THE EQUIPARTITION OF ENERGY AND THE HORIZON SPECTRUM

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***ABSTRACT*:** *A holographic version of the equipartition of energy was derived for a general diffeomorphism invariant theory by Padhmanabhan* (2010). *We discerned that Hawking's temperature is the equipartition law, or vice versa, granted that the horizon area is measured in Planck units* (2017). *Under the equipartition law, we wish to point out that the semi-classical holographic setup, including the Bekenstein-Hawking entropy, is consistent only when the horizon is quantized in exact Planck units. This is in clear contrast to the previously obtained values of the minimal area from theories such as loop quantum gravity and others. It becomes evident that the previously determined values though good for the entropy formula, are inconsistent with the equipartition law, hence with Hawking's temperature.*

Keywords: Equipartition, quantum gravity, black hole, loop quantum gravity, Immirzi parameter, holographic principle

INTRODUCTION

The striking resemblance between the laws of black hole mechanics and those of thermodynamics [1, 2] led Bekenstein to conjecture that a black hole should possess entropy proportional to its horizon area measured in Planck units $G\hbar$ [3]. We will assume $c=1$ $k_B=1$ and throughout the paper. Investigating quantum fields in a Schwarzschild metric, Hawking discovered that a non-zero temperature

$$T=\frac{\hbar}{8\pi GM} \qquad (1)$$

should be attributed to the horizon. The requirement that the area-mass relation $A=16\pi(GM)^2$ leads to the first law $\delta M = T\delta S$ the formula for the black hole entropy could then be deduced as

$$S=\frac{A}{4G\hbar}. \qquad (2)$$

This confirmed Bekenstein's conjecture and fixed the proportionality constant at $1/4$,

't Hooft elevated the area law of entropy to the status of a general rule, termed the holographic principle, according to which all the three-dimensional information about a gravitational system in a spatial region is represented by its boundary with a finite microscopic degree of freedom proportional to the area of the boundary in units of the Planck area [4]. Evidence for the holographic principle came from the AdS/CFT correspondence [5].

The deep connection between gravity and thermodynamics was further unveiled by few authors. Jacobson derived Einstein's field equations from the laws of thermodynamics on the horizon [6]. Padhmanabhan demonstrated that combined with the holographic principle the field equations of general relativity reduce to the equation of state of a macroscopic thermodynamic system [7]. Furthermore, it was shown by Verlinde that Newton's gravity emerges as an entropic force in a holographic setup [8]. These observations hint towards reclassifying gravity as an emergent phenomenon—in much the same manner as the macroscopic thermodynamic variables, such as temperature and entropy, emerge from the dynamics of a large number of underlying microscopic degrees of freedom. If we extrapolate this, the spacetime must also consist of some microscopic degrees of freedom. One may call these microstates "atoms of spacetime", the correct dynamics of which will be governed by a much-awaited theory of quantum gravity.

A crucial relation that connects the microscopic degrees of freedom with macroscopic thermodynamic variables of an ensemble is the law of equipartition of energy. Remarkably, a holographic version of such a relation was shown to exist in any diffeomorphism invariant theory of gravity [9, 10]. For a general horizon at temperature T, as perceived by a local Rindler observer, the total equipartition energy contributed by all quanta of space on the boundary is equal to the active gravitating mass M enclosed by the boundary, i.e.,

$$M=\frac{1}{2}nT. \qquad (3)$$

Here $n=A/G\hbar$ is a natural number giving a (large but) a finite number of degrees of freedom on the horizon. Each degree of freedom contributes a patch of Planck area and energy $T/2$. The essential input in this derivation was the Davies-Unruh temperature [11, 12].

In a previous communication, we pointed out that Hawking's temperature (1) and the holographic form of the equipartition of energy (3) are the same equation provided the horizon area is measured in Planck units [13]. This indicates that Hawking's temperature agrees to the quantization of the horizon in discrete units of the Planck area. It turns out that the whole holographic setup is consistent only if the horizon is quantized in Planck's units. This observation has a direct impact on the horizon spectra predicted by loop quantum gravity (LQG) and other theories. In LQG, for instance, the spacing in the area spectrum happens to be a "constant multiple" of the Planck area [14-16]. This comes after fixing its inherent ambiguity in the geometric spectra, represented by the unknown Immirzi parameter [17]. The actual problem we deal with is that LQG sets the minimal area by appealing to the entropy formula (2)—a component of the holographic setup—which by itself favors the quantum of the area to be none other than the Planck unit. In addition, while one expects the quantum of the area to be consistent with all the holographic equations, the result predicted by LQG does not comply with the equipartition rule, or equivalently, Hawking's temperature. Thus, we face an apparent inconsistency. Another example in which such a situation arises is a rough theory of quantum gravity postulated by Bekenstein and Mukhanov in an attempt to quantize the black hole [18, 19]—here it will be referred to as BM theory in short. In this letter, we show that a simple rearrangement of the holographic relations yields quantization of the horizon in units of Planck area and that the quantum of the area from LQG (or the BM theory and the like) does not consistently obey the holographic equations.

