k-FORCING NUMBER OF SOME CYCLE-RELATED GRAPHS

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ABSTRACT: *Let $G=(V,E)$ be a graph and k be a positive integer. A set $S(\subseteq V)$ is a k-forcing set if its vertices are initially colored, while the remaining vertices are initially non-colored, and the graph is subjected to the following color change rule such that all of the vertices in G will eventually become colored. A colored vertex with at most k non-colored neighbors will cause each non-colored neighbor to become colored. The k-forcing number of G, denoted by $F_k(G)$, is the minimum cardinality of a k-forcing set.*

This study gave the k-forcing number of sun graphs, and sunlet graphs.

Keywords: *k*-forcing number, sun graph, sunlet graph, cycles

1. INTRODUCTION

A subset *S* of vertices of a graph is a *k*-forcing set if its vertices are initially colored, while the remaining vertices are initially non-colored, and the graph is subjected to the following color change rule until all the vertices will eventually become colored. A colored vertex with at most *k* non-colored neighbors will cause each non-colored neighbor to become colored. The *k*-forcing number of a graph, denoted by $F_k(G)$, is the cardinality of a smallest *k*-forcing set.

For example, consider graph *G* in Figure 1. Then $S_1 = \{a\}$ is a 2-forcing set, while $S_2 = \{b\}$ is not. The 2-forcing number of *G* is 1.

To see this, we note that *a* can 2-forces *b* and *f*, *b* can 2-forces *c* and *e*, *c* can 2-forces *d*. Hence, all the vertices of *G* will eventually be colored. Thus, $S_1 = \{a\}$ is a 2-forcing set.

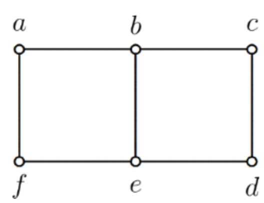

Figure 1. The graph *G*

On the other hand, we observe that *b* can not 2-force either *a*, *e* and *c*. Hence, color change cannot take effect. This shows that $S_2 = \{b\}$ is not a 2-forcing set.

Clearly, $S_1 = \{a\}$ is a minimum 2-forcing set. Thus, $F_2(G) = 1$.

The *k*-forcing concept is a generalization of the concept *zero forcing number* of a graph (the zero forcing number is actually the 1-forcing number). The concept was introduced by Barioli et al. [2] and independently, by Burgarth et al. [4]. These concepts were studied in [1–22].

2. PRELIMINARY RESULTS

In this section, we present some of the general properties of the *k*-forcing number found in [23]. Clearly the *k*-forcing number of a graph cannot exceed its order. This observation is more formally stated in the next lemma.

Lemma 2.1. *Let G be a graph of order n. Then $F_k(G) \le n$ for all $k \in \mathbb{N}$.*

The following remark says that a *k*-forcing set is also a *k*+1-forcing set. This idea is utilized in Corollary 2.3.

Remark 2.2. *Let G be a graph. Then every k-forcing set in G is also a k +1-forcing set.*

Corollary 2.3. *Let G be a graph. Then $F_k(G) \ge F_{k+1}(G)$ for all $k \in \mathbb{N}$.*

3. MAIN RESULTS AND DISCUSSIONS

Definition 3.1. *The **sun** graph, denoted by S_n, is the graph of order 2n obtained from the cycle $C_n = [v_1, v_1, . . . , v_n]$ by adding vertices u_i joined by edges to vertices v_i and $v_{i+1(mod\ n)}$ for i = 1, 2, . . . , n.*

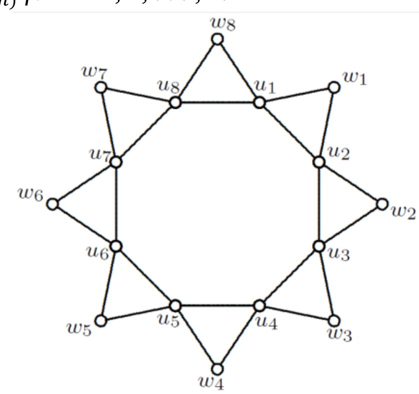

Figure 2. The sun graph S_8

Theorem 3.2. *Let S_n be the sun graph of order 2n. If k = 1, then $F_k(S_n) = n$.*

Proof: Let S_n be the sun graph of order 2*n* obtained from $C_n = [v_1, v_1, . . . , v_n]$ by adding vertices u_i joined by edges to vertices v_i and $v_{i+1(mod\ n)}$ for *i* = 1, 2, . . . , *n*. Let $S = \{v_1, v_1, . . . , v_n\}$. Then for each *i* = 1, 2, . . . , *n*, v_i can 1-force u_i . Hence, all the vertices of S_n will eventually be colored. Hence, *S* is a 1-forcing set. Note that a 1-forcing set of S_n cannot have less than *n* elements. Therefore, $F_1(S_n)$ = n. □

Theorem 3.3. *Let S_n be the sun graph of order 2n. If $k \ge 2$, then $F_k(S_n) = 1$.*

Proof: Let S_n be the sun graph of order 2*n* obtained from $C_n = [v_1, v_1, . . . , v_n]$ by adding vertices u_i joined by edges to vertices v_i and $v_{i+1(mod\ n)}$ for *i* = 1, 2, . . . , *n*. Let *S* = $\{u_1\}$ and consider the following cases.

