r-DYNAMIC CHROMATIC NUMBER OF CYCLE-RELATED GRAPHS

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ABSTRACT: *Let $G=(V,E)$ be a graph. An r-dynamic k-coloring of G is a function f from V to a set C of colors such that (1) f is a proper coloring, and (2) for all vertices v in V , $|f(N(v))| \geq \min\{r,\deg(v)\}$. The r-dynamic chromatic number of a G, denoted by $\chi_r(G)$, is the smallest k such that f is an r-dynamic k-coloring of G. This study gave the r-dynamic chromatic number of cycle-related graphs.*

Keywords: coloring, *r*-dynamic coloring, *r*-dynamic chromatic number, cycle-related graphs

1. INTRODUCTION

As mentioned in [11], the earliest results about graph coloring is on planar graphs motivated by a problem regarding the coloring of maps. While coloring a map of the counties of England, Francis Guthrie presented the four color conjecture which stated that four colors were sufficient to color the map so that no regions sharing a common border received the same color. This idea may have motivated the concept proper vertex coloring. Using graph theoretic parlance, a proper vertex coloring of a graph $G=(V,E)$ is a function from *V* to a finite set *C*, whose elements are called colors, such that no two adjacent vertex are assigned the same color.

A generalization of the proper vertex coloring is the dynamic coloring of graphs. The dynamic coloring of a graph *G* is a proper coloring such that every vertex of *G* with degree at least two has at least two neighbors that are colored differently. This concept was introduced by Montgomery in [3]. Since then, the concept was studied by many authors, see for example [1], [2], [4], [5], [6] and [7].

The concept dynamic coloring was further generalized by Montgomery in [3]. The new concept is called the *r*-dynamic *k*-coloring. This concept was studied in [4], [5], and [6].

In [8] and [9], the authors investigated the general *r*-dynamic *k*-coloring of planar grids. And in [13], Inoferio and Baldado investigated the *r*-dynamic chromatic number of cycles, complete graphs, and forests.

A vertex coloring of a graph $G=(V,E)$ is a function f from *V* to a finite set *C*, whose elements are called color. A k-coloring is a vertex coloring with at most *k*-colors. In this case, we always assume that $C=\{1,2,\ldots,k\}$. A *k*-coloring may also be viewed as a vertex partition $\{V_1,V_2,\ldots,V_3\}$, where $V_i=f^{-1}(i)$ are called the color classes. A graph is k-colorable if it admits a proper vertex coloring with at most *k*-colors. The chromatic number of a graph *G*, denoted by $\chi(G)$, is the smallest *k* such that *G* is *k*-colorable.

An *r*-dynamic *k*-coloring of *G* is a proper coloring *f* such that for all vertices *v* in *V*, $|f(N(v))| \geq \min\{r,\deg(v)\}$. The *r*-dynamic chromatic number of a *G*, denoted by $\chi_r(G)$, is the smallest *k* such that *f* is an *r*-dynamic *k*-coloring
of *G*.

Definition 1.1. *The* ***sun*** *graph, denoted by S_n, is the graph of order 2n obtained from the cycle C_n = [u, u_1, . . . , u_n] by adding vertices w_i joined by edges to vertices u_i and $u_{i+1(mod\ n)}$ for i = 1, 2, . . . , n.*

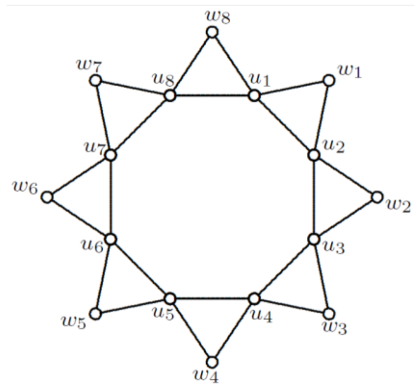

Figure 1. The sun graph S_8

Definition 1.2. *The* ***sunlet*** *graph, denoted by L_n, is the graph of order 2n obtained from the cycle C_n = [u, u_1, . . . , u_n] by attaching pendant edges u_iw_i for i = 1, 2, . . . , n.*

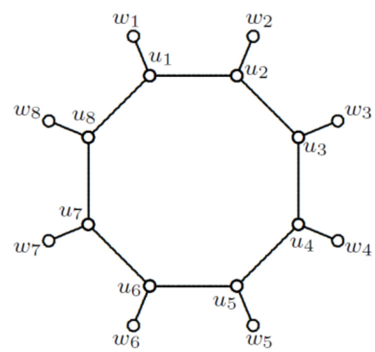

Figure 2. The sunlet graph L_8

Hereafter, please refer to [12] for concepts that were used but were not discussed in this paper.

