**A NOTE ON BI-PERMUTABLE AG-GROUPOIDS**

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***Abstract*** *The idea of J. Jezek for left (resp. right) permutable groupoids satisfying the identity (res.  is considered to introduce new subclasses of AG-groupoids as left, right and bi-permutable AG-groupoids. A method for testing an arbitrary Caley’s table for these AG-groupoids is produced, and a table of enumeration for these AG-groupoids up to order 6 is presented. Various relations of these AG-groupoids with some known subclasses of AG-groupoids are found and some basic and general properties of left, right and bi-permutable-AG-groupoids are investigated. Furthermore, bi-permutable-AG-groupoids are characterized by the property of their ideals.*

**Keywords:** AG-groupoid; bi-permutable; nuclear square AG-groupoids; T1-AG-groupoid; right alternative; self-dual AG-groupoids.

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**1. INTRODUCTION**

A groupoid  is a “left (resp. right) permutable groupoid if it satisﬁes the identity (res.  .  is called bi-permutable if it is both a left and a right permutable groupoid” [1, 2]. A groupoid S is called an AG-groupoid if it satisﬁes the left invertive law; .  is called “medial if it satisﬁes the medial law, , it is easy to prove that every AG-groupoid is medial”.  is called an AG-monoid if it contains the left identity. An AG-monoid always satisﬁes the paramedial law, .  is called AG∗ -groupoid if it satisﬁes the identity or . Recently, many new classes of AG-groupoids have been explored [3-10] that are presented with their identities in Table 2. It can be easily seen that extensive and rapidly grown research is in progress on these classes [11-14]. AG-groupoids have a variety of applications in ﬂock theory, geometry, ﬁnite mathematics and fuzzy algebra [14- 19]. Here, in this article, the idea of bi-permutable groupoids is extended to produce some other subclasses of AG-groupoids as “left permutable AG-groupoids (or simply LP-AG- groupoids), right permutable AG-groupoids (RP-AG-groupoids) and bi-permutable AG-groupoids (BP-AG-groupoids)”. It is worth mentioning that the LP-AG-groupoids exist in the literature by the name of AG\*\*-groupoids [20, 21]. Here, we call them LP-AG-groupoids to associate them to their relative class of BP-AG-groupoids. We explore various relations of these newly introduced classes of AG-groupoid with some already known subclasses of AG-groupoids.

The enumerations of various newly discovered subclasses have been made up to order 6 in a variety of papers like [9, 19, 22]. We present enumerations of non-associative LP-AG-groupoids, RP-AG-groupoids and BP-AG-groupoids up to order 6 using GAP in Table. 1

**Table 1. Enumeration of non-associative Bi-permutable AG-groupoids up to order 6.**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **AG-groupoids\Order** | **3** | **4** | **5** | **6** |
| AG-groupoid (Total) | 8 | 269 | 31467 | 40097003 |
| Left Permutable/ (AG\*\*) | 4 | 39 | 526 | 13497 |
| Right Permutable | 6 | 194 | 22276 | 34845724 |
| Bi-permutable | 2 | 23 | 306 | 6500 |

The following table contains some of the already known subclasses of AG-groupoids with their identities that will be used in the rest of this article.

**Table 2. Various AG-groupoids with their identities**

|  |  |  |  |
| --- | --- | --- | --- |
| **AG-groupoid** | **Satisfying identity** | **AG-groupoid** | **Satisfying identity** |
| Left nuclear square |  | Right nuclear square |  |
| Middle nuclear square |  | T1-AG-groupoid |  |
| T2-AG-groupoid |  | T4f-AG-groupoid |  |
| T4b-AG-groupoid |  | T4-AG-groupoid | Both T4f and T4b |
| Flexible |  | Bol\* -AG-groupoid |  |
| AG-3-band |  | Right alternative |  |
| Self-dual |  | Weak commutative |  |
| Paramedial |  | Medial |  |

**RESULTS AND DISCUSSIONS**

Next we present our results and discuss various properties of BP-AG-groupoids.

**2. PERMUTABLE AG-GROUPOIDS**

In this section, left, right and bi-permutable AG-groupoid are deﬁned as new subclasses of AG-groupoids and non-associative examples of lowest order generated by GAP are presented.

