A NOTE ON THE HOMOMORPHISM OF *g*-GROUPS

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ABSTRACT: *A nonempty set G is a g-group* [*with respect to a binary operation* $*$] *if it satisfies the following properties:* ($g1$) $a * (b * c) = (a * b) * c$ *for all* $a, b, c \in G$; ($g2$) *for each* $a \in G$, *there exists an element* $e \in G$ *such that* $a * e = a = e * a$ (*e is called an identity element of a*); *and,* ($g3$) *for each* $a \in G$, *there exists an element* $b \in G$ *such that* $a * b = e = b * a$ *for some identity element e of a. In this study, we gave some important properties of g-subgroups, homomorphism of g-groups, and the zero element. We also presented a couple of ways to construct g-groups and g-subgroups.*

Keywords: *g*-group, *g*-group homorphism, *g*-subgroups

, kernel

1. INTRODUCTION

A binary operation $*$ on a set G is a function from $G \times G$ to G. The image of (a, b) under $*$ will be denoted by $a * b$. A nonempty set G is a g-group with respect to a binary operation $*$ if it satisfies the following properties: ($g1$) $a * (b * c) = (a * b) * c$ for all $a, b, c \in G$ (in this case, we say that $*$ is associative); (g2) for each $a \in G$, there exists an element $e \in G$ (called an identity element) such that $a * e = a = e * a$; and, ($g3$) for each $a \in G$, there exists an element $b \in G$ (called an inverse of g) such that $a * b = e = b * a$ for some identity element e of a. In this case, we write $(G, *)$ to denote the algebraic structure. If $a * b = b * a$ for all $a, b \in G$, then we say that G is an Abelian g-group. An element with a unique identity element is called a unit, otherwise, we say that a is non-unit. The singleton sets $\{0\}$ and $\{1\}$ with respect to multiplication $\times$ are g-groups (the two are called trivial g-groups). Tables 1 and 2 may be helpful in seeing this. Also, the set $\{0, 1\}$ is also a g-group under multiplication as shown in Table 3. The introduction of the algebraic structure g-group was motivated by the intention of presenting a structure having a unique operation which generalizes the properties of the operation multiplication in a field.

×	0
0	0

Table 1. The *g*-group {0}

×	1
1	1

Table 2. The *g*-group {1}

Hereafter, please refer to [3] for the other concepts. This paper is a sequel of a previously published study [1] where a new algebraic structure called g-group was introduced. Additional properties of such structure is presented and shown in this paper. In the early twentieth century, algebra had evolved into a study of axiomatic systems. It was then referred to as abstract algebra [5]. Since then, mathematicians have introduced and explored various algebraic structures. Some were found related to another and others were found to be entirely different. One particular concept that captured the attention of many researchers is that of groups. Several group-related structures such as quasi-groups, generalized groups and similar structures became the favorite topic of algebra enthusiasts [6, 2, 7]. Findings from these studies were found to be applicable in other branches of mathematics such as Number Theory, Geometry, Analysis [4], Computer Science [1], etc. Unlike in groups, distinct elements of a g-group may have different identity elements. Also, the identity element as well as the inverse may not be unique. A g-group is generally not a group, but groups are necessarily g-groups.

The structure g-group may have important applications in microprocessor design. Specifically, it can be used to minimize digital circuits which uses *AND* gates only. For example, consider the digital circuit with three inputs, A, B, and C, given by $(A \vee B) \vee (A \vee C)$. By inspection, the expression $(A \vee B) \vee (A \vee C)$ suggest that a digital circuit needs three *AND* gates to give the desired output. However, using some properties of the g-group $(\square, \cdot)$, the circuit can be minimized as follows. Identifying $\cdot$ with $\vee$, we have $(A \vee B) \vee (A \vee C) = (A \cdot B) \cdot (A \cdot C) = [(A \cdot B) \cdot A] \cdot C = [A \cdot (B \cdot A)] \cdot C = [A \cdot (A \cdot B)] \cdot C = [(A \cdot A) \cdot B] \cdot C = (A \cdot B) \cdot C = (A \vee B) \vee C$. Note that the expression $(A \vee B) \vee C$ uses two *AND* gates only, but still performs the same function as $(A \vee B) \vee (A \vee C)$. This simplifies the design of the circuit.

Inthis paper, we introduce the concept of the kernel of g-group homomorphisms. Some properties of this concept were shown. We are also able to present another means of constructing g-subgroups using the concept of kernels of g-group homomorphisms.

Fraleigh (2003) defines the kernel of a group homomorphism as the set of pre-images of the identity element in the co-domain. This definition generally does not apply to g-groups since for g-groups, elements may have identity elements different from those of other elements. Also, an element in a g-group may have more than one identity element.

The next statement defines what the kernel of a g-group homomorphism is.

