p-REINFORCEMENT NUMBER OF SUN AND SUNLET GRAPHS

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ABSTRACT*: A graph* $G=(V,E)$ *is an ordered pair where V is a non-empty set whose elements are called vertices, and E is a set of 2-element subsets of V called edges. Let p be a positive integer. A p-dominating set of G is a subset D of G such that* $|N_G(x)\cap D|\geq p$ *for all* $x\in V\backslash D$. *The p-domination number of G, denoted by* $\gamma_p(G)$, *is the minimum cardinality among the p-dominating sets of G. The p-reinforcement number of G, denoted by* $r_p(G)$, *is the smallest number of edges of* G^c *that have to be added to G in order to reduce* $\gamma_p(G)$. *This study presented bounds and exact values on the p-domination number of sun and sunlet graphs. The concepts* η_p-set *and p-private neighbourhood used by Lu et al. [11] were also utilized in proving some of the p-reinforcement number of sun and sunlet graphs. This study can be a guide in the creation of new results and will be helpful in our transportation, security, and networking.*

Keywords: *p*-domination number, *p*-reinforcement number, sun graph, sunlet graph

1. INTRODUCTION

Let $C_n=[u_1,u_2,\ldots,u_n]$ be a cycle of order *n*. The *sun graph* S_n of order *2n* obtained by adding a vertex w_i joined by edges to vertices u_i and $u_{i+1(mod n)}$ for every $i=1,2,\ldots,n$. The sunlet graph L_n of order *2n* is the graph obtained from C_n by attaching pendant edges u_iw_i for each $i=1,2,\ldots,n$. The neighborhood of *x* is the set $N(x)$ consisting of all vertices *y* which are adjacent to *x*, that is, $N(x)=\{y\in V: xy\in E\}$. The elements $y\in N(x)$ are called neighbors of *x*. The closed neighborhood of *S* in *G* is the set $N_G[S]=S\cup N_G(S)$. Let $G=(V,E)$ be a graph and *p* a positive integer. A *p*-dominating set of *G* is a subset *D* of *G* such that $|N_G(x)\cap D|\geq p$ for all $x\in V\backslash D$. The *p-domination number* $\gamma_p(G)$ is the minimum cardinality among the *p*-dominating sets of *G*. The *p*-reinforcement number $r_p(G)$ of a graph *G* is the smallest number of edges of G^c that have to be added to *G* in order to reduce $\gamma_p(G)$, that is, $r_p(G)=min\{|B|: B\subseteq E(G^c)\; with\; \gamma_p(G+B)<\gamma_p(G)\}$.

Moreover, Lu et al. [11] introduced the following concepts and notations. For a subset $X\subseteq V(G)$,

$$\eta_p(x,X,G)=\begin{cases} p-|N_G(x)\cap X| \; if\; x\in X^* \\ 0\; if\; otherwise\end{cases} \quad for\; x\in V(G),$$

$$\eta_p(S,X,G)=\sum_{x\in S}\eta_p(x,X,G)\; for\; S\subseteq V(G),$$

$$\eta_p(G)=min\{\eta_p(V(G),X,G): |X|<\gamma_p(G)\}.$$

A subset $X\subseteq V(G)$ is called an η_p-set of G if $\eta_p(G)=\eta_p(V(G),X,G)$. Clearly, for any two subsets S', $S\subseteq V(G)$ and two subsets X', $X\subseteq V(G)$,

$$\eta_p(S',X,G)\leq\eta_p(S,X,G), S'\subseteq S$$

$$\eta_p(S,X,G)\leq\eta_p(S,X',G), |X'|\leq|X|.$$

Let $X\subseteq V(G)$ and $x\in X$. A vertex $y\in\bar{X}$ is called a *p*-private neighbor of *x* with respect to *X* if $xy\in E(G)$ and $|N_G(y)\cap X|=p$. The private neighborhood of *x* with respect to *X* is defined as $PN_p(x,X,G)=\{y: y\; is\; a\; private\; neighbor\; of\; x\; with\; respect\; to\; X\}$.