**Deﬁnition 1.** An AG-groupoid *S* is called –

(1) – a left permutable AG-groupoid (or shortly, LP-AG-groupoid) if ,

 (2.1)

(2) – a right permutable AG-groupoid (or shortly, RP-AG-groupoid) if,

 (2.2)

(3) – a bi-permutable AG-groupoid (or shortly, BP-AG-groupoid) if it is both an LP-AG-groupoid and an RP-AG-groupoid.

**Example 1.**  We present some non-associative examples of smallest order, to support the existence of these AG-groupoids. (i). LP-AG-groupoid, (ii). RP-AG-groupoid, (iii). BP-AG-groupoid.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | 1 2 3 |  |  | 1 2 3 |  |  | 1 2 3 |
| 1 | 1 1 1 | 1 | 1 1 1 | 1 | 1 1 1 |
| 2 | 1 1 1 | 2 | 1 1 1 | 2 | 1 1 1 |
| 3 | 1 2 1 | 3 | 2 1 2 | 3 | 1 2 2 |
| (i) | | (ii) | | (iii) | |

**3. PERMUTABLE AG-TEST**

It remained always time consuming and a tough job to verify a table of groupoid for a speciﬁc algebraic property. In this section, we discuss various methods to verify any ﬁnite arbitrary Caley’s table of our newly introduced AG-groupoids.

**3.1 Right Permutable AG-Test**

For a ﬁnite groupoid  Abel-Grassmann’s test has been described in [20] by introducing new operations for some , as follows;

 (3.1)

 (3.2)

 is an AG-groupoid if  and  coincide for all . In similar lines, we present a test to check whether an AG-groupoid  is RP-AG-groupoid or not, for this we deﬁne the following binary operations for some ﬁxed  and .

 (3.3)

The identity (2.2) holds if,

 (3.4)

To construct the table for operation  for some ﬁxed , we use the procedure of [20]. While the table of the operation  is constructed by multiplying a ﬁxed element  by the elements of the  table from the left. If the tables for  and  coincides for all , then the identity (3.4) holds. In the extended table of Example 2 given below, the tables on the right of the  table are constructed for operation  and the downwards tables are constructed for the operation .

**Example 2.** Using the above test and check the following AG-groupoid for RP- AG-groupoid.

|  |  |
| --- | --- |
|  | 1 2 3 |
| 1  2  3 | 1 1 1  1 1 1  2 1 2 |

We extend the given table in the way as prescribed above, to get the following table.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | 1 2 3 | **1** | **2** | **3** |
| 1  2  3 | 1 1 1  1 1 1  2 1 2 | 1 1 1  1 1 1  1 1 1 | 1 1 1  1 1 1  1 1 1 | 1 1 1  1 1 1  1 1 1 |
| **1**  **1**  **2** | 1 1 1  1 1 1  1 1 1 |  | | |
| **1**  **1**  **1** | 1 1 1  1 1 1  1 1 1 |
| **1**  **1**  **2** | 1 1 1  1 1 1  1 1 1 |

It is clear from the extended table that the downward tables and the tables on the right coincide for all . So *G* is an RP-AG-groupoid.

**3.2 Left Permutable AG-test**

The procedure for LP-AG-Test (or AG\*\*-Test) is given in [20] and is shortly described as follows;

 (3.5)

 (3.6)

for some .  is an LP-AG (AG\*\*)-groupoid if  and  coincide for all .

N.B. In the following example, we construct tables on the right to the  table for the operation  and the downward tables are constructed for the operation .

**Example 3.** Check the following AG-groupoid for LP-AG-groupoid.

|  |  |
| --- | --- |
|  | 1 2 3 |
| 1  2  3 | 1 1 1  1 1 1  1 2 1 |

Extend the given table in the way as prescribed above to get the following;

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | 1 2 3 | **1 1 1** | **1 1 1** | **1 2 1** |
| 1  2  3 | 1 1 1  1 1 1  2 1 2 | 1 1 1  1 1 1  1 1 1 | 1 1 1  1 1 1  1 1 1 | 1 1 1  1 1 1  1 1 1 |
|  | | 1 1 1  1 1 1  1 1 1 | 1 1 1  1 1 1  1 1 1 | 1 1 1  1 1 1  1 1 1 |

It is clear from the extended table that  and coincide for every , so  is an LP-AG-groupoid.

**3.3 Bi-Permutable AG-test**

An AG-groupoid  is BP-AG-groupoid if it satisﬁes the identities (3.1, 3.2) and (3.5, 3.6).

