SOLUTION OF CERTAIN QUARTIC EQUATIONS

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ABSTRACT. *The reduction for both quartic equations $x^4 + 2^{2k}y^4 = z^2$ and $x^4 + (2^{2m} + 2)x^2y^2 + y^4 = z^2$ into the system of two quardratic equations by using Pythagorean result and Diophantine equation of three variables has been done in this note. Moreover, it is shown that both the quartic equations have trivial solution by using a system of equations. The natural extension of all the results of the paper [1] has also been given.*

Key Words: Diophantine equation, system of equations, Pythagorean triple, primitive Pythagorean triple, trivial solution.

1. INTRODUCTION

Mordell discussed the triviality of the solution of $x^4 + y^4 = z^2$ and $x^4 + 6x^2y^2 + y^4 = z^2$ subject to constraint $(x; y) = 1$ over the set of integers [2]. Later in [1], Fulix showed the equivalence trivially solutions of the equations $x^4 + 4y^4 = z^2$ and $x^4 + 6x^2y^2 + y^4 = z^2$ by reducing these equations into system of equations of second degree which he named "resolvent". We will follow the same terminology and extended his results. This paper determines the equivalence between other quartic equations using "resolvent", which is basically a second order system of equations obtained algebraically from the original Diophantine equations. A proposed form of this resolvent should be:

$$a\bar{X}^2 + b\bar{Y}^2 = c\hat{X}^2 + d\hat{Y}^2;\ \hat{X}\hat{Y} = \overline{XY},$$
$$\gcd(\hat{X},\hat{Y}) = 1 = \gcd(\bar{X},\bar{Y}).$$

Up till now this result has been useful in generating relationship between 4th-order Diophantine equations over integers and some results for factorial expansions of integers (e.g. $Z[2]$ and also $Z[i]$). We shall now elaborate the above results with example to show the triviality and equality of the solutions of both the equations.

The rest of this note is organized as follows: Section 2 recalls some basic definitions and notations for general properties of the ordinary simple graphs. In section 3, we present our main results of this paper. In particular, we reduced both quartic equations $x^4 + 2^{2k}y^4 = z^2$ and $x^4 + (2^{2m} + 2)x^2y^2 + y^4 = z^2$ into the system of two quardratic equation by using Pythagorean result and Diophantine equation of 3 variable in this paper. Moreover, it is shown that both the quartic equations have trivial solution by using the system of equations and generalized all the results of the paper [1]. We finally end up with a conclusion.

2. MATERIALS AND METHODS

The following definition and theorems are taken from ([3], [4], [5] and [6]). One can see the proof of these theorems in these references.

Definition: A Pythagorean triple is a set of three integer $x; y$ and z such that $x^2 + y^2 = z^2$ and the triple is said to be primitive if $gcd(x; y; z) = 1$.

Lemma 2.1: If $x; y; z$ is a primitive Pythagorean triple then one of the integer x or y is even while other is odd.

Theorem 2.2: All solution of the Pythagorean equation

$$x^2 + y^2 = z^2$$

satisfying the conditions

$$gcd(x; y: z) = 1, 2|x, \qquad x; y; z > 0$$

are given by formulas

$$x = 2st; y = s^2 - t^2; z = s^2 + t\text{^}2$$

for integers s > t > 0 such that $gcd(s; t) = 1$and s is not congruent $t(mod 2)$.

Theorem 2. 3: The only solutions of the Diophantine equation $x^2 + 2y^2 = z^2$ with $gcd(x; y; z) = 1$are given by formulas

$$y = 2st$$
$$x = \pm(2s^2 - t^2)$$
$$z = 2s^2 + t^2$$

The next theorem is proof in [9].

Theorem 2.4: If k and n are positive integer in 3-variables Diophantine equation

$$xy = kz^n$$

Then all positive integer solutions of this equation can be described by the parametric formulas (two parameters$t_1; t_2$) $x = at_1^n;y = bt_2^n;\ z = t_1t_2$ such that $gcd(a; b) = 1 = gcd(t_1; t_2);\ k = ab$.

3. MAIN RESULTS

In this section, we will give our main results of this note.

