# FUZZY CONNECTEDNESS AND COMPACTNESS IN FUZZY TOPOLOGICAL SPACES ON FUZZY SPACE

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***ABSTRACT:*** *In this paper we introduce and characterized the notion of fuzzy connectedness and compactness in fuzzy topological spaces of fuzzy subspaces and obtain a relation between this new approach of defining fuzzy connectedness and compactness and previous approaches of fuzzy connectedness and compactness in fuzzy topological spaces.*

**Keyword** Fuzzy space, fuzzy topology, fuzzy function, connectedness, compactness.

## 1. INTRODUCTION

The study of fuzzy topological spaces was started with the introduction of the concept of fuzzy topology by Chang [1] in 1968, later Lowen [5] redefined it which is known as fully stratified fuzzy topology. The main problem in fuzzy mathematics is how to carry out ordinary concepts to the fuzzy case. The difficulty lies in how to pick out the rational generalization from the large number of available approaches. In his Remarkable paper Dib [3] remarked the absence of the fuzzy universal set and discussed some problems in the classical fuzzy approaches to define fuzzy groups. Its absence has strong effect on the introduced structure of fuzzy theory. A new approach to define and study fuzzy groups, fuzzy subgroups and fuzzy topology is given in [3, 4], which depends on the concept of fuzzy space which serves as the concept of the universal set in the ordinary set theory. This approach can be considered as a generalization and a new formulation of other classical approaches.

The study of fuzzy topological spaces is an interesting research topic of fuzzy sets. Therefor in this paper, we continue Dib's ork on fuzzy topology of fuzzy subspaces to define fuzzy connectedness and fuzzy compactness.

## 2. Preliminaries

In this section, we summarize the preliminary definitions and results required in the sequel.

Chang [1] defined fuzzy topology and fuzzy compact spaces in the following manner

A family $T$ of fuzzy sets in $X$ which satisfies the following conditions:

1. $0,1 \in T$ ,
2. If $A, B \in T$ , then $A \cap B \in T$ ,
3. If $A_i \in T$ for each $i \in I$ , then $\bigcup_I A_i \in T$ .

The ordered pair $(X, T)$ is a called a fuzzy topological space. Members of $T$ are called fuzzy open sets while fuzzy closed sets are defined as the complement of members of $T$ .

A family $\mathcal{A}$ of fuzzy sets is a cover of fuzzy a fuzzy set $B$ iff $B \subset \bigcup\{A : A \in \mathcal{A}\}$. It is an open cover iff each member of $\mathcal{A}$ is an open fuzzy set. A subcover of $\mathcal{A}$ is a subfamily of $\mathcal{A}$ which is also a cover. A fuzzy topological space $(X, T)$ is compact iff each open cover has a finite subcover. Fuzzy topology and fuzzy compact topological space introduced by Chang will be denoted by C-fuzzy topology and C-compact fuzzy topological space respectively.

In [6] and [7] Zheng introduced the notion of fuzzy connected space by starting with a fuzzy set $D$ in a fuzzy topological space (in the sense of Cheng) $(X, \tau)$ and calling the fuzzy set $D$ a disconnected fuzzy set if there are a non-empty fuzzy sets $A$ and $B$ in the subspace $(D_0, \tau_0)$ such that $A$ and $B$ are separated and $A \cup B = D$ . Therefore a fuzzy set $D$ is said to be connected in the fuzzy topological space $(X, \tau)$ if it is not disconnected. A connected fuzzy topological space defined by Zheng will be denoted by Z-connected space.

The concept of fuzzy space $(X, I)$ was introduced and discussed by Dib [3], where $(X, I)$ is the set of all ordered pairs $(x, I); x \in X$ ; i.e. $(X, I) = \{(x, I); x \in X\}$, where $(x, I) = \{(x, r); r \in I\}$. The ordered pair $(x, I)$ is called a fuzzy element in the fuzzy space $(X, I)$ .

A fuzzy subspace $U$ of the fuzzy space $(X, I)$ is the collection of all ordered pairs $(x, u_x)$, where $x \in U_\circ$ for some $U_\circ \in X$ and $u_x$ is a subset of $I$ , which contains at least one element beside the zero element. If it happens that $x \notin U_\circ$ then $u_x = 0$ . An empty fuzzy subspace is defined as $\{(x, \phi_x); x \in \phi\}$.