Hence, consider the following concepts:

$$\mu_p(x,X,G)=|PN_p(x,X,G)|+max\{0,p-|N_G(x)\cap X|\}$$

$$\mu_p(X,G)=min\{\mu_p(x,X,G): x\in X\}$$

$$\mu_p(G)=min\{\mu_p(X,G): X\; is\; a\; \gamma_p-set\; of\; G\}.$$

2. *p*-DOMINATION NUMBER OF SUN AND SUNLET GRAPHS

In this section, we presented the exact values of the p-domination number of the sun and sunlet graphs. Moreover, the authors also gave their observations.

Theorem 2.1. *Let* $S_n(n\geq 3)$ *be a sun graph of order 2n. For* $p\geq 2, \gamma_p(S_n)\geq n$.

Proof: Let $p\geq 2$ be a positive integer and $S_n(n\geq 3)$ be a sun graph of order *2n*. Note that S_n consists of $C_n=[u_1,u_2,\ldots,u_n]$ and w_i joined by edges adjacent to vertices u_i and $u_{i+1(mod n)}$ for every $i=1,2,\ldots,n$. Clearly, every u_i is adjacent to w_{i-1} and w_i. Then consider $D=\{w_1,w_2,w_3,\ldots,w_n\}\cup\sum\{u_i\}$. Now, let $v=u_i$ such that $u_i\notin D$. Then $|N(v)\cap D|\geq|\{w_i,w_{i+1}\}|$. This shows that *D* is a *p*-dominating set in S_n. Hence, $\gamma_p(S_n)=|D|\geq |=\{w_1,w_2,w_3,\ldots,w_n\}|=n$. Therefore, $\gamma_p(S_n)\geq n$. *QED*

Theorem 2.2. *Let* S_n *be a sun graph of order 2n with* $n\geq 3$. Then $\gamma_2(S_n)=n$

Proof: Let $p=2$ and $S_n(n\geq 3)$ be sun graph of order *2n*. By theorem 2.1 $\gamma_p(S_n)\geq n$. Suppose that $\gamma_2(S_n)=n$. Then we let $D=\{u_1,u_2,u_3,\ldots,u_n\}\subseteq S_n$. Note that for every $v\in V(S_n)\backslash D, |N(v)\cap D|=2$. Then *D* is the minimum 2-dominating set in S_n. Therefore, $\gamma_2(S_n)=n$. *QED*

Observation 2.3 *Let* C_n *be a cycle of order n. Then* $\gamma_2(C_n)<\gamma_3(C_n)$, *that is, if D is a minimum 2-dominating set, then it cannot be a 3-dominating set.*

Theorem 2.4. *D is a minimum 3-dominating set in* $S_n(n\geq 3)$ *if and only if* $S\cup\{w_1,w_2,w_3,\ldots,w_n\}$ *where*

$$S=\begin{cases} u_1,u_4,\; u_7,\ldots,u_{n-3},u_n/u_{n-1} & if\; n\equiv 1(mod3)\\ u_1,u_4,\; u_7,\ldots,u_{n-4},u_{n-1}/u_{n-2} & if\; n\equiv 2(mod3)\\ u_1,u_4,\; u_7,\ldots,u_{n-5},u_{n-2} & if\; n\equiv 0(mod3)\end{cases}.$$

Proof: Let $D=S\cup\{w_1,w_2,\ldots,w_n\}$ be a minimum 3-dominating set in $S_n(n\geq 3)$ and $D\neq S\cup\{w_1,w_2,\ldots,w_n\}$ where

$$S=\begin{cases} u_1,u_4,u_7,\ldots,u_{n-3},u_n/u_{n-1} & if\; n\equiv 1(mod3)\\ u_1,u_4,u_7,\ldots,u_{n-4},u_{n-1}/u_{n-2} & if\; n\equiv 2(mod3)\\ u_1,u_4,u_7,\ldots,u_{n-5},u_{n-2} & if\; n\equiv 0(mod3)\end{cases}$$

Then there exists a subgraph $P=(u_i,u_{i+1(mod n)},u_{i+2(mod n)})$ or (u_{n-1},u_n) or (u_n) of

$(u_1, u_2, \ldots, u_n)$ such that $V(P) \cap D = \emptyset$. Let $v = u_{i+1}$ or u_{n-1} or u_n. Thus, $|N(v) \cap D| = 2$. This is a contradiction.