**Example 4.** Using the above tests to check the following AG-groupoid  for BP-AG-groupoid.

|  |  |
| --- | --- |
|  | **1 2 3** |
| 1  2  3 | 1 1 1  1 1 1  1 2 2 |

Extend the table as described above, we get the following;

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | **1 2 3** | **1** | **2** | **3** |
| 1  2  3 | 1 1 1  1 1 1  1 2 2 | 1 1 1  1 1 1  1 1 1 | 1 1 1  1 1 1  1 1 1 | 1 1 1  1 1 1  1 1 1 |
| **1**  **1**  **1** | 1 1 1  1 1 1  1 1 1 |  | | |
| **1**  **1**  **2** | 1 1 1  1 1 1  1 1 1 |
| **1**  **1**  **2** | 1 1 1  1 1 1  1 1 1 |

It is clear from the extended table that “ ◦ ” and “ ∗ ” coincide for every , thus  is an RP-AG-groupoid. Now, we are testing the same table for LP-AG-groupoid as discussed in 3.2.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | 1 2 3 | 1 1 1 | 1 1 1 | 1 2 2 |
| 1  2  3 | 1 1 1  1 1 1  1 2 2 | 1 1 1  1 1 1  1 1 1 | 1 1 1  1 1 1  1 1 1 | 1 1 1  1 1 1  1 2 2 |
|  | | 1 1 1  1 1 1  1 1 1 | 1 1 1  1 1 1  1 1 1 | 1 1 1  1 1 1  1 2 2 |

It is obvious that the operations “ × ” and  coincide for every , so  is an LP-AG-groupoid, and hence is BP-AG-groupoid.

**4. CHARACTERIZATION OF BP-AG-GROUPOIDS**

In this section, we various relations of RP, LP and BP-AG-groupoids with already known subclasses of AG-groupoids are investigated. A number of counterexamples are produced by Mace-4 where the relations are not established.

**4.1 Characterization of LP-AG-groupoids**

***Theorem 1.***  *Every T4-AG-groupoid is an LP-AG-groupoid.*

**Proof.**  Let S be a T4-AG-groupoid, then by deﬁnition of T4-AG-groupoid and left invertive law,



Hence  is an LP-AG-groupoid.

We give a counter example to show that every T4-AG-groupoid is not always an RP-AG-groupoid, and hence is not BP-AG-groupoid.

**Example 5.** T4-AG-groupoid that is not an RP-AG-groupoid.

|  |  |
| --- | --- |
|  | 1 2 3 4 |
| 1  2  3  4 | 2 1 4 3  3 4 1 2  4 3 2 1  1 2 3 4 |

Clearly (1 ∗ 3) ∗ 4 (1 ∗ 4) ∗ 3.

***Lemma 1.*** *In every LP-AG-groupoid the identity*  *holds.*

***Proof.***  Let  be an LP-AG-groupoid, and then,



Hence the identity holds as required.

**4.2 Characterization of RP-AG-groupoids**

***Theorem 2.***  *The following are true for every RP-AG-groupoid* *,*

(i)  is paramedial AG-groupoid.

(ii)  is LC-AG-groupoid.

(iii)  is right Cheban AG-groupoid.

(iv)  is weak commutative AG-groupoid.

***Proof.*** Let  be an RP-AG-groupoid, then 

1. By left invertive law, and equation 2.2, we have

Hence is a paramedial AG-groupoid.

1. Now for LC-AG-groupoid, using deﬁnition of RP-AG-groupoid and left invertive law, we get



Hence  is LC-AG-groupoid.

1. To show that  is right Cheban AG-groupoid, we have

 Thus. Hence  is right Cheban AG-groupoid.

1. For weak commutative AG-groupoid, by using medial law, left invertive law and repeated use of RP-AG-groupoid, we get



Thus . Hence  is a weak commutative AG-groupoid. Hence the theorem is proved.

We provide a counterexample to show that every RP-AG-groupoid is neither always a left Cheban AG-groupoid, nor an RC-AG-groupoid and hence is not a Cheban or a BC-AG-groupoid.

**Example 6.** An RP-AG-groupoid of order 3 that is neither left Cheban nor an RC-AG-groupoid.

|  |  |
| --- | --- |
|  | 1 2 3 |
| 1  2  3 | 1 1 1  1 1 1  2 1 1 |

***Lemma 2.*** *RP-AG-groupoid with left identity is commutative semigroup.*

*Proof.* Let  be an RP-AG-groupoid, and and  be the left identity element of , then  Thus  is commutative and hence is associative, as commutativity implies associativity in AG-groupoids.