Let $U = \{(x, u_x); x \in U_\circ\}$ and $V = \{(x, v_x); x \in V_\circ\}$ be fuzzy subspaces of $(X, I)$ . The union, intersection and difference between $U$ and $V$ are defined respectively as follows:

$U \cup V = \{(x, u_x \cup v_x); x \in U_\circ \cup V_\circ\}$,

$U \cap V = \{(x, u_x \cap v_x); x \in U_\circ \cap V_\circ\}$,

$U - V = \{(x, h_x); x \in U_\circ\}$, where $h_x = (u_x - v_x) \cup \{0\}$.

Clearly all of $U \cup V$ , $U \cap V$ and $U - V$ are fuzzy subspaces of the fuzzy space $(X, I)$ .

Let $A$ be a fuzzy subset of $X$ , then two main fuzzy subspaces induced by $A$ are:

$\underline{H}[A) = \{(x, [0, A(x)) : A(x) \neq 0\}$ and

$\overline{H}[A]=\{(x,\{0\}\cup[A(x),1]):x\in X\}$.

A family $\tau$ of fuzzy subspaces of the fuzzy space $(X,I)$ is called a fuzzy topology on the fuzzy space $(X,I)$, if $\tau$ satisfies the following conditions:

1. $(X,I)\in\tau$ and $\phi\in\tau$,
2. $U\cap V\in\tau$ for all $U,V\in\tau$,
3. $\bigcup_{U\in\tau_1}U\in\tau$ for every $\tau_1\in\tau$.

The ordered pair $((X,I),\tau)$ is called a fuzzy topological space. The elements of $\tau$ are called fuzzy open subspaces and $(X,I)-U$ is a fuzzy closed subspaces provided that $U$ is open fuzzy subspace.

If $((X,I),\tau)$ is a fuzzy topological space and $U$ is a fuzzy subspace of $(X,I)$ then the class $\tau_U$ consists of all intersections of $U$ with open fuzzy spaces of $((X,I),\tau)$ is a fuzzy topology on $U$. Such a class will be called the relative fuzzy topology on $U$ and $(U,\tau_U)$ will be called a fuzzy subspace of the fuzzy topological space $((X,I),\tau)$.

Let $\tau$ be a fuzzy topology in $(X,I)$, then:

1. For every $x_\circ\in X$, $\tau$ induces an ordinary topology $\tau_I(x_\circ)$ on $I$, which is defined by: $\tau_I(x_\circ)=\{u_{x_\circ}:(x_\circ,u_{x_\circ})\in U\in\tau\}\cup\phi$.
2. $\tau$ induces the family $\tau_X$ of subsets of $X$ defined by: $\tau_X=\{U_\circ:U\in\tau\}$, where $U_\circ$ is the support of $U$. $\tau_X$ is a topology on $X$ iff $\tau_X$ is closed under finite intersections.

To each fuzzy topology $\tau$ on $(X,I)$ there is associated an ordinary topology $\tau_I$ on $I$, where $\tau_I=\bigcap_{x\in X}\tau_I(x)$.

Let $A$ be a fuzzy subset of $X$, then every family of fuzzy subsets $\sigma$ of $X$ defines a family $\tau(\sigma)$ of fuzzy subspaces of $(X,I)$ as follows:

$\tau(\sigma)=\{(X,I)\}\cup\{\underline{H}[A]:A\in\sigma\}$,

$\tau(\sigma)$ is called a family of fuzzy subspaces induced by $\sigma$.

## 3. connectedness and compactness in fuzzy topological spaces of fuzzy spaces

In classical topological spaces the concepts of connectedness and compactness are two main principal topological properties that used to distinguish topological spaces. In this section we introduce the notions of connected and compact fuzzy topological spaces based on fuzzy space.

**Definition 3.1** *Two fuzzy subspaces* $U=\{(x,u_x):x\in U_\circ\}$ *and* $V=\{(x,v_x):x\in V_\circ\}$ *of the fuzzy space* $(X,I)$ *are called disjoint fuzzy subspaces iff*

$U_\circ\cap V_\circ=\phi$ and $u_x\cap v_x=\{0\}$.