Conversely, suppose that $D_1 = S \cup \{w_1, w_2, \ldots, w_n\}$ where

$$S = \begin{cases} u_1, u_4, u_7, \ldots, u_{n-3}, u_n/u_{n-1} & if\ n \equiv 1(mod3) \\ u_1, u_4, u_7, \ldots, u_{n-4}, u_{n-1}/u_{n-2} & if\ n \equiv 2(mod3) \\ u_1, u_4, u_7, \ldots, u_{n-5}, u_{n-2} & if\ n \equiv 0(mod3) \end{cases}$$

and D_1 is not a minimum 3-dominating set in $S_n (n \geq 3)$. Clearly, D_1 is a 3-dominating set. Let D be a minimum 3-dominating set. Then $|D| < |D_1|$. Consider the following cases:

Case 1: $w_i \in D$

If $w_i \in D$, then $D \backslash w_i$ is not a dominating set of $(u_1, u_2, \ldots, u_n)$. Hence, there exists $v \in \{u_1, u_2, \ldots, u_n\}$ such that $N(v) = w_i, w_{i+1}$. Thus, $|N(v) \cap D| = 2$. This is a contradiction.

Case 2: $w_i \notin D$

If $w_i \notin D$, then we note that S is a minimum dominating set of $(u_1, u_2, \ldots, u_n)$. Hence, by Observation 2.3 S cannot be a 3-dominating set of $(u_1, u_2, \ldots, u_n)$, and so is *D*. Since $w_i \notin D$, D cannot be a 3-dominating set of S_n. This is a contradiction. *QED*

Corollary 2.5 *Let S_n be a sun graph of order 2n with $n \geq 3$. Then $\gamma_3(S_n) = \left\lceil \frac{n}{3} \right\rceil + n$.*

Observation 2.6 *Let C_n be a cycle of order n. Then $\gamma_3(C_n) < \gamma_4(C_n)$ that is, if D is a minimum 3-dominating set, then it cannot be a 4-dominating set.*

Theorem 2.7 *D is a minimum 4-dominating set in $S_n (n \geq 3)$ if and only if $D = S \cup \{w_1, w_2, \ldots, w_n\}$ where*

$$S = \begin{cases} u_1, u_3, u_5 \ldots, u_{n-2}, u_n/u_{n-1} & if\ n \equiv 1(mod2) \\ u_1, u_3, u_5 \ldots, u_{n-3}, u_{n-1} & if\ n \equiv 0(mod2) \end{cases}$$

Proof: Let $D = S \cup \{w_1, w_2, \ldots, w_n\}$ be a minimum 4-dominating set in $S_n (n \geq 3)$ and $D \neq S \cup \{w_1, w_2, \ldots, w_n\}$ where

$$S = \begin{cases} u_1, u_3, u_5 \ldots, u_{n-2}, u_n/u_{n-1} & if\ n \equiv 1(mod2) \\ u_1, u_3, u_5 \ldots, u_{n-3}, u_{n-1} & if\ n \equiv 0(mod2) \end{cases}$$

Then there exists a subgraph $P = (u_i, u_{i+1(mod n)})$ or (u_n) such that $V(P) \cap D = \emptyset$. If $P = (u_i, u_{i+1(mod n)})$, then we let $v = u_i$. While, if $P = (u_n)$, then we let $v = u_n$. Thus, $|N(v) \cap D| = 2$. This is a contradiction.

Conversely, suppose that $D_1 = S \cup \{w_1, w_2, \ldots, w_n\}$ where

$$S = \begin{cases} u_1, u_3, u_5 \ldots, u_{n-2}, u_n/u_{n-1} & if\ n \equiv 1(mod2) \\ u_1, u_3, u_5 \ldots, u_{n-3}, u_{n-1} & if\ n \equiv 0(mod2) \end{cases}$$

and D_1 is not a minimum 4-dominating set in $S_n (n \geq 3)$. Clearly, D_1 is a 4-dominating set. Let D be a minimum 4-dominating set. Then $|D| < |D_1|$.