Theorem 3.1: The quartic equation

$$x^4 + 2^{2m}y^4 = z^2 \qquad \text{(A)}$$

is equivalent to the system of equations

$$\bar{X}^2 - \bar{Y}^2 = \hat{X}^2 + \hat{Y}^2, \bar{X}\bar{Y} = \hat{X}\hat{Y},$$
$$gcd(\bar{X},\bar{Y}) = 1 = gcd(\hat{\hat{X}},\hat{Y})$$

or

$$\bar{X}^2 - 2\bar{Y}^2 = \hat{X}^2 + 2\hat{Y}^2, \bar{X}\bar{Y} = \hat{X}\hat{Y},$$
$$gcd(\bar{X},\bar{Y}) = 1 = gcd(\hat{\hat{X}},\hat{Y})$$

In particularly, the equation (A) has solutions if and only if the above system of equations has solution.

Proof: Let $(x_0;\ y_0; z_0)$ be a solution of $x^4 + 2^{2m}y^4 = z^2$. Moreover, $gcd(x_0,\ 2^m y_0) = 1$; if not then one can verify by direct substitution that either equation will remain the same or the term 2^{2m} will change its position from $y\ to\ x$. Applying theorem 2.4 on equation (A) will provide the following solution

$$x_0^2 = a^2 - b^2;\ 2^m y_0^2 = 2ab;\ z_0 = a^2 + b^2 \quad (1)$$

$$\Rightarrow 2^{m-1}y_0^2 = ab \quad (2)$$

Furthermore, the ring of integers Z is a unique factorization domain and from equation(2), we have $a = \beta_1^{m-1} g_0^2$ and $b = \beta_2^{m-1} h_0^2$ such that $\gcd(g_0, h_0) = 1 = \gcd(\beta_1, \beta_2)$. Now, theorem 2.4 proposed two values for β_1 and β_2 alternatively that is 1 and 2. Using $\beta_1 = 1;\ \beta_2 = 2$ $\Rightarrow a = g_0^2$ and $b = 2^{m-1}h_0^2$, so equation (1) becomes

$$x_0^2 = g_0^4 - 2^{2m-2}h_0^4 = (g_0^2 - 2^{m-1}h_0^4)(g_0^2 + 2^{m-1}h_0^2) \quad (3)$$

where $2^{m-1}h_0^2 \equiv 0(mod 2)$. Since $gcd(g_0;\ h_0) = 1 = (g_0;\ 2h_0) \Rightarrow gcd(g_0^2 - 2^{m-1}h_0^2, g_0^2 + 2^{m-1}h_0^2) = 1$.

Equation (3) together with the fact that Z is a UFD, gives

$$\rho^2 = g_0^2 - 2^{m-1}h_0^4; \sigma^2 = g_0^2 + 2^{m-1}h_0^4 \quad (B)$$

Now, we have two cases depending on value of m

- If $m = 2l$ +1, the solution of equation (B) is

  $$g_0 = \Delta_0^2 - \nabla_0^2, \quad 2^l h_0 = 2\Delta_0\nabla_0;$$

  $$g_0 = \hat{\Delta}_0^2 + \hat{\nabla}_0^2, \quad 2^l h_0 = 2\hat{\Delta}_0\hat{\nabla}_0$$

  using theorem 2.2 and equation (3), we get the required form of the system of equations

  $$\bar{X}^2 - \bar{Y}^2 = \hat{X}^2 + \hat{Y}^2, \bar{X}\bar{Y} = \hat{X}\hat{Y},$$
  $$gcd(\bar{X}, \bar{Y}) = 1 = gcd(\hat{\bar{X}}, \hat{Y}) \quad (4)$$

- when $m = 2l$, the solution of equation (B) is

  $$g_0 = \pm(2\Delta_0^2 - \nabla_0^2), \quad 2^{l-1}h_0 = 2\Delta_0\nabla_0;$$

  $$g_0^2 = 2\hat{\Delta}_0^2 + \hat{\nabla}_0^2, \quad 2^{l-1}h_0 = 2\hat{\Delta}_0\hat{\nabla}_0$$

  using theorem 2.3 and equation (3), we get the following system of equations

  $$\bar{X}^2 - 2\bar{Y}^2 = \hat{X}^2 + 2\hat{Y}^2, \bar{X}\bar{Y} = \hat{X}\hat{Y},$$
  $$gcd(\bar{X}, \bar{Y}) = 1 = gcd(\hat{\bar{X}}, \hat{Y}) \quad (5)$$

