OPTIMAL CONTROL APPLIED TO EPIDEMIC MODEL WITH DIFFERENT OBJECTIVE FUNCTIONS

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ABSTRACT: *In this paper, I have discussed epidemic SEIR model with different objective functions. I have used optimal control techniques to solve these models. Then I have analyzed and compared their results numerically and graphically. Our main objective purpose is to minimize the infectious individuals infected from epidemic diseases along with cost of vaccination. Recovery in infectious individuals maximize the recovered individuals of the population. This may help to increase in recovery rate and demolish epidemics from population.*

Key words: SEIR Model, Optimal Control Theory, Hamiltonian, Forward Backward Sweep Method.

1. INTRODUCTION:

Mathematical models are playing vast role in natural and social sciences. Now, many problems from different fields are analyzing and solving using mathematical models and optimal control techniques. In bio-sciences, mathematical modeling has brought evolutionary change. We can construct model of diseases and find out their solutions mathematically. Epidemics are one of the major causes of immunity destruction from the last few centuries. But still these diseases are present and becoming more infective. Epidemic diseases spread rapidly and affect badly on the larger scale in community. These are infectious diseases and have very fast attack rate. Polio, cholera, Smallpox, dengue, malaria and tuberculosis are rapidly spread epidemics in our society. These are frequently spread through person to person contact and contaminated food.

The study of epidemic diseases is called epidemiology. Epidemiology is important to analyze, how disease spread and how it can be control. Modeling in this field has its roots in the early twentieth century. Deterministic and stochastic mathematical models have developed to study the spread and control of these diseases. Many mathematical models have been constructed for these diseases. SI, SIS, SIR, MSIR, SEIR etc are some epidemic models. There are many research articles regarding these models which help us for further research. Daniel Bernoulli, published the first epidemic model in 1760. Ronald Ross (1857-1932) has studied the malaria disease and developed his idea that this diseases cannot be eliminated completely by eradicating all mosquitoes. Kermack and Mckendrick presented generalized epidemic model in 1927. Many epidemic models and their theoretical studies are described by Becker (1978), Herbert (1988), Ogren (2000) and Zaman (2007). Neilan and Lenhart (2010) proposed some modification in SEIR model for optimal vaccination strategy [1,3,5,7].

In modern epidemiology, the first step is modeling the spread of disease theoretically and concluding it's certain conditions, then it will go extinct.

Optimal control theory is an important field of mathematics and its techniques are used in controlling infectious diseases. It is used to optimize the objective functions to obtained results.

In this I have discussed SEIR epidemic model and used optimal control technique to minimize cost of vaccination for an epidemic disease. I have considered a micro-parasitic infectious disease and its permanent immunity can be found through natural recovery. Numerical results of this problem have found using Runge-Kutta fourth order scheme and MATLAB.

2. SEIR MODEL FORMULATION:

A. SEIR Compartmental Model

There are four compartments in SEIR compartmental model. S,E,I,R represent susceptible, exposed, infected and recovered individuals compartments respectively. Every individual is born susceptible as immunity is not transmitted during birth. For an incubation time period, susceptible individual becomes exposed then micro-parasitic dominate and individual becomes infected. Recovered individuals are kept in recovered "R" compartment. The SEIR compartmental model is shown below:

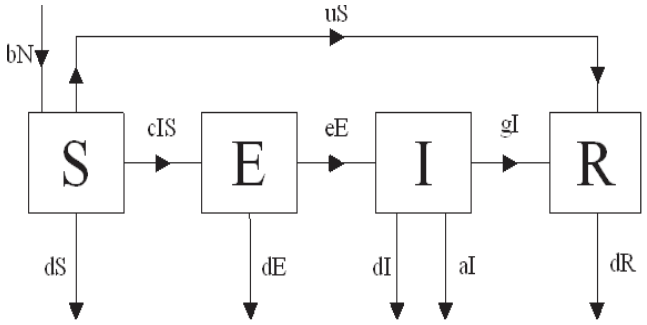

Fig. I SEIR Model

B. SEIR Mathematical Model

Let S(t), E(t), I(t) and R(t) represent number of susceptible, exposed, infected and recovered individuals at time t. These compartments are characterizing the different individuals of population. Let N(t) be the total number of individual of population. It can be represented as:

$$N(t) = S(t) + E(t) + I(t) + R(t) \qquad (1)$$

In this section, an ODE system of epidemic model is presented. As this model is characterizing the interaction between compartments. Therefore by applying the law of mass action, the ODEs of this model are:

$$S'(t) = bN(t) - dS(t) - cS(t)I(t) - u(t)S(t)$$
$$E'(t) = cS(t)I(t) - (e + d)E(t)$$

$$I'(t) = eE(t) - (g + a + d)I(t) \quad (2)$$
$$R'(t) = gI(t) - dR(i) + u(t)S(t)$$
$$N'(t) = (b - d)N(t) - aI(t)$$

