k-FORCING NUMBER OF UNIFORM *n*-STAR SPLIT GRAPHS

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***ABSTRACT**: Let $G=(V,E)$ be a graph and k be a positive integer. A set $S(\subseteq V)$ is a k-forcing set if its vertices are initially colored, while the remaining vertices are initially non-colored, and the graph is subjected to the following color change rule such that all of the vertices in G will eventually become colored. A colored vertex with at most k non-colored neighbors will cause each non-colored neighbor to become colored. The k-forcing number of G, denoted by $F_k(G)$, is the minimum cardinality of a k-forcing set.*

This study gave the k-forcing number of uniform n-star split graphs, graph SS(n,r), and graph CS(n,r).

Keywords: *k*-forcing number, uniform *n*-star split graphs, graph *SS*(*n*,*r*), graph *CS*(*n*,*r*).

1. INTRODUCTION

A subset *S* of vertices of a graph is a *k*-forcing set if its vertices are initially colored, while the remaining vertices are initially non-colored, and the graph is subjected to the following color change rule until all the vertices will eventually become colored. A colored vertex with at most *k* non-colored neighbors will cause each non-colored neighbor to become colored. The *k*-forcing number of a graph, denoted by $F_k(G)$, is the cardinality of a smallest *k*-forcing set.

For example, consider graph *G* in Figure 1. Then $S_1 = \{a\}$ is a 2-forcing set, while $S_2 = \{b\}$ is not. The 2-forcing number of *G* is 1.

To see this, we note that *a* can 2-forces *b* and *f*, *b* can 2-forces *c* and *e*, *c* can 2-forces *d*. Hence, all the vertices of *G* will eventually be colored. Thus, S_1 = {a} is a 2-forcing set.

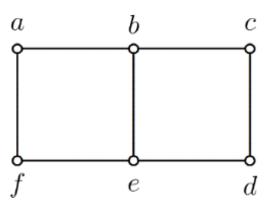

Figure 1. The graph *G*

On the other hand, we observe that *b* can not 2-force either *a*, *e* and *c*. Hence, color change cannot take effect. This shows that $S_2 = \{b\}$ is not a 2-forcing set.

Clearly, $S_1 = \{a\}$ is a minimum 2-forcing set. Thus, $F_2(G) = 1$.

The *k*-forcing concept is a generalization of the concept *zero forcing number* of a graph (the zero forcing number is actually the 1-forcing number). The concept was introduced by Barioli et al. [2] and independently, by Burgarth et al. [4]. These concepts were studied in [1–22].

2. *k*-Forcing Number of Uniform *n*-Star Split Graphs

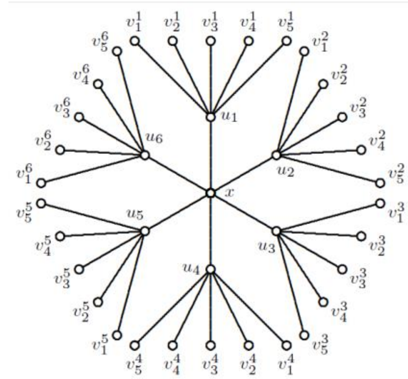

Figure 2. The graph SS_6^5

Theorem 2.1. *Let SS_n^r be a uniform n-star split graph. If k = max{r,n}, then $F_k(SS_n^r)$ = 1.*

Proof: Let SS_n^r be the uniform *n*-star split graph obtained from the star $K_{1,n} = (\{x\}, \emptyset) + (\{u_1, u_2, . . . , u_n\}, \emptyset)$ by adding stars $K_{1,r}^i = (\{u_i\}, \emptyset) + (\{v_1^i, v_2^i, . . . , v_r^i\}, \emptyset)$ for *i* = 1, 2, . . . , *n*. Let *k* = max{*r*,*n*} and *S* = {*x*}. Then simultaneously, *x* can *k*-force u_i for all *i* = 1, 2, . . . , *n*. Next, for each *i* = 1, 2, . . . , *n* u_i can *k*-force $v_j^{(i)}$ for all *j* = 1, 2, . . . , *m*. Hence, all the vertices of SS_n^r can be colored by applying the color-change rule to *S*. Thus, *S* is a *k*-forcing set. Therefore, $F_k(SS_n^r) = 1$. □

Theorem 2.2. *Let SS_n^r be a uniform n-star split graph. Then $F_1(SS_n^r) = nr$.*

Proof: Let *k* = 1, and S = $\{v_j^i : j = 1, 2, . . . , m$ and $i = 1, 2, . . . , n\}$. Then for each *i* = 1, 2, . . . , *n*, v_1^i can 1-force u_i. Next, for some *i* = 1, 2, . . . , *n* u_i can 1-force *x*. Hence, all the vertices of SS_n^r can be colored by applying the color-change rule to *S*. Thus, *S* is a i-forcing set. Therefore, $F_1(SS_n^r) \leq nr$. It can be shown that a 1-forcing set of SS_n^r cannot have less than *nr* elements. Accordingly, $F_1(SS_n^r) = nr$. □

3. *k*-Forcing Number of Graph *SS*(*n, r*)

Theorem 3.1. *Let SS*(n, r) *be the graph obtained from the star* $K_{1,n}$ = ({x}, ∅) + ({u_1, u_2, . . . , u_n}, ∅) *by adding edges* $u_i w_i$ *and stars* $K_{1,r}$ = ({w_i}, ∅) + ({v_1^i, v_2^i , . . . , v_r^i}, ∅) *for i* = 1, 2, . . . , *n*. *If k* = *max*{*r*,*n*}, *then* F_k(*SS*(n, r)) = 1.