**Definition 3.2** *A non-empty fuzzy topological space* $((X,I),\tau)$ *is said to be a disconnected fuzzy topological space if* $(X,I)$ *is the union of two non-empty disjoint open fuzzy subspaces. That is,* $(X,I)=U\cup V$ *such that* $U=\{(x,u_x):x\in U_\circ\}$ *and* $V=\{(x,v_x):x\in V_\circ\}$ *are non-empty open fuzzy subspaces of* $((X,I),\tau)$. *A fuzzy topological space* $((X,I),\tau)$ *is connected if it is not disconnected.*

Note that $(X,I)=U\cup V$ with $U=\{(x,u_x):x\in U_\circ\}$ and $V=\{(x,v_x):x\in V_\circ\}$ iff $X=U_\circ\cup V_\circ$ and $u_x\cup v_x=I$.

The next proposition re-formulate the definition of connectedness in fuzzy topological spaces.

**Theorem 3.3** *A fuzzy topological space* $((X,I),\tau)$ *is connected if and only if the only fuzzy subspaces of* $(X,I)$ *that are both open and closed in* $((X,I),\tau)$ *are* $(X,I)$ *and* $\phi$.

*proof.* If we assume the non-empty fuzzy subspace $U$ of $(X,I)$ in which its both open and closed in $(X,I)$, then the fuzzy subspaces $U$ and $V=(X,I)-U$ are open disjoint with $U\cup V=(X,I)$. That is $(X,I)$ is disconnected. On the other hand if $U\cup V=(X,I)$ where both $U,V$ are non-empty open disjoint fuzzy subspaces, then the complement of $U$ is $(X,I)-U=V$ which is a non-empty open fuzzy subspace hence $U$ is both open and close in $(X,I)$.

**Definition 3.4** *Let* $((X,I),\tau)$ *be a fuzzy topological space. A fuzzy subspace* $U$ *of* $(X,I)$ *is connected if* $(U,\tau_U)$ *is connected.*

**Example 3.5** *Consider the fuzzy space* $(X,I)$. *The trivial fuzzy topology* $\tau=\{(X,I),\phi\}$ *is a connected fuzzy topological space. While the discrete fuzzy topology* $\tau_d=\{U:U\subseteq(X,I)\}$ *is disconnected.*

**Theorem 3.6** *Let* $\tau$ *be a connected fuzzy topology on* $(X,I)$. *If the induced family* $\tau_X=\{U_\circ:U\in\tau\}$ *of subsets of* $X$ *is a topology on* $X$ *then* $\tau_X$ *is connected on* $X$.

**Theorem 3.7** *If* $\sigma$ *is a C-fuzzy topology on* $X$ *then* $\sigma$ *induces a connected fuzzy topology* $\tau(\sigma)$ *in which each open subspace is connected. That is*

1. $\tau(\sigma)$ is a connected fuzzy topology on $(X,I)$.
2. $\underline{H}[A]$ is connected in $\tau(\sigma)$ for all $A\in\sigma$.

*Proof.* (1) Recall that

$\tau(\sigma) = \{(X, I)\} \cup \{\underline{H}[A] : A \in \sigma\}$ defines a fuzzy topology by Theorem 6 [4]. Clearly $\tau(\sigma)$ is a connected fuzzy topological space since the open fuzzy subspaces have the form $\underline{H}[A] : A \in \sigma$ and $(X, I) \neq \underline{H}[A] \cup \underline{H}[B] = \underline{H}[A \vee B]$ for any $A, B \in \sigma$. (2) Clearly $\underline{H}[A]$ is connected, since there is no non-empty disjoint open fuzzy subspaces $\underline{H}[B], \underline{H}[C]; B, C \in \sigma$ such that $\underline{H}[A] = \underline{H}[B] \cup \underline{H}[C]$.

In a parallel approach to the concept of ordinary connected topological space which is defined in term of open covers we now introduce fuzzy compactness based on fuzzy space.

**Definition 3.8** *A collection* $\mathrm{C} = \{V : V \text{ isafuzzysubspaceof } (X, I)\}$ *is said to be a fuzzy covering of* $(X, I)$ *if the union of elements of* $\mathrm{C}$ *is equal to* $(X, I)$*, that is* $\bigcup_{V \in \mathrm{C}} V = (X, I)$

If the elements of $\mathrm{C}$ are open subspaces of $(X, I)$ then $\mathrm{C}$ is called an open fuzzy covering of $(X, I)$.

