{4n-3+2i-4j}{6} \right\rceil, & \text{for } \left\lfloor \dfrac{n}{2} \right\rfloor + 2 \le j \le n-1 \end{cases}$$

This labeling gives weight of the vertices as follows:

When n is odd,

$$wt(v_{1,j}) = \begin{cases} 4j+i-1, & \text{if n=1,2 and } 1 \le j \le \left\lfloor \frac{n}{2} \right\rfloor - 1 \\ 4j+i, & \text{if n=1,2 and } j = \left\lfloor \frac{n}{2} \right\rfloor \\ 4k+2\left\lfloor \frac{2j}{3} \right\rfloor + 2 - 2i, & \text{if n=1,2 and } j = \left\lfloor \frac{n}{2} \right\rfloor + 1; \\ & n \equiv 3 \pmod 6 \\ 4k+2\left\lfloor \frac{2j}{3} \right\rfloor + 1 - i, & \text{if n=1,2 and } j = \left\lfloor \frac{n}{2} \right\rfloor + 1; \\ & n \ne 3 \pmod 6 \\ (2n+i+(-1)^n) - 4(j - \left\lfloor \frac{n+3}{2} \right\rfloor) + 1, & \text{if n=1,2 and } j = \left\lfloor \frac{n}{2} \right\rfloor + 2 \\ (2n+i+(-1)^n) - 4(j - \left\lfloor \frac{n+3}{2} \right\rfloor), & \text{if n=1,2 and } \left\lfloor \frac{n}{2} \right\rfloor + 3 \le j \le n \end{cases}$$

When n is even,

$$wt(v_{1,j}) = \begin{cases} 4j+i-1, & \text{if n=1,2 and } 1 \le j \le \left\lfloor \frac{n}{2} \right\rfloor \\ 2n+4-i, & \text{if i=1,2 and } j = \left\lfloor \frac{n}{2} \right\rfloor + 1; \\ & n \equiv 0 \pmod 6 \\ 2n+i+1, & \text{if i=1,2 and } j = \left\lfloor \frac{n}{2} \right\rfloor + 1; \\ & n \ne 0 \pmod 6 \\ (2n+i+(-1)^n) - 4(j - \left\lfloor \frac{n+3}{2} \right\rfloor), & \text{if i=1,2 and } \left\lfloor \frac{n}{2} \right\rfloor + 2 \le j \le n \end{cases}$$

In both these cases It is easy to check that the weight of the vertices are different, that is $\{4, 5, ..., 2n-3, 2n-1, \ldots, 2n+3\}$. This labelling construction shows that

$$tvs\ (DL_n) \le \left\lceil \frac{2n+3}{6} \right\rceil$$

Combining with the lower bound, we conclude that

$$tvs\ (DL_n) = \left\lceil \frac{2n+3}{6} \right\rceil$$

Let TS_n be a triangular snake graph, we find the total vertex irregularity strength of such a graph in the following theorem.

Theorem 4 *The total vertex irregular strength of triangular snake graph TS_n, for $n > 7$ is*

$$\left\lceil \frac{2n+1}{5} \right\rceil$$

Proof. Recall that the vertex set of DL_n is

$$\{v_i, u_j \mid 1 \le i \le n,\ 1 \le j \le n-1\}$$

and the edge set of TS_n is

$$\{v_i v_{i+1}, v_i u_i, u_i v_{i+1} \mid 1 \le i \le n-1\}$$

The triangular snake graph TS_n has n + 1 vertices of degree 2, and n − 2 vertices of degree 4. The lower bound of the total vertex irregular strength of triangular ladder TL_n follows from (1). Thus

$$tvs\ (TS_n) \le \left\lceil \frac{2n+1}{5} \right\rceil$$

We now prove the upper bound by providing labelling construction for TS_n. Let

$$\left\lceil \frac{2n+1}{5} \right\rceil = k$$

We label the vertices and the edges of TS_n in the following way.

$$\phi(v_i) = \begin{cases} n-2k+3, & \textit{for } i = 1, 3 \le i \le k+1 \\ 2, & \textit{for } i = 2 \\ n-2k+2, & \textit{for } i = n, k+2 \le i \le 2k \\ i-4k+n+2, & \textit{for } 2k+1 \le i \le n-1 \end{cases}$$

$$\phi(u_i) = \begin{cases} k, & \textit{for } i = 1 \\ \max\{1, i-2k+1\}, & \textit{for } 2 \le i \le n-1 \end{cases}$$

$$\phi(v_i v_{i+1}) = k,\ \textit{for } 1 \le i \le n-1$$

$$\phi(u_i u_i) = \begin{cases} k, & \textit{for } i = 1 \\ 1, & \textit{for } 2 \le i \le k+1 \\ \min\{i-k, k\}, & \textit{for } \text{k+2} \le i \le n-1 \end{cases}$$

$$\phi(\mathrm{v}_{i+1} u_i) = \begin{cases} n-2k+1, & \textit{for } i = 1 \\ \min\{i-1, k\}, & \textit{for } 2 \le i \le n-1 \end{cases}$$