Furthermore, it is assumed that all parameters are positive constraints.

$$S(0) = S_0 \geq 0, E(0) = E_0 \geq 0, I(0) = I_0 \geq 0, R(0) = R_0 \geq 0, N(0) = N_0 \quad (3)$$

Many and parameters are used in above model described in the given table

Table 1. Description of terms used

Terms	Description
b	Natural exponential birth rate
d	Natural exponential death rate
c	Disease incidence rate
e	rate at which exposed individual become infectious
g	rate at which infectious individuals recover
a	death rate due to disease in infectious individuals

3. THE OPTIMAL CONTROL PROBLEM:

A. Minimize Objective Function

In this section, the SEIR mathematical model (2) has solved using following objective function.

$$\min_u \int_0^T AI(t) + u(t)^2 \, dt \quad (4)$$

Where A ≥ 0 is a positive parameter and u(t) is the control of the function. It is percentage of susceptible that are vaccinated per unit time. The Hamiltonian [5] of this objective function using ODE system (2) is:

$$H = AI + u^2 + \varphi_S(bN - dS - cSI - uS) + \varphi_E(cSI - (e + d)E) + \varphi_I(eE - (g + a + d)I) + \varphi_N\big((b - d)N - aI\big) \quad (5)$$

Let $\varphi_S, \varphi_E, \varphi_I . \varphi_N$ are the adjoint of S, E, I and N resp. Then associated adjoint and transversality equations from (5) are

$$\varphi_S' = \varphi_S(d + cI(t) + u) - \varphi_E cI(t); \quad \varphi_S(T) = 0$$

$$\varphi_E' = \varphi_E(e + d) - \varphi_I e \; ; \quad \varphi_E(T) = 0$$

$$\varphi_I' = -A + (\varphi_S - \varphi_E)cS(t) + \varphi_I(g + a + d)I(t) + \varphi_N a \quad \varphi_I(T) = 0$$

$$\varphi_N' = -\varphi_S b - \varphi_N(b - d) \; ; \quad \varphi_N(T) = 0$$

Optimality condition:

$$\left.\frac{\partial H}{\partial u}\right|_{u=u^*} = 2u - \varphi_S S = 0$$

$$u^* = \frac{(\varphi_S(t)).S(t)}{2} \quad (6)$$

$$0 \leq u(t) \leq 0.9$$

B. Maximize Objective Function

Now, SEIR model is solved using the following objective function.

$$\max_u \int_0^T AN(t) - u(t)^2 \, dt \quad (7)$$

Then the Hamiltonian using above objective function is;

$$H = AN - u^2 + \varphi_S(bN - dS - cSI - uS) + \varphi_E(cSI - (e + d)E) + \varphi_I(eE - (g + a + d)I) + \varphi_N\big((b - d)N - aI\big) \quad (8)$$

And its associated adjoint and transversality equations are:

$$\varphi_S' = \varphi_S(d + cI(t) + u) - \varphi_E cI(t); \quad \varphi_S(T) = 0$$

$$\varphi_E' = \varphi_E(e + d) - \varphi_I e \; ; \quad \varphi_E(T) = 0$$

$$\varphi_I' = (\varphi_S - \varphi_E)cS(t) + \varphi_I(g + a + d)I(t) + \varphi_N a \quad \varphi_I(T) = 0$$

$$\varphi_N' = -A - \varphi_S b - \varphi_N(b - d) \; ; \quad \varphi_N(T) = 0$$

Optimality condition is:

$$u^* = -\frac{(\varphi_S(t)).S(t)}{2} \quad (9)$$

$$0 \leq u(t) \leq 0.9$$

4. NUMERICAL METHOD:

There are many methods and techniques are available for the programing and solving above optimization problems. Here an iterative method with Runge-Kutta fourth order scheme is used which is also known as Forward Backward Sweep Method (FBSM).