2. PRELIMINARY RESULTS

Remark 2.1. *Let G be graph. Then $\chi_r(G) \geq \chi(G)$.*

This remark follows from the fact that an *r*-dynamic *k*-coloring must be a proper *k*-coloring.

Lemma 2.2. *Let G be graph that contains C_3. Then $\chi_r(G) \geq 3$ for all r.*

Proof: Let $C_3=[a,b,c]$ and *f* be a proper coloring of *G*. Then $f(a)$, $f(b)$, and $f(c)$ must be distinct. Hence, by Remark 2.1 $\chi_r(G) \geq \chi(G) \geq 3$. QED

Lemma 2.3. *Let G be graph and f be an r-dynamic k-coloring of G. Then $k > \min\{r,\Delta\}$ where Δ is the maximum degree of G.*

Proof: Let f be an r-dynamic k-coloring of G and v be a vertex with $\deg(v)=\Delta$. Suppose that $k \le \min\{r,\Delta\}$. Then $|f(N(v))| < k \le \min\{r,\Delta\} = \min\{r,\deg(v)\}$. This is a contradiction. QED

Lemma 2.4. *Let G be graph. If $r \ge \Delta$, then every r-dynamic k-coloring of G is an $r+1$-dynamic k-coloring of G.*

Proof: Let f be an r-dynamic k-coloring of G with $r \ge \Delta$. Then $|f(N(v))| \ge \min\{r,\deg(v)\} = \deg(v) = \min\{r+1,\deg(v)\}$ for all vertex v. Hence, f is an r+1-dynamic k-coloring of G. QED

3. *r*-DYNAMIC CHROMATIC NUMBER OF SUN GRAPHS

Theorem 3.1. *Let S_n be the sun graph of order $2n$,* with $n \equiv 0 \pmod 3$. *Then* $\chi_2(S_n)=3$.

Proof: Let S_n be the sun graph of order $2n$ obtained from $C_n = [u_1, u_1, \ldots, u_n]$ by adding vertices w_i joined by edges to vertices u_i and $u_{i+1(mod\ n)}$ for i = 1, 2,…, n. Let $f: V(S_n) \to \{1,2,3\}$ be given by

$$f(u_i)=\begin{cases}1 & \text{, if } i\equiv 1 \pmod 3\\ 2 & \text{, if } i\equiv 2 \pmod 3\\ 3 & \text{, if } i\equiv 0 \pmod 3\end{cases}$$

$$f(w_i)=\begin{cases}3 & \text{, if } i\equiv 1 \pmod 3\\ 1 & \text{, if } i\equiv 2 \pmod 3\\ 2 & \text{, if } i\equiv 0 \pmod 3\end{cases}.$$

Let $x \in V(S_n)$, and consider the following cases:

Case 1. $x = u_i$, for $i = 1,2,\ldots,n$

If $x=u_i$ for $i=1,2,\ldots,n$, then

$$\begin{aligned}|f(N(u_i))| &= \left|f\left(\{u_{i-1(\text{mod } n)}, w_{i-1(\text{mod } n)}, u_{i+1(\text{mod } n)}, w_i\}\right)\right|\\ &= \left|\left\{f\left(u_{i-1(\text{mod } n)}\right), f\left(w_{i-1(\text{mod } n)}\right), f\left(u_{i+1(\text{mod } n)}\right), f(w_i)\right\}\right|\\ &= \left|\left\{f\left(u_{i-1(\text{mod } n)}\right), f\left(u_{i+1(\text{mod } n)}\right)\right\}\right|\\ &= 2\\ &\ge \min\{2,4\}\\ &= \min\{r,\deg(u_i)\}.\end{aligned}$$

Case 2. $x = w_i$, for $i=1,2,\ldots,n$

If $x = w_i$ for $i=1,2,\ldots,n$, then

$$\begin{aligned}|f(N(w_i))| &= \left|f\left(\{u_i, u_{i+1(\text{mod } n)}\}\right)\right|\\ &= \left|\left\{f(u_i), f\left(u_{i+1(\text{mod } n)}\right)\right\}\right|\\ &= 2\\ &\ge \min\{2,2\}\\ &= \min\{r,\deg(w_i)\}.\end{aligned}$$

Thus, f is a 2-dynamic 3-coloring. Hence, $\chi_2(S_n) \le 3$. But by Lemma 2.2 $\chi_2(S_n) \ge 3$. Therefore, $\chi_2(S_n)=3$. QED

Theorem 3.2. *Let S_n be the sun graph of order $2n$, with $n \equiv 0 \pmod 3$. Then $\chi_3(S_n)=4$.*