Definition 1: Let G_1 and G_2 be g-groups and $\phi: G_1 \to G_2$ be a homomorphism. Also, let $g \in G_2$ and E be the set of all identity elements of g, that is, $E = \{e \in G_2 \mid eg = ge = g\}$. Define the **kernel of ϕ with respect to g**, denoted $\ker[\phi_g]$, as the subset of G_1 whose elements are those whose images are in E, that is, $\ker[\phi_g] = \{a \in G_1 \mid \phi(a) \in E\}$.

2. RESULTS

In Lemma 1, we show that all identity elements of the elements in ker[ϕ_g] are also in ker[ϕ_g].

Lemma 1. Let G_1 and G_2 be g-groups and ϕ: $G_1 \to G_2$ be a homomorphism. Also, let $g \in G_2$. If $a \in$ ker[ϕ_g] and e is an identity of a, then $e \in$ ker[ϕ_g].

Proof: Since $a \in$ ker[ϕ_g], $\phi(a)g = g\phi(a) = g$. Thus, $g\phi(e) = g\phi(a)\phi(e) = g\phi(ae) = g\phi(a) = g$. Similarly, $\phi(e)g = g$. *QED*

The next lemma shows that the binary operation in G_1 is induced in *ker*[ϕ_g] whereas Lemma 3 shows that the inverses of the elements in *ker*[ϕ_g] are also in *ker*[ϕ_g].

Lemma 2. Let G_1 and G_2 be g-groups and ϕ: $G_1 \to G_2$ be a homomorphism and let $g \in G_2$. If $a,b \in$ ker[ϕ_g], then $ab \in$ ker[ϕ_g].

Proof: Observe that $\phi(ab)g = [\phi(a)\phi(b)]g = \phi(a)[\phi(b)g] = \phi(a)g = g$. Similarly, $g\phi(ab) = g$. *QED*

Lemma 3. Let G_1 and G_2 be g-groups and ϕ: $G_1 \to G_2$ be a homomorphism and let $g \in G_2$. If $a \in$ ker[ϕ_g] and a^{-1} is the inverse of a for its identity e, then $a^{-1} \in$ ker[ϕ_g].

Proof: Note that $\phi(a^{-1})g = \phi(a^{-1})[\phi(a)g] = [\phi(a^{-1})\phi(a)]g = \phi(a^{-1}a)g = \phi(e)g = g$. *QED*

The following theorem shows another means of generating a g-subgroup using the concept of the kernel of a g-group homomorphism.

Theorem 1. Let G_1 and G_2 be g-groups and ϕ: $G_1 \to G_2$ be a homomorphism. Also, let $g \in G_2$. If ker[ϕ_g] $\neq \varnothing$, then ker[ϕ_g] is a g-subgroup of G_1.

Proof: This result follows directly from Lemmas 1, 2 and 3 and from the fact that ker[ϕ_g] $\subseteq G_1$. *QED*

Theorem 2. Let G_1 and G_2 be g-groups and ϕ: $G_1 \to G_2$ be a homomorphism. Also, let f, $g \in G_2$. Then ker[ϕ_f] $\cap$ ker[ϕ_g] $\subseteq$ ker[ϕ_{fg}].

Proof: Suppose $a \in$ ker[ϕ_f] $\cap$ ker[ϕ_g]. Then $\phi(a)[fg] = [\phi(a)f]g = fg$. Similarly, $[fg]\phi(a) = fg$. Thus $a \in$ ker[ϕ_{fg}]. *QED*

Corollary 1. Let G_1 and G_2 be Abelian g-groups and ϕ: $G_1 \to G_2$ be a homomorphism. Also, let f, g be units G_2. Then ker[ϕ_f] $\cap$ ker[ϕ_g] = ker[ϕ_{fg}].

This result is a direct consequence of the hypotheses and Theorem 2.

Remark 8. Although intersections of g-subgroups are not necessarily g-subgroups, we can see in this case that an intersection of g-subgroups may also be a g-subgroup.

Theorem 3. Let G_1 and G_2 be g-groups and ϕ: $G_1 \to G_2$ be a homomorphism. If G_2 has a zero element 0, then ker[ϕ_0] = G_1.

Proof: Note that if G_2 has a zero element 0, then $g0 = 0g = 0$ for all elements g in G_2, that is, all elements in G_2 are identity elements of 0. Hence, the result follows. *QED*

Corollary 2. Let G_1 and G_2 be g-groups and ϕ: $G_1 \to G_2$ be a homomorphism. If g is a zero divisor in G_2, then there is an element $f \neq 0$ in G_2 such that ker[ϕ_{fg}] = G_1.

Proof: Suppose g is a zero divisor in G_2. Then there exists $f \neq 0$ in G_2 such that $fg = gf = 0$. Hence, by Theorems 2 and 3, ker[ϕ_{fg}] = G_1.*QED*

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