This method is applicable as, first make an initial value guess then solve state system forward in time and adjoint equation backward in time.

The numerical results of these problems have shown using MATLAB.

5. Numerical Simulations of (3.A):

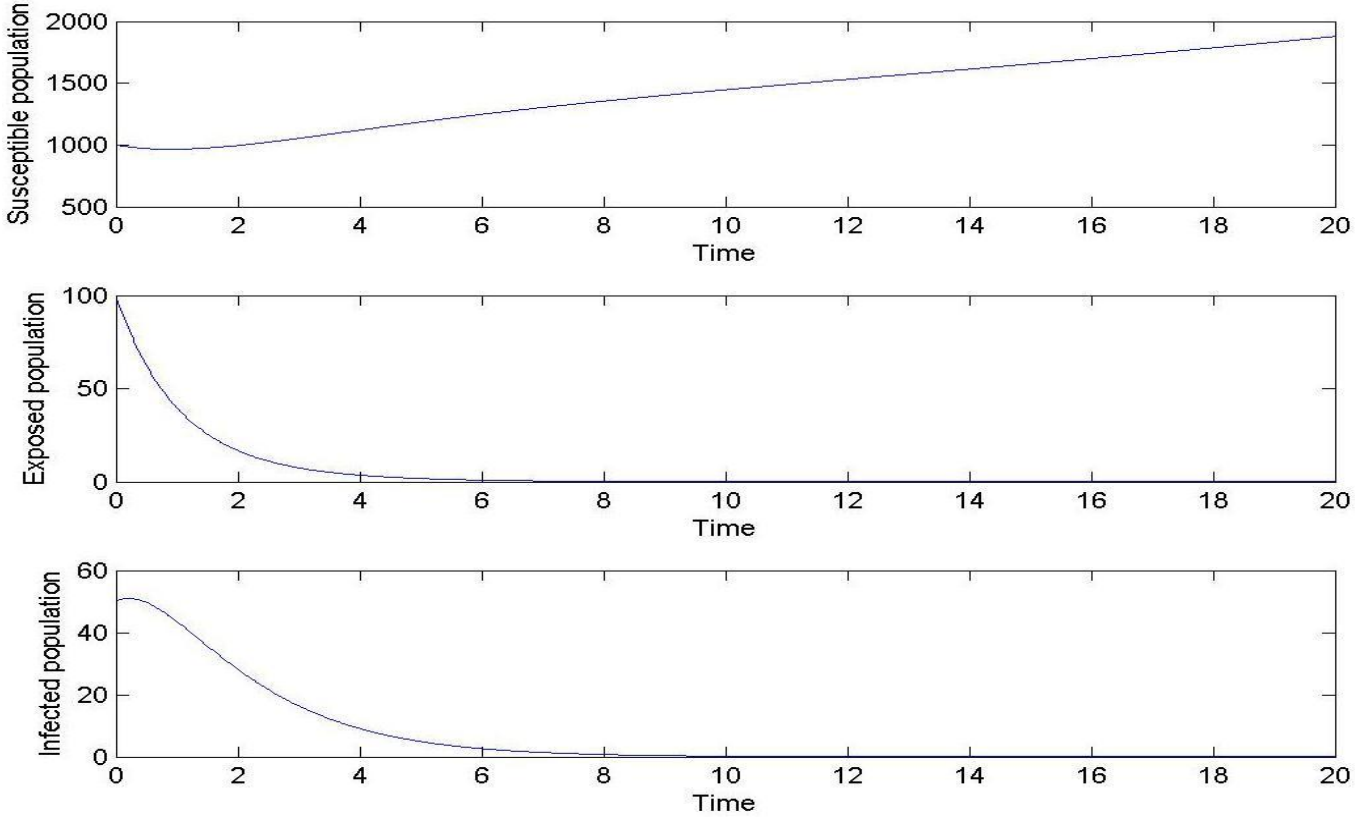

Fig. 1

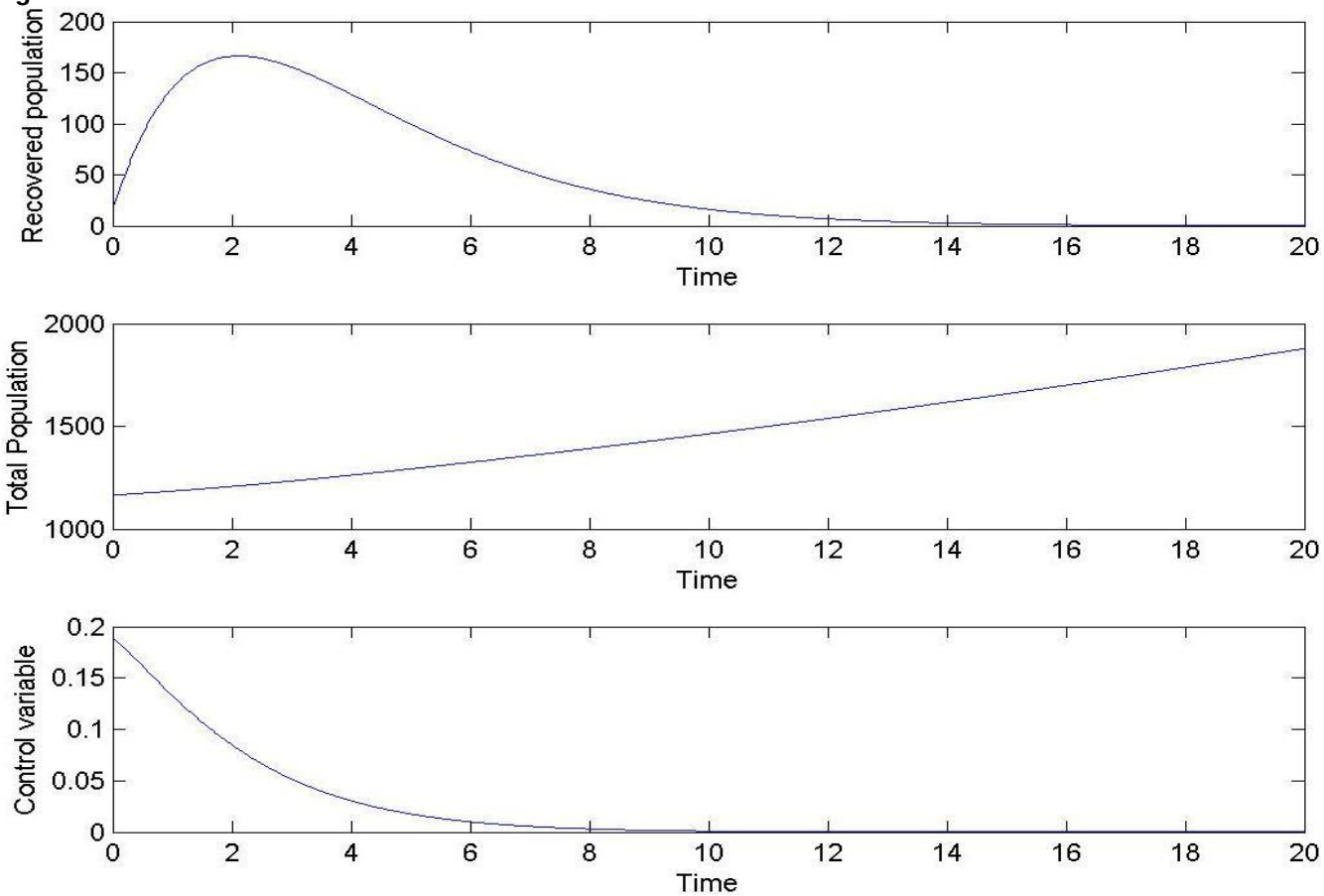

Fig. II Simulation Results 3.A

6. Numerical Simulations of (3.B):

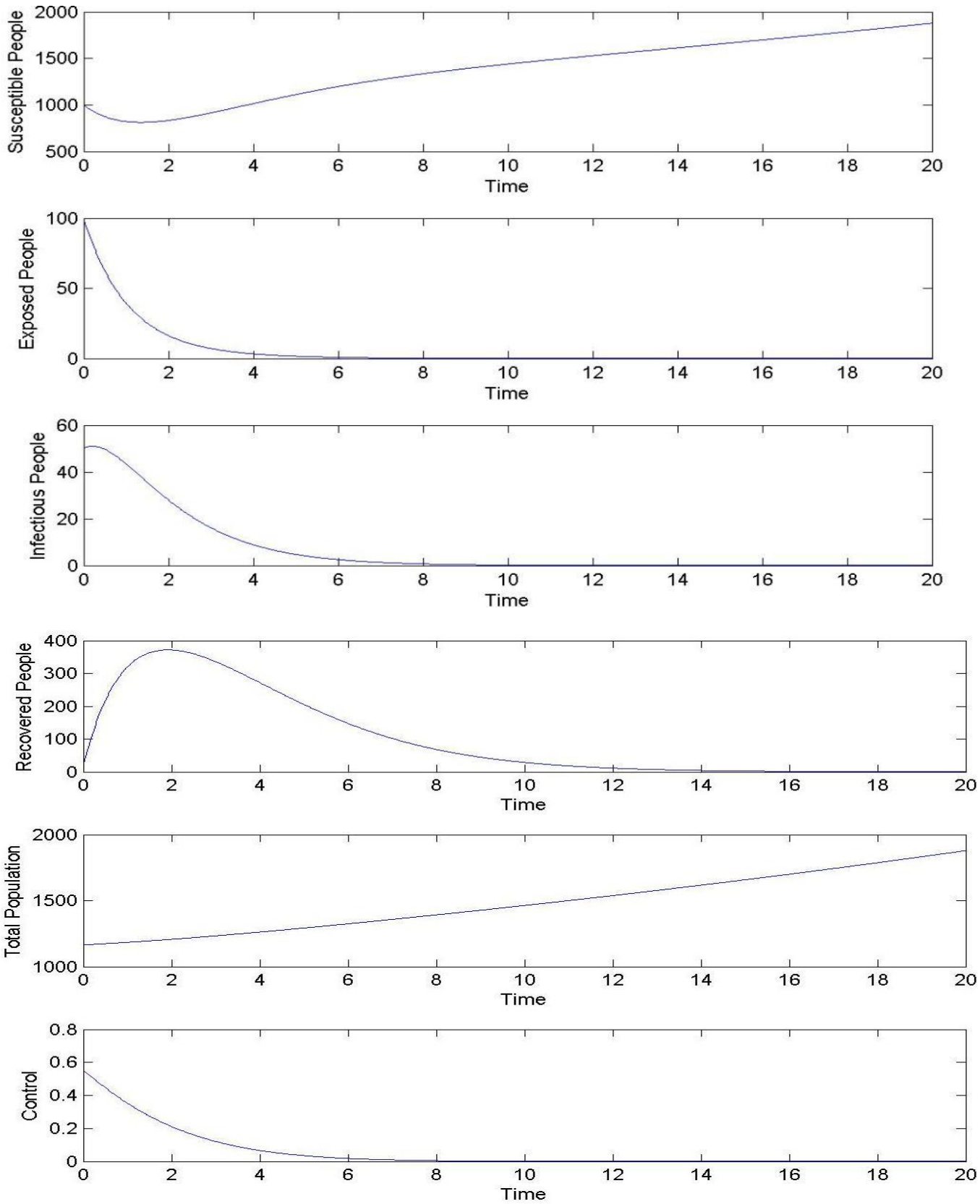

Fig. III Simulation Results 3.B

CONCLUSION:

In this, simulation of disease with a low incidence measure has been done. The optimal vaccination schedule is one of containment. An early round of vaccinations is used to shield the susceptible population from initially significant exposed and infectious populations. This, combined with the low incidence level, results in the virtual end of the disease spread. Exposed and infectious populations quickly disappear (through and recovery). The recovered people increases rapidly at first due to vaccinations, but slowly disappears when vaccination ends. By the end of the time period, susceptible people make up almost the entire population.

In 3.A, we need to minimize the infectious people and in 3.B we need to maximize the total population. In both the cases the incidence level is very low, and susceptible people take the entire population. In 3.A simulation, the susceptible population decreases slightly in the beginning time interval and at that interval vaccination rate is greater than overall growth. But in second case, initially there is more decrease in susceptible people as compared to the first case. Exposed and infectious populations quickly disappear (through and recovery). Almost in 6-8 years the disease is essentially wiped out and vaccination ends.

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