Consider the following cases:

Case 1: $w_i \in D$

If $w_i \in D$, then $D \backslash w_i$ is not a 2-dominating set of $(u_1, u, \ldots, u_n)$. Hence, there exists $v \in \{u_1, u_2, \ldots, u_n\}$ such that $N(v) = u_i, w_i, w_{i+1}$. Thus, $|N(v) \cap D| = 3$. This is a contradiction.

Case 2: $w_i \notin D$

If $w_i \notin D$, then we note that S is a minimum 2-dominating set of $(u_1, u_2, \ldots, u_n)$. Hence, by Observation 2.6 S cannot be a 4-dominating set of $(u_1, u_2, \ldots, u_n)$, and so is *D*. Since $w_i \notin D$, D cannot be a 3-dominating set of S_n. This is a contradiction. *QED*

Corollary 2.8 *Let S_n be a sun graph of order 2n with $n \geq 3$. Then $\gamma_4(S_n) = \left\lceil \frac{n}{2} \right\rceil + n$.*

Corollary 2.9 *For every w_i in G such that $|deg_G(w_i)| = \delta(G) < p$, then $\gamma_p - set\ D$ contains the vertex w_i.*

Corollary 2.10 *For every $x \in D$ and $y \notin D$ in S_n, $|N(y)| = |u_i \cap D)| + \delta(S_n)$.*

The following are the results of Theorems 2.4, 2.7 and Corollaries 2.9, 2.10.

Corollary 2.11 *Let $L_n = C_n \circ K_1$ be a sunlet graph of order 2n. Then $\gamma_2(L_n) = \left\lceil \frac{n}{3} \right\rceil + n$.*

Corollary 2.12 *Let $L_n = C_n \circ K_1$ be a sunlet graph of order 2n. Then $\gamma_3(L_n) = \left\lceil \frac{n}{2} \right\rceil + n$.*

3. *p*-REINFORCEMENT NUMBER OF SUN AND SUNLET GRAPHS

In this section, we used concepts and some results from Lu et al. (2015) in proving of the exact values and bounds of p-reinforcement number of sun and sunlet graphs.

Lemma 3.1. $\gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil$ *and* $\gamma_2(C_n) = \left\lceil \frac{n}{2} \right\rceil$.

Lemma 3.2. *Let $p \geq 1$ be an integer and G be a graph with $\gamma_p(G) > p$. If $\Delta(G) < p$, then $r_p(G) = p - \Delta(G)$.*

Lemma 3.3. *If X is an η_p −set of a graph G, then $|X| = \gamma_p(G) - 1$.*

Lemma 3.4. *For any graph G and positive integer p, $r_p(G) = \eta_p(G)$ if $r_p(G) = p$.*

Remark **3.5.** *If $\gamma_p(C_n) > p$, then*

$$r_2(C_n) = \begin{cases} 2\ if\ n\ is\ odd \\ 4\ if\ n\ is\ even \end{cases}.$$

Observation 3.6. *Let S_n be a sun graph of order 2n. For $D *= D \backslash v$, $N(v) = max\{y : y \epsilon D\}$.*

Theorem 3.7. *Let p=2,3,4 and $G = S_n$ be a sun graph of order 2n. Then $r_p(S_n) \leq p$ if $|N_G(v) \cap D| \leq p$ for all $v \in V(S_n) \backslash D$.*