In the following section, for relevance, we briefly review how the elemental area is fixed in LQG and the BM theory. After the following section, we show that the holographic equations altogether require the horizon to be quantized in exact Plank units. Here we also point out that the previously determined horizon spectra do not encompass Hawking's temperature which is at the root of the holographic setup. Finally, we conclude the paper.

The minimal area elements in quantum theories of gravity

Loop quantum gravity (LQG) is a canonical quantization of gravity that has produced results that geometrical quantities such as area and volume are quantized [20-22]. The state-space of LQG is spanned by those of spin networks. Spin networks are graphs with edges each of which carries with it a spin $j \in \mathbf{N}/2$ labeling irreducible representations of $SU(2)$ which serves as the gauge group of the theory. The area of a given region of space has a discrete spectrum in such a way that if a surface is punctured by a set of spin network edges $\{j_i\}$ the surface acquires an area

$$A(j_i) = 8\pi G\hbar\gamma \sum_i \sqrt{j_i(j_i+1)}. \qquad (4)$$

The theory, however, carries the burden of the undetermined free parameter γ, called the Immirzi parameter [17]. This curious parameter is absent in classical gravity but appears unavoidably in the quantized version. The physical significance of the parameter is obscured but its value is usually fixed by the requirement that the LQG computation produces the Bekenstein–Hawking entropy (2).

The area of the horizon, being an eigenvalue of the area operator, is considered to be a consequence of a large number of edges embedded in the boundary. The black hole entropy is then calculated according to the usual definition as the logarithm of the dimension of the boundary Hilbert space $\mathscr{H}_{\text{boundary}}$ [14-16],

$$S = \ln\left(\prod_i^N \dim\mathscr{H}_{j_i}\right). \qquad (5)$$

Here $\dim\mathscr{H}_j\,(2j+1)$ is for a puncture with spin j N and is the number of edges puncturing the horizon. The leading contribution to the entropy comes from that configuration in which the $j_{\min}$ edges are dominant [15]. Therefore, the maximum entropy can be written as

$$S = N_{j_{\min}} \ln(2j_{\min}+1). \qquad (6)$$

Here $N_{j_{\min}} = A/A_{j_{\min}}$ is the number of edges, all with spins $j_{\min}$, whereas $A_{j_{\min}}$ represents the minimal area element. It is assumed that $A(j_{\min,i}) \approx A$. The entropy (6), with $j_{\min} = 1/2$, is then equated to the Bekenstein-Hawking formula to extract the value of the Immirzi parameter $\ln 2/\pi\sqrt{3}$.

But, by considering the entropy formula along with the quasi-normal mode (QNM) spectrum of a Schwarzschild black hole [23-25], Dreyer came up with a different value, $\gamma = \ln 3/2\pi\sqrt{2}$ [26]. Concurrently, this approach also suggested that the dominant contribution should come from edges with $j_{\min} = 1$ and that the true gauge group of the theory should therefore be considered as $SO(3)$ rather than $SU(2)$.

Based on a combinatoric formulation of the black hole entropy it was claimed that contribution from all values j should be taken into account [27, 28]. These studies resulted in different values of γ. The controversy over the group structure of LQG and the counting of the microstates has provoked considerable attention. See, for instance, references [29-36].

To summarize, after fixing γ, the minimal area element or the lower bound on the increase in the horizon area is given by

$$A_{j_{\min}} = 4G\hbar \ln(2j_{\min}+1). \qquad (7)$$

Here $j_{\min}$ is 1/2 or 1 depending on whether $SU(2)$ or $SO(3)$ is chosen as the gauge. So, there is still a lack of agreement over the exact spacing in the area spectrum of the horizon.

A similar result for the spacing of the area spectrum also follows in the BM theory [18, 19]. In a quantum theory of black holes, one would be interested in finding the mass and areal spectra required to determine the quantum and statistical mechanical properties of black holes. In the BM theory, the authors assumed that the horizon of a neural and non-rotating Schwarzschild black hole is quantized in equal steps of a constant multiple of the Planck area,

$$A = n\alpha G\hbar. \qquad (8)$$

Here n is a natural number and α is an arbitrary scale factor. This will allow, semi-classically, a discrete mass spectrum and hence a discrete mass emission. Assuming that all the Planck scale degrees of freedom on the boundary are equally probable and that each one is k-fold degenerate, the entropy is written as

$$S = \frac{A}{\alpha G\hbar}\ln k \qquad (9)$$

The unknown α is fixed at $4\ln k$ by comparing (9) with (2). This gives the elemental area as

$$A_{\min} = 4G\hbar \ln k, \qquad (10)$$

which is similar in form to (7). Choosing $k=2$ makes consecutive energy levels differ in entropy of exactly one bit. However, considering the QNM spectrum of the black hole and the BM theory, Hod [24] suggested the area spacing of a quantum black hole to be $4G\hbar\ln 3$.

