**FIXED POINT THEOREMS FOR INTEGRAL TYPE CONTRACTION ON**

**SPACES WITH TWO METRICS**

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***ABSTRACT****. In this paper, we extend the notion of generalized --contractive mapping of integral type on spaces with two metrics. Our results generalize many existing results of the literature. An example is also constructed to support our results.*

***Keywords***: fixed point theorems; complete metric space; contractive mapping; gauge functions

1. **INTRODUCTION AND PRELIMINARIES**

The most important and well-known metric fixed point theorem is the Banach fixed point theorem [1], also known as contraction mapping principle, which guarantees the existence and uniqueness of fixed point for a contraction mapping from a complete metric space to itself. A number of authors extended and generalized Banach contraction principle in many different ways, see for example [2-16] and references therein. Samet *et al*. [10] introduced the concept of --contractive mapping and obtained some fixed point results for such mapping on complete metric space. Branciari [7] generalized Banach contraction principle by introducing contraction condition of integral type. Karapinar *et al*. [11] introduced the notion of generalized --contractive mappings of integral types and proved some fixed point theorems for these contractions on complete metric spaces.

In this manuscript, we extend notion of generalized --contractive mappings of integral types on spaces with two metrics. Our results generalize some fixed point results on spaces with two metrics for example the fixed point results of Agarwal and O'Regan [4] and Kiran and Kamran [5] can be easily deduced from our fixed point theorems in main results section as well as some fixed point results associated with --contractive type mappings can also be obtained. For completeness, we recollect the following definitions and results:

**Definition 1.1** Let be a family of functions such that is non-decreasing and

, for each

**Definition 1.2** Define = { such that is nonnegative, summable (with finite integral), Lebesgue integrable, and satisfies

0, > 0

**Definition 1.3** [10] Let be a complete metric space and be a mapping. We say that is -contractive mapping if there exist two functions and such that

**Definition 1.4** [10] Let and be two mappings. We say that is -admissible, if Karapinar et al. [11] introduced two new classes of generalized --contractive mappings of integral types in the following way:

**Definition 1.5 [11]** A mapping is said to be generalized --contractive of integral type I, if

(1)

Where

and .

**Definition 1.6 [11]** A mapping is said to be generalized --contractive of integral type II, if

(2)

Where

and .

**Definition 1.7 [12]** Let , we say that is -transitive on , if

In particular, we say that is transitive if it is 1-transitive, *i.e.*

and

Karapinar *et al.* [11] gave the following result:

**Theorem 1.1** [11] Let be a complete metric space and be a transitive mapping. Suppose that is generalized --contractive mapping of integral type I or type II and satisfies the following conditions:

is -admissible; There exists such that ; is continuous.Then has a fixed point.

2. Main Results

Now we state and prove our first main result.

**Theorem 2.1** Let be a complete metric space and be another metric on . Let and be such that is transitive and is generalized --contractive mapping of integral type I with respect to , and satisfies the following conditions

is -admissible and there exists such that ; If assume is uniformly continuous from to ; If assume is continuous from to Assume that is also continuous if . Then there exists such that.

**Proof** Let be an arbitrary point of such that . We construct an iterative sequence in starting from as for all {0}. If for some {0} then is the fixed point of . From now on, we suppose that {0}. Then from (i) and the admissibility of , we infer that

Proceeding inductively, we get

{0}. (3)

By taking and , we deduce from (1) that

(4)

Where

(5)

Thus

(6)

Otherwise, we have a contradiction. Proceeding inductively, we have

, (7)

for all . Letting in (7) and using the property of, we get  
.

Now, since is nonnegative this implies that

. (8)

We claim that is a Cauchy sequence in . Suppose, on contrary, that is not Cauchy sequence and there exists an > 0 and such that the following inequalities hold (9)

Then using triangular inequality, we get

This further shows that

(10)

Letting in (10) and using (8) and (9), we deduce that

(11)

Therefore

Then

.