Proof: Let $p \leq 4$ be a positive integer and *G* be a graph of order *n*. Let *D* be a *p*-dominating set in *G*. This implies that $D' = V(G) \backslash D$. Let $|D^*| = \gamma_p(G) - |v| = |D| - 1$. Then $v \in V(G) \backslash D^* = D"$. Hence, we construct G' from *G* for each $y \in D'$, by adding $\eta_p(y, D^*, G)$ edges of G^c to G joining y to $\eta_p(y, D^*, G)$ vertices in *S*. Clearly, D^* is a *p*-dominating set of *G*, that is, $\gamma_p(G) \leq |D^*|$. By Lemma 3.2.4 $r_p(G) = \eta_p(G)$. Then

$$r_p(G) = \eta_p(G) = |B| = \sum_{y \in D"} (p - |N_G(y) \cap D^*|)$$

Note that $(p - |N_G(y) \cap D^*| = 0)$ if y is not a *p*-private neighborhood of *v*. Then

$$\begin{aligned} r_p(G) &= \sum_{y \in D"} (p - |N_G(y) \cap D^*|) \\ &= (p - |N_G(y_1) \cap D^*|) + (p - |N_G(y_2) \cap D^*|) + \cdots \\ &\quad + (p - |N_G(y_{n-1}) \cap D^*|) + (p - |N_G(y_n) \cap D^*| \\ &= (p - |N_G(y_1) \cap D^*|) + (p - |N_G(y_2) \cap D^*|) + \\ &\quad 0 + \cdots + 0 + (p - |N_G(y_n) \cap D^*|) \end{aligned}$$

Observe that $|N_G(y_1) \cap D^*| \geq 2$ and for any *p*-private neighborhood of y_n, $(p - |N_G(y_1) \cap D^*|) = 1$. Hence, $p - 2 \geq p - 3 \geq p - \Delta(G) = 0$. We have

$$r_p(G) = (p - |N_G(y_1) \cap D^*|) + (p - |N_G(y_2) \cap D^*|) + 0 + \cdots + 0 + (p - |N_G(y_n) \cap D^*| \le (p-2) + 1 + 1 \le p$$

Therefore, $r_p(G) \le p$. QED

Corollary 3.8. *For $p = 2,3,4$ and $|N_G(v) \cap D^*| = p$ for all $v \in V \backslash S_n \backslash D$. Then $r_p(S_n) = p$.*

Theorem 3.9. *Let S_n be a sun graph of order 2n. If $n \ge 3$, then*

$$r_3(S_n) = \begin{cases} 1 & if\ n = 3s+1 \\ 2 & if\ n = 3s+2 \\ 3 & if\ n = 3s \end{cases}$$

Proof: By corollary 3.1.5, $r_3(S_n) = \left\lceil \frac{n}{3} \right\rceil + n$. Then we let, $D = S \cup \{w_1, w_2, \ldots, w_n\}$ where $S = \bigcup_{i=1}^{m} \{u_{3i-2}\}$ and $m = \frac{n+2}{3}, \frac{n+1}{3}, \frac{n}{3}$, respectively. Then consider the following cases:

Case 1: $n = 3r + 1$

Let $v = u_{n-1} \in D$ or $v = u_n \in D$. By definition of p-reinforcement, let $D^* = D \backslash \{u_{n-1}\}$ or $D^* = D \backslash \{u_n\}$. Without laws of generality, let $D^* = D \backslash \{u_n\}$, then we have $\eta_3(V(S_n), D^*, S_n) = \eta_3(u_{n-1}, D^*, S_n) = 1$. So

$$r_3(S_n) = \eta_3(V(S_n), D, S_n) \le \eta_3(V(S_n), D^*, S_n) = 1.$$

Since D^* is a minimum 3-dominating set in $S_n + w_1u_{n-1}$. Hence, $r_3(S_n) = 1$.

Case 2: $n = 3r + 2$

Without laws of generality, let $v = u_{n-1} \in D$ and $D^* = D \backslash \{u_{n-1}\}$. Then it is easy to find that $\eta_3(V(S_n), D^*, S_n) = \eta_3(u_{n-2}), D^*, S_n) + \eta_3(u_{n-1}), D^*, S_n) = (p-2) + (p-2) = 2$. So,

$$r_3(S_n) = \eta_3(V(S_n), D, S_n) \le \eta_3(V(S_n), D^*, S_n) = 2.$$

Suppose that $r_p(S_n) = 1$. Partition $V(S_n)$ as $A = \{w_1, w_2, \ldots, w_n\}$ and $B = \{u_1, u_2, \ldots, u_{3s+2}\}$. Note that the observation 3.2.6, $A \subseteq D^*$. Since S is not a 3-dominating set in C_n, then there must be two adjacent vertices, denoted by u_i and u_{i+1} of C_n not in S. This means that $\eta_3(u_i, D^*, C_n) \ge 1$ and $\eta_3(u_{i+1}, D^*, C_n) \ge 1$. So

$$r_3(S_n) = \eta_3(V(S_n), D^*, S_n) \ge (u_i, D^*, C_n) + \eta_3(u_{i+1}, D^*, C_n) = 2$$