Conversely, suppose that

$$G_0 = \Delta_0^2 - \nabla_0^2 = \hat{\Delta}_0^2 + \hat{\nabla}_0^2$$
$$H_0 = 2\Delta_0\nabla_0 = 2\hat{\Delta}_0\hat{\nabla}_0$$

Let us define the following complete squares

$$\Gamma^2 = G_0^2 + 2^{k-1}H_0^2$$

$$\Sigma^2 = G_0^2 - 2^{k-1}H_0^2$$

Now, multiplying above equalities, we obtain

$$x_0^2 = \Gamma^2\Sigma^2 = G_0^4 - 2^{2k-2}H_0^4$$

Adding, subtracting and multiplying Γ^2 and Σ^2, we have

$$2^k y_0^2 = \frac{(\Gamma^2 + \Sigma^2)(\Gamma^2 - \Sigma^2)}{2} = \frac{(G_0^2 + 2^{k-1}H_0^2)(G_0^2 - 2^{k-1}H_0^2)}{2} = 2^k G_0^2 H_0^2$$

Now, squaring and adding the last two equations we get the required Diophantine equations

$$x_0^4 + 2^{2m}{y_0}^4 = {z_0}^2;\ \text{where } gcd(x_0;\ 2y_0) = 1.$$

Theorem 3.2: An equation $x^4 + (2^{2k} + 2)x^2y^2 + y^4 = z^2$ is equivalent to system of equations

$$\bar{X}^2 - \bar{Y}^2 = \hat{X}^2 + \hat{Y}^2, \bar{X}\bar{Y} = \hat{X}\hat{Y},$$
$$gcd(\bar{X}, \bar{Y}) = 1 = gcd(\hat{\bar{X}}, \hat{Y})$$

or

$$\bar{X}^2 - 2\bar{Y}^2 = \hat{X}^2 + 2\hat{Y}^2, \bar{X}\bar{Y} = \hat{X}\hat{Y},$$
$$gcd(\bar{X}, \bar{Y}) = 1 = gcd(\hat{\bar{X}}, \hat{Y})$$

Proof: Let $(x_0;\ y_0; z_0)$ be a solution of

$${x_0}^4 + (2^{2k} + 2){x_0}^2{y_0}^2 + {y_0}^4 = {z_0}^2 \quad (6)$$

such that $gcd(x_0;\ y_0) = 1$. Replacing $\alpha_0 = {x_0}^2 + {y_0}^2 + 2x_0y_0$ and $\beta_0 = 2^k x_0 y_0$ in the equation (6), then it will become the standard Fermat equation ${\alpha_0}^2 + {\beta_0}^2 = z^2$. By using theorem 2.2, we get the solutions of the form

$$\alpha_0 = a_0^2 - b_0^2, \beta_0 = 2a_0b_0$$

Substituting backward the values of α_0 and β_0. We have same equation (A) and apply the same technique as in the proof of theorem 3.1 we will get the same system as required.

Now, for converse, let us consider the following equation:

$$f(\bar{X}, \bar{Y}) = \bar{X}^2 - 2^{2k-2}\bar{Y}^2 - (\hat{X}^2 + \hat{Y}^2) \quad (7)$$

Since $\bar{X}\bar{Y} = \hat{X}\hat{Y} \Rightarrow \bar{Y}^2 = \left(\frac{\hat{X}\hat{Y}}{\bar{X}}\right)^2$, so, equation (7) implies that

$$g(\bar{X}) = \bar{X}^2 - 2^{2k-2}\left(\frac{\hat{X}\hat{Y}}{\bar{X}}\right)^2 - (\hat{X}^2 + \hat{Y}^2)$$

$$h(\bar{X}) = \bar{X}^2 g(\bar{X}) = \bar{X}^4 - \bar{X}^2(\hat{X}^2 + \hat{Y}^2) - 2^{2k-2}(\hat{X}\hat{Y})^2$$

But the above equation is quadratic in $\bar{X}^2$ so discriminants is given as

$$D^2 = \left(\hat{X}^2 + \hat{Y}^2\right)^2 + 2^{2k}(\hat{X}\hat{Y})^2$$
$$= \hat{X}^4 + (2^{2k}+2)\hat{X}^2\hat{Y}^2 + \hat{Y}^4,$$

which is the required form of Diophantine equation.