**Lemma 3.** In every RP-AG-groupoid , the identity holds.

**Proof.** Let  be an RP-AG-groupoid, and  then



Hence the required identity holds in .

**Theorem 3.** Every RP-AG-groupoid having either a right or left cancellative element is a commutative semigroup.

**Proof.** Let  be an RP-AG-groupoid, and  be a right cancellative element of S, then by left invertive law, right cancellativity and RP-AG-groupoid, we have



Again, let  be a left cancellative element, then by left invertive law, Lemma (3), left cancellativity and RP-AG-groupoid, we have

Hence,  is commutative semigroup as, a commutative AG-groupoids is always associative. Hence, the theorem is proved.

The following counter example shows that a cancellative LP-AG-groupoid is neither associative nor commutative always.

**Example 7.** Cancellative LP-AG-groupoid that is neither associative nor commutative.

|  |  |
| --- | --- |
|  | 1 2 3 |
| 1  2  3 | 1 2 3  3 1 2  2 3 1 |

In Example (1), Part (ii) is RP-AG-groupoid which is neither self-dual nor right nuclear square. In the following example, it is also shown that self-dual AG-groupoid is not a right nuclear square AG-groupoid. However, the self-dual AG-groupoid and RP-AG-groupoid gives us right nuclear square AG-groupoid as proved in the following theorem.

**Example 8.** self-dual AG-groupoid whicht is not a right nuclear square.

|  |  |
| --- | --- |
|  | 1 2 3 4 |
| 1  2  3  4 | 1 3 4 2  4 2 1 3  2 4 3 1  3 1 2 4 |

As (1 ∗ 2) · 32  1 ∗ (2 ∗ 32), thus the AG-groupoid is not right nuclear square.

***Theorem 4.***  *Self-dual RP-AG-groupoid is right nuclear square.*

**Proof.**  Let  be a self-dual RP-AG-groupoid, and  then,



**Example 9.** In the following counter examples, neither T1-AG-groupoid (see table (i)) nor RP-AG-groupoid (see table (ii)) is RC-AG-groupoid as in (i), 1 · (2 · 3)  1 · (3 · 2), and in table (ii) 3 · (1 · 3)  3 · (3 · 1), left Cheban AG-groupoid as in table (i), 1 · ((2 · 3) · 3)  (3 · 1) · (2 · 3), and in table (ii) 3 · ((3 · 3) · 3)  (3 · 3) · (3 · 3), right nuclear square AG-groupoid as in table (i), (1 · 2) · 32 1 · (2 · 32), and in table (ii) (3 · 2) · 32 3 · (2 · 32), and middle nuclear square AG-groupoid as in table (i) (2 · 12) · 3  2 · (12· 3) and in table (ii) (3 · 22) · 3  3 · (22· 3).

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | 1 2 3 |  |  | 1 2 3 |
| 1  2  3 | 1 2 3  3 1 2  2 3 1 | 1  2  3 | 1 1 1  1 1 1  2 1 2 |
|
|
| (i) | | (ii) | |

However, we have the following theorem.

***Theorem 5.*** *Let*  *be a T1-RP-AG-groupoid, then the following holds;*

(1)  is RC-AG-groupoid

(2)  is left Cheban AG-groupoid

(3)  is right nuclear square AG-groupoid

(4)  is middle nuclear square AG-groupoid

**Proof.** Let  be a T1-RP-AG-groupoid, and , then

1. By assumption and Theorem (2, Part (ii)), we have



*(2)* Again by assumption and Theorem (2, Part (ii)), we have

*(3)*  For right nuclear square AG-groupoid, by Theorem (2, Part (ii)) we have,



Hence is right nuclear square AG-groupoid.

(4) For middle nuclear square AG-groupoid, using Theorem (2, Part (ii)) and

the assumption we get,



Hence  is middle nuclear square AG- groupoid.

(5) Now for self-dual AG-groupoid, we have

 Hence the result is proved.

**4.3 Characterization of BP-AG-groupoids**

***Theorem 6.*** *Every BP-AG-groupoid is Bol\*.*

**Proof.** Let  be a BP-AG-groupoid, and let  Then



Hence  is Bol\* -AG-groupoid.