This shows that $r_3(S_n) = 2$.

Case 3: $n = 3r$

Let $D' = V(S_n) \backslash D$. Note that for all $y \in D'$, $|N_G(y) \cap D^*| = p$. Hence, by corollary 3.2.8, $r_3(S_n) = 3$. QED

Theorem 3.10. Let $S_n (n \ge 3)$ be a sun graph. Then $r_4(S_n) = r_2(C_n)$.

Proof: Let S_n be a sun graph of order *2n* with $n \ge 3$. Note that S_n consists of $C_n = [u_1, u_2, \ldots, u_n]$ and w_i joined by edges to vertices u_i and $u_{i+1(mod n)}$ for every $i = 1,2, \ldots, n$. By **theorem 3.1.7**. D is a minimum 4-dominating set in S_n if and only if $D = S \cup \{w_1, w_2, \ldots, w_n\}$ where

$$S = \begin{cases} u_1, u_3, u_5, \ldots, u_{n-2}, u_n/u_{n-1}\ if\ n = 2r+1 \\ u_1, u_3, u_5, \ldots, u_{n-3}, u_{n-1}\ if\ n = 2r \end{cases}$$

Without laws of generality, let $v = u_{n-1} \in D$ and consider the following cases: If n is odd, then $D^* = D \backslash u_{n-1}$ is a 4-dominating set in $S_n + B$ where $B = \{w_1u_n, w_nu_{n-1}\}$. While, if n is even, then $D^* = D \backslash u_{n-1}$ is a 4-dominating set in $S_n + B$ where $B = \{w_nu_{n-1}, w_1u_{n-1}, w_nu_{n-2}, w_1u_n\}$. Hence, by **remark 3.2.5**, $r_4(S_n) \le r_2(C_n)$. Suppose that $r_4(S_n) < r_2(C_n)$ for $n \ge 3$. Then $D \backslash u_{n-1}$ must be a 4-dominating set in $S_n \cup B'$ such that $|B'| < |B|$. This is a contradiction. Thus, $r_4(S_n) = r_2(C_n)$. QED

Theorem 3.11. *Let L_n be a sunlet graph of order 2n with $n \ge 3$. Then $r_p(L_n) = p - 1$.*

Proof: Let $2 \le p \le \Delta(L_n)$ and $L_n = C_n \circ K_1$ be a sunlet graph of order *2n*. Note that L_n consists of $C_n = [u_1, u_2, \ldots, u_n]$ and isolated vertices w_1 attached to each u_i. By **corollaries 3.2.11, 3.2.12,** and **lemma 3.1.1**, $|D| = \gamma_p(L_n) = \gamma_{p-1}(C_n) + n$. Without laws of generality, let $D^* = D \backslash w_i$. Note that $0 = p - \Delta(L_n) < p - 2 < p - 1 < p$. By **lemma 3.2.4**, we have

$$r_p(L_n) \le |B| = \sum_{y \in D^*} (p - |N_G(y) \cap D|) = (p - |N_G(y) \cap D|) + 0 + 0 + \cdots = p - 1.$$

Since D^* is not a p-dominating set in L_n. Then there exists $|N(w_i) \cap D^*| = 1$, that is, $N(w_i) \in D^*$. Hence, $\eta_p(w_i, X, L_n) = p - 1$. So, $r_p(L_n) = \eta_p(V(L_n), D^*, L_n) \ge \eta_p(w_i, D^*, L_n) = p - 1$. Thus, $r_p(L_n) = p - 1$. QED

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