This labeling gives weight of the vertices as follows:

$$wt(v_i) = \begin{cases} n+2, & \textit{if } i = n \\ n+2+i, & \textit{if } 1 \le i \le n-1 \end{cases}$$

$$wt(u_i) = \begin{cases} n+1, & \textit{if } i = 1 \\ i+1, & \textit{if } 2 \le i \le n-1 \end{cases}$$

It is easy to check that the weight of the vertices are different, that is $\{3, 5, \ldots, 2n+1\}$. This labelling construction shows that

$$tvs(TS_n) \le \left\lceil \frac{2n+1}{5} \right\rceil$$

Combining with the lower bound, we conclude that

$$tvs(TS_n) = \left\lceil \frac{2n+1}{5} \right\rceil$$

Let DTS_n be a double triangular snake graph, we find the total vertex irregularity strength of such a graph in the following theorem.

Theorem 5 *The total vertex irregular strength of double triangular snake graph DTS_n, for $n \geq 4$ is*

$$\left\lceil \frac{2n}{3} \right\rceil$$

Proof. Recall that the vertex set of DTS_n is

$$\{v_i, u_j, w_j \,|\, 1 \leq i \leq n,\ 1 \leq j \leq n-1\}$$

and the edge set of DTS_n is

$$\{v_i v_{i+1}, v_i u_i, u_i v_{i+1}, v_i w_i, w_i v_{i+1} \,|\, 1 \leq i \leq n-1\}$$

Thus the double triangular snake graph DTS_n has $2n-2$ vertices of degree 2, 2 vertices of degree 3 and $n-2$ vertices of degree 6. The smallest weight of DTS_n must be 3, so the largest weight of vertices of degree 2 is at least $2n$, the largest weight of vertices of degree 3 is at least $2n+2$. Moreover, the largest weight of $n-2$ vertices of degree 6 is $3n$. Consequently, the largest label of one of vertices or edges of DTS_n is at least $\max\left\{\left\lceil \frac{2n}{3} \right\rceil, \left\lceil \frac{n+1}{2} \right\rceil, \left\lceil \frac{3n}{7} \right\rceil\right\} = \left\lceil \frac{2n}{3} \right\rceil$ for $n \geq 4$. Thus

$$tvs(DTS_n) \geq \left\lceil \frac{2n}{3} \right\rceil$$

We now prove the upper bound by providing labelling construction for DTS_n.

Let

$$\left\lceil \frac{2n}{3} \right\rceil = k$$

We label the vertices and the edges of DTS_n in the following way.

$$\phi(v_i) = \begin{cases} 2n-2k, & \text{for } i = 1 \\ \max\{1, i-k\}, & \text{for } 2 \leq i \leq n-1 \\ 2n+2-3k, & \text{for } i = n \end{cases}$$

$$\phi(u_i) = \max\{1, i-k+1\} \text{ for } 1 \leq i \leq n-1$$

$$\phi(w_i) = \begin{cases} i+1, & \text{for } 1 \leq i \leq k-1 \\ i-2k+n+1, & \text{for } k \leq i \leq n-1 \end{cases}$$

$$\phi(v_i u_i) = \begin{cases} 1, & \text{for } 2 \leq i \leq k \\ k, & \text{for } k+1 \leq i \leq n-1 \end{cases}$$

$$\phi(v_{i+1} w_i) = \begin{cases} 1, & \text{for } 1 \leq i \leq k-1 \\ k, & \text{for } k \leq i \leq n-1 \end{cases}$$

$$\phi(v_{i+1} u_i) = \min\{i, k\},\ \text{for } 1 \leq i \leq n-1$$

$$\phi(v_i v_{i+1}) = \phi(v_i w_i) = k,\ \text{for } 1 \leq i \leq n-1.$$

This labeling gives weight of the vertices as follows:

$$wt(v_i) = \begin{cases} 2n+1, & \text{if } i = 1 \\ 2n+2, & \text{if } i = n \\ 3k+2+i, & \text{if } 2 \leq i \leq k \\ i+5k, & \text{if } k+1 \leq i \leq n-1 \end{cases}$$

$$wt(u_i) = \begin{cases} i+2, & \text{if } 1 \leq i \leq k \\ i+k+1, & \text{if } k+1 \leq i \leq n-1 \end{cases}$$

$$wt(w_i) = \begin{cases} i+k+2, & \text{if } 1 \leq i \leq k-1 \\ i+n+1, & \text{if } k \leq i \leq n-1 \end{cases}$$

It is easy to check that the weight of the vertices are different. This labelling construction shows that

$$tvs(DTS_n) \leq \left\lceil \frac{2n}{3} \right\rceil$$

Combining with the lower bound, we conclude that

$$tvs(DTS_n) = \left\lceil \frac{2n}{3} \right\rceil$$

The proof is now complete.

4. REFERENCES

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