It can be easily proved from Theorem 6 that:

***Proposition 1.*** *Every BP-AG-groupoid S is left nuclear square.*

***Lemma 4.***  *Every BP-AG-groupoid is middle nuclear square AG-groupoid.*

**Proof.** Let  be a BP-AG-groupoid, and  Then



Hence  is middle nuclear square AG-groupoid.

***Theorem 7.*** *Every BP-AG-3-band is commutative semigroup.*

**Proof.** Let  be a bi-permutable AG-3-band, and  Then



Thus,  is commutative and hence associative. Therefore,  is a commutative semigroup.

***Theorem 8.*** *Every BP-AG-groupoid S is ﬂexible if and only if*  *is right alternative AG-groupoid.*

Proof. Let  be a BP-AG-groupoid, and  Assume ﬁrst that  is ﬂexible. Then Thus,  is right alternative AG-groupoid.

Conversely, assume that  is right alternative. Then



Hence  is ﬂexible and the result is proved.

**Example 10.** We list the following as counterexamples (i). BP-AG-groupoid of order 3 that is not right nuclear square AG-groupoid. (ii). ﬂexible AG-groupoid of order 4 that is not right nuclear square AG-groupoid.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| \* | 1 2 3 4 |  | \* | 1 2 3 4 |
| 1 2 3 4 | 1 1 1 1  1 1 1 1  1 1 1 1  1 1 3 3 | 1 2 3 4 | 1 1 1 1  1 1 1 3  2 1 2 2  1 2 3 4 |
| (i) | | (ii) | |

However, we have the following;

***Theorem 9.***  *Every ﬂexible BP-AG-groupoid is nuclear square.*

**Proof.**  Let  be a ﬂexible BP-AG-groupoid and  Then



Thus  is right nuclear square AG-groupoid and by Lemma (4) and Proposition (1) the result holds.

***Theorem 10.*** *A self-dual bi-permutable AG-groupoid is nuclear square AG-groupoid.*

**Proof.** Let  be a self-dual bi-permutable AG-groupoid, and  Then



Thus,  is right nuclear square AG-groupoid and hence by Lemma (4) and Proposition (1), S is nuclear square AG-groupoid.

**5. IDEALS IN BP-AG-GROUPOIDS**

In this section, the ideals in LP-AG-groupoids and RP-AG-groupoids are discussed.  subset  of an AG-groupoid  is said to be a left (resp. right) ideal of  if (resp. ). We proceed with the following remarks:

**Remark 1.** If S is an RP-AG-groupoid and a ∈ S, then by left invertive law, we have



For any  we conclude that 

**Remark 2.** If  is an LP-AG-groupoid and , then by Identity (2.2), it follows that:



Hence, in general  for any 

**Remark 3.** If  is an RP-AG-groupoid and , then by Identity (2.3), we have



Thus ** for any **

***Theorem 11.*** *Let* *be an RP-AG-groupoid then for each*  *the set*  *is a right ideal of* 

**Proof.** Let , then by *Remark 1, 3,* we have



Hence the result follows.

***Theorem 12.*** *Let*  *be an RP-AG-groupoid then for each*  *if*  *has the left identity, then the set*  *is a left ideal of* *containing* *.*

**Proof.** Let , then by use of left invertive law, Remark (2) and left identity, we have



Hence,  is a left ideal of .

**6. CONCLUSIONS**

The concept of right (left) permutable groupoid (by J. Jezek 1984) satisfying the identity  is extended to right (left) permutable AG-groupoid. Various results have been produced, such that T4-AG-groupoid is an LP-AG-groupoids. RP-AG-groupoid is paramedial, left commutative, right Cheban, weak commutative AG-goupoid. RP-AG-groupoid is a commutative semigroup if and only if it contains a left identity element and cancellative element. T1-RP-AG-groupoid is right commutative, left Cheban, right nuclear square, middle nuclear square AG-groupoid. BP-AG-groupoid is Bol\*, middle nuclear square AG-groupoid. It has also been proved that BP-AG-groupoid is flexible if and only if it is the right alternative AG-groupoid. A simple verification test is included in this paper for testing any arbitrary AG-groupoid as BP-AG-groupoid. Many examples and counterexamples have been produced using the modern computational techeniques of Mace4 to support the results included in the paper. All the introduced classes are also characterized by the properties of their ideals.

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