Horizon area quantization in the semi-classical theory

As opposed to equations (6) and (9), the Bekenstein-Hawking entropy (2) was not derived, in the first place, by counting degenerate microstates on the horizon. This is why it is free of the logarithm factor. The only inputs in its derivation were Hawking's temperature, the classical area-mass relation, and the first law of thermodynamics at the horizon. Hawking's temperature, which is fundamental to the entropy formula (2), by itself resulted from low energy quantum field theory on a black hole metric. But a simple rearrangement of these semi-classical equations on the boundary shows that the horizon is quantized in the standard Planck units $G\hbar$. This is in contrast to the result from LQG (or the BM theory) which uses the semi-classical formula (2) to fix the minimal area.

To derive area quantization in the semi-classical theory it is important to note that the Hawking temperature formula (1) can be written precisely as the equipartition law (3), or *vice versa*, provided the horizon area is measured in Planck units [13]. This implies that Hawking's temperature by itself requires the horizon to be quantized in equal Planck's steps. To derive the area spectrum explicitly, one combines (1) with (3) to get

$$16\pi\left(GM\right)^{2}=nG\hbar, \qquad (11)$$

Alternatively, from the equipartition formula (3), one can write

$$\delta M=\frac{\hbar}{8\pi GM}\delta\left(\frac{n}{4}\right). \qquad (12)$$

Here one notices that the correct first law of thermodynamics (with the correct entropy) follows only if the horizon area is measured in standard Planck units.

That the Planck area is the actual operational unit in the holographic setup can also be seen from the following argument. One can rewrite (12) or the area-mass relation $A=16\pi\left(GM\right)^{2}$ as

$$\delta M=\frac{1}{8\pi G^{2}M}\delta\left(\frac{A}{4}\right). \qquad (13)$$

It can be noted that one can multiply and divide (13) by $\beta G\hbar$, any free constant β; however, the correct first law $\delta M=T\delta S$ that follows is free of β. Therefore, any constant that multiplies the Planck unit cancels out in the holographic equations altogether.

For LQG (or the BM theory) to be physically viable, the minimal area should be fixed only once and has to be consistent with all the relevant equations. But we see that this is not the case. For instance, although the quantum of area (7) accounts for the correct entropy, it fails to reproduce Hawking's temperature through the equipartition law (3). If the LQG result is to be made consistent with (1), one has to choose γ such that the minimal area element on the boundary is the Planck area. This may be possible if a version of the equipartition rule is imposed on the $j_{\min}$ edges such that the Hawking temperature is obtained—of course, one may be tempted to think of the equipartition to hold for homogeneously distributed $j_{\min}$ edges forming a spherical boundary. But having γ adjusted in this way the entropy produced by LQG will not match the Bekenstein-Hawking formula as long as we stick to the usual definition of entropy. Thus, LQG cannot produce both the Hawking temperature and the black hole entropy consistently. The same reasoning also applies to the BM theory.

CONCLUSION

The semi-classical holographic formulation uncovers the horizon area to be quantized in exact Planck units. The impetus in this proof has been the holographic version of equipartition law (3) and the clue that this law represents Hawking's temperature. Comparison of the entropy from LQG (and the BM theory) with formula (2) leads to the quantum of the area that is inconsistent with the equipartition law (3), or equivalently, Hawking's temperature. The minimal area element should be adjusted once in a way as to be consistent with all the relevant laws.

Could it be that the free parameter γ of LQG is a free factor like β that should disappear in the low energy holographic relations though it may well be significant at the deep quantum gravity level? The parameter γ can be seen to cancel out in the actual definition of the number of edges on the surface, $N_{j_{\min}}=A(j_{\min,i})/A_{j_{\min}}$, before the coarse-graining approximation, $A(j_{\min,i})\approx A$. This approximation seems to be crucial to determining $N_{j_{\min}}$, yet it comes at the price of bringing the ambiguity γ back to the scene, which is then made to compensate for the constant and the logarithm factor by equating (6) to the known semi-classical formula (2). This implies that LQG cannot find the black hole entropy unless the answer is known from elsewhere. Besides this, it is not obvious how trivially one could make correspondence between LQG and a low energy semi-classical theory of the horizon.

The constant α in the BM theory was associated with the microscopic unit of the area since it was not realized at the time that the true quantum of the area according to the holographic equations is the exact Planck unit. In the same theory, the degeneracy of the microstates was assumed to account for the entropy directly, but this was done without realizing that the resulting minimal area is inconsistent with Hawking's temperature.

Astonishingly, many authors have studied the equipartition law in the holographic perspective (see, for instance, [8, 10, 37]), but no one has even suspected the law is exactly the Hawking temperature formula. The equipartition rule for the minimal spin edges on a spherically symmetric holographic boundary was also discussed by Smolin whilst dealing with the derivation of Newton's gravity as an entropic force in the LQG framework [38].

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