Proof: Let S_n be the sun graph of order $2n$, with $n \equiv 0 \pmod 3$, obtained from $C_n = [u_1, u_1, \ldots, u_n]$ by adding vertices w_i joined by edges to vertices u_i and $u_{i+1(mod\ n)}$ for i = 1, 2,…, n. Let $f: V(S_n) \to \{1,2,3,4\}$ be given by

$$f(u_i)=\begin{cases}1 & \text{, if } i\equiv 1 \pmod 3\\ 2 & \text{, if } i\equiv 2 \pmod 3\\ 3 & \text{, if } i\equiv 0 \pmod 3\end{cases}$$

and $f(w_i)=4$. Let $x \in V(S_n)$, and consider the following cases:

Case 1. $x = u_i$, for $i=1,2,\ldots,n$

If $x = u_i$ for $i=1,2,\ldots,n$, then

$$\begin{aligned}|f(N(u_i))| &= \left|f\left(\{u_{i-1(\text{mod } n)}, w_{i-1(\text{mod } n)}, u_{i+1(\text{mod } n)}, w_i\}\right)\right|\\ &= \left|\left\{f\left(u_{i-1(\text{mod } n)}\right), f\left(w_{i-1(\text{mod } n)}\right), f\left(u_{i+1(\text{mod } n)}\right), f(w_i)\right\}\right|\\ &= \left|\left\{f\left(u_{i-1(\text{mod } n)}\right), f\left(u_{i+1(\text{mod } n)}\right), f(w_i)\right\}\right|\\ &= 3\\ &\ge \min\{3,4\}\\ &= \min\{r,\deg(u_i)\}.\end{aligned}$$

Case 2. $x = w_i$, for $i=1,2,\ldots,n$

If $x = w_i$ for $i=1,2,\ldots,n$, then

$$\begin{aligned}|f(N(w_i))| &= \left|f\left(\{u_i, u_{i+1(\text{mod } n)}\}\right)\right|\\ &= \left|\left\{f(u_i), f\left(u_{i+1(\text{mod } n)}\right)\right\}\right|\\ &= 2\\ &\ge \min\{3,2\}\\ &= \min\{r,\deg(w_i)\}.\end{aligned}$$

Thus, f is a 3-dynamic 4-coloring. Hence, $\chi_3(S_n) \le 4$. But by Lemma 2.3 $\chi_4(S_n) > 3$. Therefore, $\chi_3(S_n)=4$. QED

Theorem 3.3. *Let S_n be the sun graph of order $2n$,* with $n \equiv 0 \pmod 6$. *Then* $\chi_4(S_n)=5$.

Proof: Let S_n be the sun graph of order $2n$, with $n \equiv 0 \pmod 6$, obtained from $C_n = [u_1, u_1, \ldots, u_n]$ by adding vertices w_i joined by edges to vertices u_i and $u_{i+1(mod\ n)}$ for i = 1, 2,…, n. Let $f: V(S_n) \to \{1,2,3,4,5\}$ be given by

$$f(u_i)=\begin{cases}1 & \text{, if } i\equiv 1 \pmod 3\\ 2 & \text{, if } i\equiv 2 \pmod 3\\ 3 & \text{, if } i\equiv 0 \pmod 3\end{cases}$$

$$f(w_i)=\begin{cases}4 & \text{, if } i\equiv 1 \pmod 2\\ 5 & \text{, if } i\equiv 0 \pmod 2\end{cases}.$$

Then clearly f is a 4-dynamic 5-coloring. Hence, $\chi_4(S_n) \le 5$. But by Lemma 2.3 $\chi_4(S_n) > 4$. Therefore, $\chi_4(S_n)=5$. QED

Corollary 3.4. *Let S_n be the sun graph of order $2n$,* with $n \equiv 0 \pmod 6$. *Then for all* $r \ge 4$, $\chi_r(S_n)=5$.

Proof: By Theorem 3.3 $\chi_4(S_n)=5$. Since $\Delta(S_n)=4$, by Lemma 2.4. $\chi_r(S_n)=5$ for all $r \ge 4$. QED

4. *r*-DYNAMIC CHROMATIC NUMBER OF SUNLET GRAPHS

Theorem 4.1. *Let L_n be the sunlet graph of order $2n$. Then* $\chi_2(L_n)=3$.