Theorem 3.3: The equation $x^4 + 2^{2m}y^4 = z^2$ and $x^4 + (2^{2m}+2)x^2y^2 + y^4 = z^2$ has only trivial solution.

Proof: Let us define $\mu_q(a)$ as the valuation at prime q in the positive integer a, then

$$\mu(a) = \sum_{q\in\Pi} \mu_q(a)$$

Suppose $(\bar{X}_0; \bar{Y}_0; \hat{X}_0; \hat{Y}_0)$ is a solution to equation (4). A logical projection of equation (4) onto modulo 2 and 3 respectively, infers that $\mu(\bar{X}_0, \bar{Y}_0) \geq 2$. Assuming $\mu(\bar{X}_0, \bar{Y}_0)$ is minimal (i.e. there does not exist any other $(\bar{X}_1; \bar{Y}_1; \hat{X}_1; \hat{Y}_1)$ so that it satisfies equation (5) and $\mu(\bar{X}_1, \bar{Y}_1) < \mu(\bar{X}_0, \bar{Y}_0)$: Since we have $\bar{X}_0\bar{Y}_0 = \hat{X}_0\hat{Y}_0$ where$(\bar{X}_0, \bar{Y}_0) = 1 = (\hat{X}_0, \hat{Y}_0)$; the following factorization is a possibility:

$$\bar{X}_0 = (\bar{X}_0, \hat{X}_0)(\bar{X}_0, \hat{Y}_0)$$

$$\bar{Y}_0 = (\bar{Y}_0, \hat{X}_0)(\bar{Y}_0, \hat{Y}_0)$$

$$\bar{\bar{X}}_0 = (\hat{X}_0, \bar{X}_0)(\hat{X}_0, \bar{Y}_0)$$

$$\hat{Y}_0 = (\hat{Y}_0, \bar{X}_0)(\hat{Y}_0, \bar{Y}_0)$$

Reffering back to equation (4), the first line can be re-written as follows:

$$\left[(\bar{X}_0, \hat{X}_0)(\bar{X}_0, \hat{Y}_0)\right]^2 - \left[(\bar{Y}_0, \hat{X}_0)(\bar{Y}_0, \hat{Y}_0)\right]^2 = \left[(\hat{X}_0, \bar{X}_0)(\hat{X}_0, \bar{Y}_0)\right]^2 + \left[(\hat{Y}_0, \bar{X}_0)(\hat{Y}_0, \bar{Y}_0)\right]^2 \quad (8)$$

A re-arrangement of the terms to have products implies the following:

$$\left[(\bar{X}_0, \hat{X}_0)^2 - (\bar{Y}_0, \hat{Y}_0)^2\right](\bar{X}_0, \hat{Y}_0)^2 = \left[(\hat{X}_0, \bar{X}_0)^2 + (\hat{Y}_0, \bar{Y}_0)^2\right](\hat{X}_0, \bar{Y}_0)^2 \quad (9)$$

Since we gain
$(\bar{X}_0, \hat{Y}_0)$ and $(\hat{X}_0, \bar{Y}_0)$ relatively prime, and $(\bar{X}_0, \hat{X}_0)^2 - (\bar{Y}_0, \hat{Y}_0)^2$ is relatively prime with$(\hat{X}_0, \bar{X}_0)^2 + (\hat{Y}_0, \bar{Y}_0)^2$; the equation can only stand true if:

$$\left(\bar{X}_0, \hat{Y}_0\right)^2 = (\hat{X}_0, \bar{X}_0)^2 + \left(\hat{Y}_0, \bar{Y}_0\right)^2\left(\hat{X}_0, \bar{Y}_0\right)^2$$
$$= (\bar{X}_0, \hat{X}_0)^2 - \left(\bar{Y}_0, \hat{Y}_0\right)^2$$