Letting in the above inequality and using (8), we get that

(12)

From the transitivity of , we get

(13)

From (1), (9), (12) and (13), we infer that

Which leads to a contradiction due to the fact that . Therefore is a Cauchy sequence in . From the definition of Cauchy sequence, for each > 0, there exists natural number such that

We claim that is also Cauchy sequence with respect to If then our claim is trivially true. Next assuming let be any constant then uniform continuity in (ii) guarantees that there exists =() > 0, such that

Since, is Cauchy sequence in considering the last two inequalities, we get

Which proves our claim, that is Cauchy sequence with respect tonow since is complete, so there exists such that as We further claim that is the fixed point of that is First consider the case when . By using triangular inequality, we have

Let in the above inequality then the continuity hypothesis in (iii) assures that

This shows that is the fixed point of i.e. . Next assume the case when then from triangular inequality and the continuity hypothesis in (iv), we infer that . Hence in both the cases is the fixed point of that is

**Theorem 2.2** Let be complete metric space and be another metric on . Suppose that be a transitive mapping and is generalized --contractive mapping of integral type II with respect to and satisfy the following conditions:

is -admissible and there exists such that ;

1. If assume is uniformly continuous from to ;
2. If assume is continuous from to
3. Assume that is also continuous if .

Then there exists such that

**Proof** Let be such that and be the iterative sequence starting from and defined as , {0}. By hypothesis (i) and from the admissibility of , we have

(14)

Taking and in (2), we have

(15)

Where

We claim that to see this, consider the following three cases:

**Case (1):** If then our claim is trivially true.

**Case (2):**  then from (15) and and since is nonnegative, so we have

and so we have that

This implies that

This further shows that

,

This is a contradiction, since is the maximum.

**Case (3):** Finally if then from triangular inequality, we get that

Which is again a contradiction to the definition of . Hence our claim is true in all possible cases. Rest of the proof of this Theorem follows on the same line as the proof of Theorem 2.1 is done. Thus there exists such that

To assure the uniqueness of the fixed point of , we add the following hypothesis to conditions of the above two Theorems.

**(U):** For all there exists such that and

**Theorem 2.3** Adding hypothesis **(U)** to the conditions of Theorem 2.2 (resp. Theorem 2.1) we obtain the unique fixed point of .

**Proof** Let and be two fixed points of, then from **(U)** we have that for there exists such that and . Consider a sequence such that. Then by -admissibility of we conclude that and. Taking and then from (2) we have

Where

We claim that and for this consider the following three possible cases:

**Case (i):** If then our claim is trivially true.

**Case (ii):** If then using triangular inequality we get

Which is a contradiction to the definition of .

**Case (iii):**  then from (16) and since , we have that

Since is nonnegative, so we deduce from the last inequality that

The above inequality further gives

Which is not possible, since is the maximum. Thus in all three cases our claim is true and from (16), we infer that

Proceeding inductively, we get

Letting in the above inequality and because, we get that,

This further gives

(17)

Similarly for and and considering (2), we have

(18)

Using triangular inequality and considering (17) and (18), we get that

.

This further gives

but, from which we conclude that that is is the unique fixed point of .

Following the idea of [15], we have given the following example to support our results.

**Example 2.1** Let be a metric space with and . Let be another metric on defined as follows:

Clearly is complete with respect to. Let , and then all the conditions of **Theorem 2.1** together with **(U)** are satisfied. Thus has a unique fixed point .

**Remark 2.1** In this paper, Popa and Mocanu [16] proved that forms a symmetric. Hence according to Popa and Mocanu [16], fixed point theorems involving integral type contractions on complete metric spaces can be obtained from the fixed point theorems proved on symmetric spaces. Our results are new, because corresponding fixed point theorems on two symmetric spaces are not available in the literature as for as we know.

**3 Consequences**

Now, we shall list some of the existing results in the literature that can be easily deduced from our theorems.

**Corollary 3.1** (Banach [1]) let be a complete metric space and be a mapping satisfying

Where then has a unique fixed point.

**Proof** Let , , , where and in Theorem 2.3. Then all the conditions of Theorem 2.3 are satisfied together with the uniqueness hypothesis **(U)** and so has a unique fixed point.

**Corollary 3.2** (Samet *et al.*[10]) let be a complete metric space and be an --contractive mapping satisfying the following conditions together with the hypothesis **(U)**;

1. is -admissible;
2. There is an such that ;
3. is continuous.

Then has unique fixed point.

**Proof** This result can easily be deduced from our Theorem 2.3 by simply taking, and .

**Corollary 3.3** (Branciari [7]) let be a complete metric space, and let be a mapping such that for all the following condition is satisfied

Where. Then has a unique fixed point.

**Proof** This result can be obtained from Theorem 2.3 by considering, and . Note that so condition **(U)** is satisfied and has a unique fixed point.

**Remark 3.1** Note that [Theorem 2.1, 4], [Theorem 2.2, 4], [Theorem 2.3, 4], [Theorem 2.3, 10], [Theorem 2.5, 10], [Theorem 2.2, 11] and [Theorem 2.3, 11] are the special cases of results.

**Conflicts of Interest:** The authors declare that there is no conflict of interests regarding the publication of this article.

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