ON TRANSNORMAL SURFACES IN THE EUCLIDEAN SPACE R^n

Huthaifa Alaroud and Kamal Al-Banawi
Department of Mathematics and Statistics, Mu'tah University,
Al-Karak, P.O. Box 7, Jordan
e-mail: h_alaroud@yahoo.com, e-mail: kbanawi@mutah.edu.jo

ABSTRACT: *In this paper we study transnormal surfaces in the Euclidean space R^n. We use the distance function and the Euler characteristic to deduce some properties regarding transnormal surfaces in R^n.*

Keywords: Transnormal surface, generating frame, distance function, critical point, Euler characteristic.

1. INTRODUCTION

The concept of transnormality, which is due to Robertson [8], is a generalization of the concept of constant width referred to Mellish [6]. A surface of constant width can be formulated as follows. Let S be a smooth compact connected surface without boundary that is smoothly embedded in R^n. A chord of S is *normal* if it is normal to S at one of its endpoints and *binormal* if it is normal to S at both end points. The surface S is of constant width if and only if every normal chord of S is binormal to S. Each point of the endpoints is called the *opposite* of the other. The differential geometry of curves and surfaces of constant width was studied by Al-Banawi [1]. Now for each point $p \in S$, there exists a unique tangent plane T_p tangent to S at p with dimension 2 and a unique normal plane N_p normal to S at p with dimension $n-2$. Thus, there are maps T and N with $T(p)=T_p$ and $N(p)=N_p$.

Definition 1. [8] The surface S is transnormal in R^n iff $\forall p,\ q \in S$, if $q \in N(p)$, then $N(q)=N(p)$.

Let W be the space of normal planes of S, so $W=N(S)$. Then $N: S \to W$ is a covering map [9]. If S is a transnormal surface in R^n and the order of N as a covering map is r, then S is called an $r-$transnormal surface.

Definition 2. [8] Let S be a transnormal surface in R^n. The *generating frame* of S at p is $\phi(p)=S \cap N(p)$. If S is r-transnormal, then $|\phi(p)|=r$ where $|\ |$ is the cardinality.

Now any normal plane of a compact $r-$transnormal surface S cuts S transversally at exactly r points. Also there are planes of the same dimension as these normal planes that don't meet S at all. The mod 2 intersection number of each plane with S is therefore equal to zero. Since any two planes of the same dimension can be moved onto one another, it follows that the mod 2 intersection number of any normal plane with S is also r. Thus, for a compact $r-$transnormal surface, r is even.

The work in this article is built on the main theorems on transnormality due to Robertson [8,9,10] and is a continuation of the work of Al-Banawi and Carter in [2,3] on transnormal compact curves and transnormal partial tubes. In [4], Al-Banawi studied transnormal surfaces in R^3 and R^4. He classified compact and noncompact transnormal surfaces in R^3 and R^4 as well their orders and illustrated how the usual torus is not transnormal in R^3 while the product one is transnormal in R^4.

2. The Euler characteristic of compact transnormal surfaces

Now let S be an $r-$transnormal surface in R^n. Assume that $F: S \to R^n$ is a smooth embedding of S into R^n. For $p \in S$, let $\Lambda_p : S \to R^+ \cup \{0\}$ be the distance function defined by $\Lambda_p(u,v)=\|F(u,v)-p\|^2$. A point $p \in S$ is a *critical point* of Λ_p if $\frac{\partial \Lambda_p}{\partial x}=\frac{\partial \Lambda_p}{\partial y}=0$. A critical point p is *nondegenerate* iff the Hessian matrix
$$\begin{pmatrix} \frac{\partial^2 \Lambda_p}{\partial x^2} & \frac{\partial^2 \Lambda_p}{\partial x \partial y} \\ \frac{\partial^2 \Lambda_p}{\partial x \partial y} & \frac{\partial^2 \Lambda_p}{\partial y^2} \end{pmatrix}$$
is nonsingular. If p is a nondegenerate critical point of Λ_p, then its *index* is the number of negative eigenvalues of the Hessian matrix at p.

The Euler characteristic of a surface is an alternating sum over the cells (vertices, edges and faces) of the surface. To have a simple look, for a surface S the Euler characteristic is defined by $\chi(S)=a-b+c$ where a is the number of vertices of S, b is the number of edges of S, and c is the number of faces of S.

If S is compact and C_i denotes the number of critical points of Λ_p of index i, then the Euler characteristic of S is given by $\chi(S)=C_0-C_1+C_2$ [7]. Since $N: S \to W$ is an r-fold covering, then $\chi(S)=r\chi(W)$ [5]. Also $C_0>0$ so that $-r<\chi(S)\le r$, and hence $-1<\chi(W)\le 1$. So $\chi(W)=0$ or $\chi(W)=1$. Thus, for a compact $r-$transnormal surface, either $\chi(S)=0$ or $\chi(S)=r$.