Proof: Let L_n be the sunlet graph of order $2n$ obtained from $C_n = [u_1, u_2, \ldots, u_n]$ by attaching pendant edges w_iu_i for $i = 1, 2, \ldots, n$. If n is even, we let $f: V(L_n) \to \{1,2,3\}$ be given by

$$f(u_i)=\begin{cases}1 & \text{, if } i\equiv 0 \pmod 2\\ 2 & \text{, if } i\equiv 1 \pmod 2\end{cases}$$

and $f(w_i)=3$. While, if n is odd we let $f: V(L_n)\to\{1,2,3\}$ be given by

$$f(u_i)=\begin{cases}1 & \text{, if } i\equiv 0 \pmod 2,\ i=1,2,\ldots,n-2\\ 2 & \text{, if } i\equiv 1 \pmod 2,\ i=1,2,\ldots,n-3,n\\ 3 & \text{, if } i=n-1\end{cases}$$

$$f(w_i)=\begin{cases}3 & \text{, if } i=1,2,\ldots,n-2\\ 1 & \text{, otherwise .}\end{cases}$$

Let $x\in V(L_n)$, and consider the following cases:

Case 1. $x=u_i$, for $i=1,2,\ldots,n$

If $x=u_i$ for $i=1,2,\ldots,n$, then

$$\begin{aligned}\left|f(N(u_i))\right| &= \left|f\left(\left\{u_{i-1(\bmod n)}, u_{i+1(\bmod n)}, w_i\right\}\right)\right|\\ &= \left|\left\{f\left(u_{i-1(\bmod n)}\right), f(w_i)\right\}\right|\\ &=2\\ &\geq \min\{2,3\}\\ &= \min\{r, \deg(u_i)\}.\end{aligned}$$

Case 2. $x=w_i$, for $i=1,2,\ldots,n$

If $x=w_i$ for $i=1,2,\ldots,n$, then

$$\begin{aligned}\left|f(N(w_i))\right| &= \left|f(\{u_i\})\right|\\ &= \left|\{f(u_i)\}\right|\\ &=1\\ &\geq \min\{2,1\}\\ &= \min\{r, \deg(w_i)\}.\end{aligned}$$

Thus, f is a 2-dynamic 3-coloring. Hence, $\chi_2(L_n)\leq 3$. But by Lemma 2.2 $\chi_2(L_n)\geq 3$. Therefore, $\chi_2(L_n)=3$. QED

Theorem 4.2. *Let L_n be the sunlet graph of order $2n$, with* $n\equiv 0 \pmod 3$. *Then* $\chi_3(L_n)=4$.

Proof: Let L_n be the sunlet graph of order $2n$, *with* $n\equiv 0\pmod 3$, obtained from $C_n = [u_1, u_2, \ldots, u_n]$ by attaching pendant edges w_iu_i for $i = 1, 2, \ldots, n$. Let $f: V(L_n)\to\{1,2,3,4\}$ be given by

$$f(u_i)=\begin{cases}1 & \text{, if } i\equiv 1 \pmod 3\\ 2 & \text{, if } i\equiv 2 \pmod 3\\ 3 & \text{, if } i\equiv 0 \pmod 3\end{cases}$$

and $f(w_i)=4$. Let $x\in V(L_n)$, and consider the following cases:

Case 1. $x=u_i$, for $i=1,2,\ldots,n$

If $x=u_i$ for $i=1,2,\ldots,n$, then

$$\begin{aligned}\left|f(N(u_i))\right| &= \left|f\left(\left\{u_{i-1(\bmod n)}, u_{i+1(\bmod n)}, w_i\right\}\right)\right|\\ &= \left|\left\{f\left(u_{i-1(\bmod n)}\right), f\left(u_{i+1(\bmod n)}\right), f(w_i)\right\}\right|\\ &=3\\ &\geq \min\{3,3\}\\ &= \min\{r, \deg(u_i)\}.\end{aligned}$$

Case 2. $x=w_i$, for $i=1,2,\ldots,n$

If $x=w_i$ for $i=1,2,\ldots,n$, then

$$\begin{aligned}\left|f(N(w_i))\right| &= \left|f(\{u_i\})\right|\\ &= \left|\{f(u_i)\}\right|\\ &=1\\ &\geq \min\{3,1\}\\ &= \min\{r, \deg(w_i)\}.\end{aligned}$$

Thus, f is a 3-dynamic 4-coloring. Hence, $\chi_3(L_n)\leq 4$. But by Lemma 2.3 $\chi_3(L_n)\geq 4$. Therefore, $\chi_3(L_n)=4$. QED

Corollary 4.3. *Let L_n be the sunlet graph of order $2n$,* with $n\equiv 0\pmod 3$. *Then for all* $r\geq 3$, $\chi_r(L_n)=3$.

Proof: By Theorem 4.2 $\chi_3(L_n)=4$. Since $\Delta(L_n)=3$, by Lemma 2.4. $\chi_r(L_n)=3$ for all $r\geq 3$. QED

5. CONCLUSION

The important concepts and results presented in this paper supported, and intertwined with, those obtained by other authors, making this article very interesting. The construction of the different theorems were realized using the definition and properties of sun graphs, sunlet graphs, *r*-dynamic chromatic coloring and *r*-dynamic chromatic number. Thus, some properties focusing on the *r*-dynamic chromatic numbers of sun graphs and sunlet graphs were realized.

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