Proof: Let *k*=*max*{*r*,*n*} and *S* = {*x*}. Then, *x* can *k*-force u_i for all *i* = 1, 2, . . . , *n*. Next, for each *i* = 1, 2, . . . , *n*, u_i can *k*-force $v_j^{(i)}$ for all *j* = 1, 2, . . . , *m*. Hence, all the vertices of *SS*(*n*, *r*) can be colored by applying the color-change rule to *S*. Thus, *S* is a *k*-forcing set. Therefore, F_k(*SS*(*n*, *r*)) = 1. □

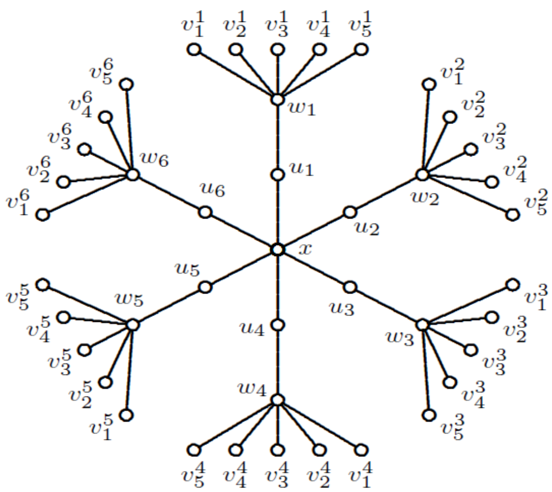

Figure 3. The graph *SS*(6,5)

Theorem 3.2. *Let SS*(n, r) *be the graph obtained from the star* $K_{1,n}$ = ({x}, ∅) + ({u_1, u_2, . . . , u_n}, ∅) *by adding edges* $u_i w_i$ *and stars* $K_{1,r}$ = ({w_i}, ∅) + ({v_1^i, v_2^i , . . . , v_r^i}, ∅) *for i* = 1, 2, . . . , *n*. *Then* F_1(*SS*(n, r)) = *nr*.

Proof: Let *k* = 1, and *S* = {v_j^i : *j* = 1, 2, . . . , *m* and *i* = 1, 2, . . . , *n*}. Then for each *i* = 1, 2, . . . , *n*, v_1^i can 1-force u_i. Next, for each *i* = 1, 2, . . . , *n* u_i can 1-force w_i . Hence, all the vertices of CS_n^r can be colored by applying the color-change rule to *S*. Thus, *S* is a 1-forcing set. Therefore, F_1(*SS*(*n*, *r*)) ≤ *nr*. It can be shown that a 1-forcing set of *SS*(*n*, *r*) cannot have less than *nr* elements. Accordingly, F_1(*SS*(*n*, *r*)) = *nr*. □

4. *k*-Forcing Number of Graph *CS*(*n, r*)

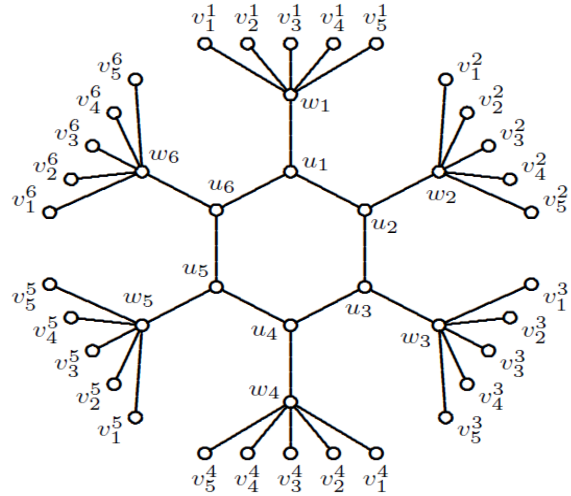

Figure 4. The graph *CS*(6,5)

Theorem 4.1. *Let CS*(n, r) *be the graph obtained from* C_n = [u_1, u_2, . . . , u_n] *by adding edges* $u_i w_i$ *and stars* $K_{1,r}$ = ({w_i}, ∅) + ({v_1^i, v_2^i , . . . , v_r^i}, ∅) *for i* = 1, 2, . . . , *n*. *Then* F_1(*CS*(n, r)) = *nr*.

Proof: Let *k* = 1, and *S* = {v_j^i : *j* = 1, 2, . . . , *m* and *i* = 1, 2, . . . , *n* }. Then for each *i* = 1, 2, . . . , *n*, v_1^i can 1-force u_i. Next, for each *i* = 1, 2, . . . , *n* u_i can 1-force w_i. Hence, all the vertices of CS_n^r can be colored by applying the color-change rule to *S*. Thus, *S* is a 1-forcing set. Therefore, $F_1(CS_n^r)$ ≤ *nr*. It can be shown that a 1-forcing set of CS_n^r cannot have less than *nr* elements. Accordingly, $F_1(CS_n^r)$ = *nr*. □

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