Example 1. Let $F(u,v)=(\cos u, \sin u, \cos v, \sin v)$, $p=(1,0,1,0)$. Then

$$\Lambda_p(u,v)=\|F(u,v)-p\|^2$$

$$= (\cos u - 1)^2 + (\sin u - 0)^2 (\cos v - 1)^2 + (\sin v - 0)^2$$
$$= \cos^2 u - 2\cos u + 1 + \sin^2 u + \cos^2 v - 2\cos v + 1 + \sin^2 v$$
$$= 4 - 2\cos u - 2\cos v.$$

Now

$$\frac{\partial \Lambda_p}{\partial u} = 2\sin u, \frac{\partial \Lambda_p}{\partial v} = 2\sin v.$$

Hence for the critical points, we have $u = \{0, \pi\}, v = \{0, \pi\}.$ Thus, Λ_p has four nondegenerate critical points, namely $(1,0,1,0)$, $(1,0,-1,0)$, $(-1,0,1,0)$, $(-1,0,-1,0).$

Now

$$\frac{\partial^2 \Lambda_p}{\partial^2 u} = 2\cos u \text{ and } \frac{\partial^2 \Lambda_p}{\partial^2 v} = 2\cos v,$$

$$\frac{\partial^2 \Lambda_p}{\partial u \partial v} = \frac{\partial^2 \Lambda_p}{\partial v \partial u} = 0.$$

Thus, the Hessian matrix is $H = \begin{pmatrix} 2\cos u & 0 \\ 0 & 2\cos v \end{pmatrix}$, and so

at $(u,v) = (0,0),\ H_1 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$

at $(u,v) = (0,\pi),\ H_2 = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix},$

at $(u,v) = (\pi,0),\ H_3 = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix},$

at $(u,v) = (\pi,\pi),\ H_4 = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}.$

Hence H_1 has no negative eigenvalues, and so index(H_1) $= 0$. Similarly, index$(H_2) =$ index$(H_3) = 1$, index $(H_4) = 2$. Thus, $C_0 = 1,\ C_1 = 2,\ C_2 = 1.$

$$\chi(S) = C_0 - C_1 + C_2 = 1 - 2 + 1 = 0.$$

Theorem 1. Let S be a surface in R^n. Then the following are equivalent:

(1) The surface S is $r-$transnormal.

(2) The distance function on S has r nondegenerate critical points.

Proof. $(a \Rightarrow b)$ Assume that S is $r-$transnormal. Then $N : S \rightarrow W$ is a covering map with order r. Let $\Lambda_p : S \rightarrow R^+ \bigcup \{0\}$ be the distance function defined by $\Lambda_p(u,v) = \lVert F(u,v) - p \rVert^2$ where $p \in S$ and (u,v) are local coordinates of S. Of course, S is the image of F. Now we show that Λ_p has a nondegenerate critical point at q iff $q - p$ is in $N(q)$. For, $\Lambda_p(u,v) = (F(u,v) - p)\cdot(F(u,v) - p),$ and so

$$\frac{\partial \Lambda_p}{\partial u}(q) = 2\frac{\partial F}{\partial u}(q)\cdot(q - p) \text{ and}$$

$$\frac{\partial \Lambda_p}{\partial v}(q) = 2\frac{\partial F}{\partial v}(q)\cdot(q - p)$$

Hence $\frac{\partial \Lambda_p}{\partial u}(q) = \frac{\partial \Lambda_p}{\partial v}(q) = 0$ iff $q - p \perp \frac{\partial F}{\partial u}(q)$ and $q - p \perp \frac{\partial F}{\partial v}(q)$ iff $q - p \in N(q).$ Thus, the set of critical points of Λ_p is the generating frame $\varphi(p)$. While Robertson in [9] had proved that a transnormal surface does not meet its sets of focal points, one can conclude that p is not a focal point of S, and so q is a nondegenerate critical point of Λ_p.

$(b) \Rightarrow (a)$ Let $p \in S$. Then Λ_p has a set Δ of r critical points including p itself. Since $\forall q \in \Delta$, $(q - p).\frac{\partial \Lambda_p}{\partial u}(q) = (q - p).\frac{\partial \Lambda_p}{\partial v}(q) = 0,$ it is clear that $q \in N(p)$. By symmetry, $p \in N(q)$. Hence $N(p) = N(q),$ and so S is transnormal. But $|\Delta| = r.$ Hence S is $r-$transnormal.