However, these are again primitive Pythagorean triplets, so the following representation must be a possibility for some g_0; h_0; $\hat{g}_0$; $\hat{h}_0$.

$$(\bar{X}_0, \hat{X}_0) = g_0^2 - h_0^2$$

$$\left(\bar{Y}_0, \hat{Y}_0\right) = 2g_0h_0$$

$$(\bar{X}_0, \hat{X}_0) = \hat{g}_0^2 + \hat{h}_0^2$$

$$\left(\bar{Y}_0, \hat{Y}_0\right) = 2\hat{g}_0\hat{h}_0$$

This evidently comes out to be:

$$g_0^2 - h_0^2 = \hat{g}_0^2 + \hat{h}_0^2$$

$$g_0h_0 = \hat{g}_0\hat{h}_0$$

$$(g_0, h_0) = 1 = (\hat{g}_0, \hat{h}_0)$$

or be a solution of equation (4) where $\mu(g_0, h_0) < \mu(\hat{g}_0, \hat{h}_0)$ a contradiction to minimality. Thus equation (4) has only trivial solution.

4. DISCUSSION

The experimental data that was collected has actually produced table 1 which shows that a few other quartics are also equivalent to the very resolvent. Nevertheless it is obvious to remark that there are equations, such as $x^4 + 2y^4 = z^2$ (whose triviality of solutions is shown in [7]), which don't belong to the below mentioned resolvents.

$\bar{X}^2 - \bar{Y}^2 = \hat{X}^2 + \hat{Y}^2$, $\bar{X}\bar{Y} = \hat{X}\hat{Y}$, $gcd(\bar{X}, \bar{Y}) = 1 = gcd(\hat{\bar{X}}, \hat{Y})$	$x^4 - y^2 = z^2$; $x^4 + 4y^2 = z^2$; $x^4 + 64y^2 = z^2$; $x^4 \pm 6x^2y^2 + y^4 = z^2$; $x^4 \pm 18x^2y^2 + y^4 = z^2$
$\bar{X}^2 - 2\bar{Y}^2 = \hat{X}^2 + 2\hat{Y}^2$, $\bar{X}\bar{Y} = \hat{X}\hat{Y}$, $gcd(\bar{X}, \bar{Y}) = 1 = gcd(\hat{\bar{X}}, \hat{Y})$	$x^4 + y^2 = z^2$; $x^4 - 4y^2 = z^2$; $x^4 + 16y^2 = z^2$; $x^4 - y^2 = z^2$; $x^4 \pm 12x^2y^2 + y^4 = z^2$;
Both Resolvent	$x^4 + y^4 = z^2$

5. CONCLUSION

For algebraic extension Z[i], Cross using infinite descent over norms implies that the equation $x^4 + y^4 = z^2$ continues to have trivial solution [8]. In this situation, it is credible to believe that the technique developed in this paper for rational integers can also be efficaciously applied to algebraic extensions.

Remark 3.5: Similarly, for $x^8 + 2^{2k}y^8 = z^4$ and $x^8 + (2^{2k}+2)x^4y^4 + y^8 = z^8$ these equation can be converted to the following system of equations

$$\bar{X}^4 - \bar{Y}^4 = \hat{X}^4 + \hat{Y}^4, \bar{X}\bar{Y} = \hat{X}\hat{Y},$$
$$gcd(\bar{X}, \bar{Y}) = 1 = gcd(\hat{\bar{X}}, \hat{Y})$$

or

$$\bar{X}^4 - 2\bar{Y}^4 = \hat{X}^4 + 2\hat{Y}^4, \bar{X}\bar{Y} = \hat{X}\hat{Y},$$
$$gcd(\bar{X}, \bar{Y}) = 1 = gcd(\hat{\bar{X}}, \hat{Y})$$

Remark 3.6; If k = 1, then the main results of [1] become the corollaries of our main results.

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