**Definition 3.9** *A fuzzy topological space* $((X, I), \tau)$ *is said to be a compact fuzzy topological space if every open fuzzy covering* $\mathrm{C}$ *of* $(X, I)$ *has a finite subcollection that covers* $(X, I)$.

**Definition 3.10** *Let* $((X, I), \tau)$ *be a fuzzy topological space. A fuzzy subspace* $U$ *of* $(X, I)$ *is compact if* $(U, \tau_U)$ *is compact.*

If $\tau$ is a fuzzy topology $(X, I)$ then $\tau$ induces the family $\tau_X$ of subsets of X defined by $\tau_X = \{U_0 : U \in \tau\}$. Moreover $\tau_X$ is a topology on $X$ iff $\tau_X$ is closed under finite intersections. The proof of this statement can be found in [4].

Based on the above statement we have the following theorem which conclude that fuzzy compactness defined earlier is a hereditary fuzzy topological property only in context of induced fuzzy spaces but not on any fuzzy subspace of $(X, I)$.

**Theorem 3.11** *Let* $\tau$ *be a compact fuzzy topology on* $(X, I)$*. If the induced family* $\tau_X = \{U_\circ : U \in \tau\}$ *of subsets of* $X$ *is a topology on* $X$ *then* $\tau_X$ *is compact on* $X$.

The proof is straightforward by noticing that every open cover for $\tau_X$ shall have a finite subcover since any open cover of $\tau_X$ must be a fuzzy subcover of any open cover of the compact fuzzy topological space $(X, I)$.

**Theorem 3.12** *If* $\sigma$ *is a compact C-fuzzy topology on* $X$ *and* $X \in \sigma$ *then*

1. $\tau(\sigma)$ is a compact fuzzy topology on $(X, I)$.
2. $\underline{H}[A]$ is compact in $\tau(\sigma)$ for all $A \in \sigma$.

*Proof*. (1) Is directly obtained using Theorem 3.11.

(2) Assume that $\mathrm{C}$ is any open cover for $\underline{H}[A] = \{(x, [0, A(x)) : A(x) \neq 0\}$ in $\tau(\sigma)$ and since $\mathrm{C}$ is a subset of every open cover of the fuzzy space $(X, I)$ and hence a subset of any open cover of the induced fuzzy space $\tau(\sigma)$ which is compact then $\mathrm{C}$ must have a finite subcover of $\underline{H}[A]$.

From the above discussions regarding fuzzy connectedness and compactness based on fuzzy space and with connection to fuzzy connectedness in the sense of Zheng and fuzzy compactness in the sense of Chang we get the following conclusions:

We conclude that the difference between a fuzzy connected topological space defined based on fuzzy spaces and the fuzzy Z-connected topological space $(X, \sigma)$ has appeared due to the absence of fuzzy universal set in the fuzzy Z-connected topology. For instance let $X$ be a non empty set and $\sigma = \{\phi, X\} \cup \{\mu_\alpha\}_{\alpha \in \Delta}$ where $\mu_\alpha$ is any fuzzy subset of $X$ satisfying $\frac{1}{2} < \mu_\alpha(x) < 1, \forall x \in X$ then $\sigma$ is a a C-fuzzy topology which is not a Z-connected fuzzy topological space while the induced fuzzy topology $\tau(sigma)$ is connected. This difference appeared due the absence of fuzzy universal set in C-fuzzy topological spaces which rely on an ordinary set $X$ to start with and then pick proper fuzzy subset of $X$ while in the case of connectedness in fuzzy spaces we start by a fuzzy space $(X . I)$ which will be the fuzzy universal set in which fuzzy subspaces are chosen from. A similar argument can be made regarding the notions of fuzzy compactness based on fuzzy space and C-compact fuzzy topology.

Applying the notion of fuzzy space and using it as universal set in fuzzifying ordinary topological concepts (such as connectedness, compactness and convergence) corrects the deviation occur while introducing these concepts without a fuzzy universal set. Also making fuzzy space act as a universal set allow fuzzy topological concepts to be presented and defined in a similar and parallel way to ordinary classical concepts.

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