Corollary 1. If S is an $r-$transnormal surface in R^n, then the order of the covering map $N : S \rightarrow W$ is the number of nondegenerate critical points of the distance function that is defined on S.

A function whose all critical points are nondegenerate is called a Morse function [7]. So good news are guaranteed by the next corollary.

Corollary 2. Any transnormal surface admits a Morse function on itself, in particular the distance function.

Theorem 2. The index of p as a minimum point of Λ_p is the number of negative eigenvalues of the first fundamental matrix evaluated at p.

Proof. Recall that

$$\frac{\partial \Lambda_p}{\partial u} = 2F_u.(F - p) \text{ and } \frac{\partial \Lambda_p}{\partial v} = 2F_v.(F - p).$$

Thus,

$$\frac{\partial^2 \Lambda_p}{\partial^2 u} = 2F_{uu}.(F - p) + 2F_u.F_u = 2F_{uu}.(F - p) + 2E,$$

$$\frac{\partial^2 \Lambda_p}{\partial^2 v} = 2F_{vv}.(F - p) + 2F_v.F_v = 2F_{vv}.(F - p) + 2G,$$

$$\frac{\partial^2 \Lambda_p}{\partial u \partial v} = 2F_{uv}.(F - p) + 2F_u.F_v = 2F_{uv}.(F - p) + 2F.$$

Now if q is a nondegenerate critical point, then $\Lambda_u(q) = \Lambda_v(q) = 0,$ and

$\det\begin{pmatrix}\Lambda_{uu}(q) & \Lambda_{uv}(q)\\ \Lambda_{vu}(q) & \Lambda_{vv}(q)\end{pmatrix}\neq 0.$ For the choice of q to be p,

$$\det\begin{pmatrix}2E(p) & 2F(p)\\ 2F(p) & 2G(p)\end{pmatrix} = 4(E(p)G(p)-F^2(p)) = 4\det\Gamma.$$

Also the Hessian matrix of Λ has no negative eigenvalues, since p is a minimum point.

Also, we show in the next example that the above result only works for the minimum point.

Example 2. Let $F(u,\mathrm{v}) = (\cos u, \sin u, \cos v, \sin v).$ Then the first fundamental matrix is $\Gamma = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$.

Take $p=(1,0,1,0).$ Then by Example 1, Λ_p has four nondegenerate critical points, namely $(1,0,1,0)$, $(1,0,-1,0)$, $(-1,0,1,0)$, $(-1,0,-1,0).$

For $p=(1,0,1,0),$ index $=0=$ index $\Gamma(p).$

For $q=(-1,0,-1,0),$ index $=2\neq$ index $\Gamma(p).$

For $s_1=(1,0,-1,0),$ index $=1\neq$ index $\Gamma(p).$

For $s_2=(-1,0,1,0),$ index $=1\neq$ index $\Gamma(p).$

Now we explain the difference in behavior between H and Γ of Λ_p at other critical points.

For $q=(-1,0,-1,0),$

$$\frac{\partial^2\Lambda_p}{\partial^2 u} = 2(1,0,0,0).(-2,0,-2,0)+2E$$
$$=(-4)+2E,$$
$$\frac{\partial^2\Lambda_p}{\partial^2 v} = (-4)+2G,$$
$$\frac{\partial^2\Lambda_p}{\partial u\partial v} = \frac{\partial^2\Lambda_p}{\partial v\partial u} = 0+2F.$$

For $s_1=(1,0,-1,0),$

$$\frac{\partial^2\Lambda_p}{\partial^2 u} = 2(0,0,0,0).(0,0,-2,0)+2E$$
$$=0+2E,$$
$$\frac{\partial^2\Lambda_p}{\partial^2 v} = (-4)+2G,$$
$$\frac{\partial^2\Lambda_p}{\partial u\partial v} = \frac{\partial^2\Lambda_p}{\partial v\partial u} = 0+2F.$$

For $s_2=(-1,0,1,0),$

$$\frac{\partial^2\Lambda_p}{\partial^2 u} = 2(1,0,0,0).(-2,0,0,0)+2E$$
$$=(-4)+2E,$$
$$\frac{\partial^2\Lambda_p}{\partial^2 v} = 0+2G,$$
$$\frac{\partial^2\Lambda_p}{\partial u\partial v} = \frac{\partial^2\Lambda_p}{\partial v\partial u} = 0+2F.$$

3. CONCLUSION

The distance function on a transnormal surface is a good tool to extract information regarding the behavior of a transnormal surface in R^n. Here the Hessian matrix of the distance function is compared with the first fundamental matrix of the surface itself.

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