Case 1. *n* is even

If *n* is even, then: u_1 can 2-force v_1 and v_2; v_1 can 2-force u_n and v_n, and v_2 can 2-force u_2 and v_3; v_n can 2-force u_{n-1} and v_{n-1}, and v_3 can 2-force u_3 and v_4; and so on. Until eventually, $v_{(n+4)/2}$ can 2-force $u_{(n+2)/2}$, and $v_{n/2}$ can 2-force $u_{n/2}$ and $v_{(n+2)/2}$.

Case 2. *n* is odd

If *n* is odd, then: u_1 can 2-force v_1 and v_2; v_1 can 2-force u_n and v_n, and v_2 can 3-force u_2 and v_3; v_n can 2-force u_{n-1} and v_{n-1}, and v_3 can 2-force u_3 and v_4; and so on. Until eventually, $v_{(n+3)/2}$ can 2-force $u_{(n+1)/2}$, and $v_{\lfloor n/2 \rfloor}$ can 2-force $u_{\lfloor n/2 \rfloor}$ and $v_{(n+1)/2}$.

In any case, all the vertices of S_n will eventually be colored. Hence, *S* is a 2-forcing set. Thus, $F_2(S_n) = 1$. By Corollary 2.11, $F_k(S_n) = 1$ for all positive integer $k \geq 2$. □

Definition 4.1. *The* ***sunlet*** *graph, denoted by* L_n, *is the graph of order 2n obtained from the cycle* $C_n = [v_1, v_1, . . . , v_n]$ *by attaching pendant edges* $v_i u_i$ *for i = 1, 2, . . . , n.*

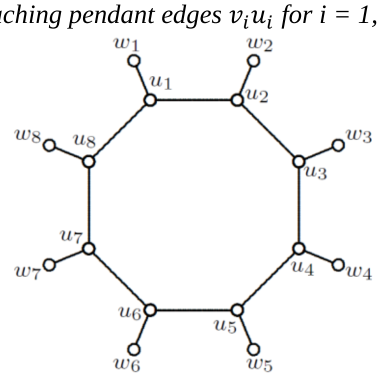

Figure 3. The sunlet graph L_8

Theorem 4.2. *Let* L_n *be the sunlet graph of order* 2*n*. *If* k = 1, *then* $F_k(L_n) = \lceil n/2 \rceil$.

Proof: Let L_n be the sunlet graph of order 2*n* obtained from $C_n = [v_1, v_1, . . . , v_n]$ by attaching pendant edges $v_i u_i$ for *i* = 1, 2, . . . , *n*. Let $S = \{v_i : i \equiv 1 \pmod 4 \text{ or } i \equiv 2 \pmod 4\}$. Without loss of generality, assume that *n* is even. Then for each *i* with $i \equiv 1 \pmod 4$ or $i \equiv 2 \pmod 4$, u_i can 1-force v_i ; and, v_i can 1-force the vertex in N(v_i)\S. Thus, all the vertices of L_n will eventually be colored. Hence, *S* is a 1-forcing set. Thus, $F_1(L_n) \leq \lceil n/2 \rceil$. Note that a 1-forcing set of L_n cannot have less than $\lceil n/2 \rceil$ elements. Therefore, $F_1(L_n) = \lceil n/2 \rceil$. □

Theorem 4.3. *Let* L_n *be the sunlet graph of order* 2*n*. *If* k ≥ 2, then $F_k(L_n) = 1$.

Proof: Let L_n be the sunlet graph of order 2*n* obtained from $C_n = [v_1, v_1, . . . , v_n]$ by attaching pendant edges $v_i u_i$ for *i* = 1, 2, . . . , *n*. Let $S = \{u_1\}$ and consider the following cases.

Case 1. *n* is even

If *n* is even, then: u_1 can 2-force v_1; v_1 can 2-force v_n and v_2; v_2 can 2-force u_2 and v_3, and v_n can 2-force u_n and u_{n-1}; v_3 can 2-force u_3 and v_4, and v_{n-1} can 2-force u_{n-1} and v_{n-2}; and so on. Until eventually, $v_{(n+2)/2}$ can 2-force $u_{(n+2)/2}$.

Case 2. *n* is odd

If *n* is even, then: u_1 can 2-force v_1; v_1 can 2-force v_n and v_2; v_2 can 2-force u_2 and v_3, and v_n can 2-force u_n and v_{n-1}; v_3 can 2-force u_3 and v_4, and v_{n-1} can 2-force $u_{n-1}$and v_{n-2}; and so on. Until eventually, $v_{\lceil n/2 \rceil}$ can 2-force $u_{\lceil n/2 \rceil}$.

In any case, all the vertices of L_n will eventually be colored. Hence, *S* is a 2-forcing set. Thus, $F_2(L_n) = 1$. By Corollary 2.11, $F_k(L_n) = 1$ for all positive integer $k \geq 2$. □

4. CONCLUSION

The important concepts and results presented in this paper supported, and intertwined with, those obtained by other authors, making this article very interesting. The construction of the different theorems were realized using the definition and properties of *k*-forcing set and *k*-forcing number. Also, some properties focusing on generalizing zero forcing set in graph theory were realized.

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