ON NON-QUASIDIAGONAL OPERATORS

Shiv Kumar Sahdev
shiv_sahde@yahoo.co.in
Department of Mathematics, Shivaji College, University of Delhi, Delhi, India

ABSTRACT: In this paper we study a sub-class of the class of non-quasidiagonal operators T on a Hilbert Space H for which $qd(T)/Qd(T) > 1/2$, whereqd(T) denotes the modulus of quasi-diagonality of the operator T and

$$Qd(T) = \overline{\lim_{p\in p(H)}} \|TP - PT\|.$$

Where P(H) denotes the set of all finite rank projections on the Hilbert Space H.

AMS Subject Classification: Primary 47A66, Secondary 47A30.

Key Words: Quasidiagonal, non-quasidiagonal and thin operators.

1. INTRODUCTION

An operator T on a Hilbert Space H is said to be quasidiagonal (resp. quasitriangular) if there exists an increasing sequence $\{P_n\}_{n=1}^{\infty}$ of finite rank orthogonal projections such that $P_n \to I$, the identity operator, strongly and $\|TP_n - P_nT\| \to 0$ (resp. $\|(I-P_n)TP_n\| \to 0$) as $n \to \infty$. T is said to be thin if it is of the form $\lambda I + K$, for a scalar λ and for a compact operator K on H. The notion of quasidiagonality was intoroduced by P.R. Halmos [6] in the year 1970. He also initiated the study of quasitriangular operators [5]. C. Apostal [1] introduced the notion of modulus of quasitrinagluarity q(T) of any operators T on the space H as

$$q(T) = \lim_{P\in P(H)} \|(I-P)TP\|$$

Where P(H) denotes the directed set of all finite rank (orthogonal) projections on H under the usual ordering. From [5, Theorem 2], it follows that, "T is quasitriangular if and only if q(T)=0". D.A. Herrero [7] defined the notion of modulus of quasidiagonalityqd(T) as

$$qd(T) = \lim_{P\in P(H)} \|TP - PT\|.$$

From [6, Page 902] it follows that, "T is quasidiagonal if and only if qd(T) = 0". Herrero also established [7, Theorem 6.13] that "qd(T) is the distance of T from the class of all quasidiagonal operators of T from the class of all quasidiagonal operators on H". C. Foias and L. Zsido in [4] defined another distance function Q(T) as

$$Q(T) = \overline{\lim_{P\in P(H)}} \|(I-P)TP\|$$

and studied its properties. Motivated by this study we defined the notion of modulus of thinalityQd(T) as

$$Qd(T) = \overline{\lim_{P\in P(H)}} \|TP - PT\|$$

and established that for every operator T on a Hilbert Space H, $Qd(T) = Q(T)$, and also "that T is thin if and only if Qd(T) = 0", and $Qd(T) = d(T,[T])$, where $d(T,[T])$ denotes the distance of T from the c*-algebra [T] of all thin operators on H (see [2], [3]). The purpose of the present paper is to study the structure of a sub-class of the class of those non-quasidiagonal operators T for which $qd(T)/Qd(T) > \frac{1}{2}$.

2. NOTATIONS

Throughout the paper H denotes an infinite-dimensional (complex) separable Hilbert space and $B(H)$, the set of all bounded linear operators on H. K(H) denotes the ideal of compact operators on H, and π the natural mapping of B(H) onto the quotient algebra B(H)/K(H). The null space, the spectrum and the point spectrum of an operator T are denoted by N(T), $\sigma(T)$ and $\sigma_p(T)$ respectively.

By [QD] we mean the class of all quasidiagonal operators in $B(H)$.

For any non-quasidiagonal operator T on the space H, denote J(T) as

$$J(T) = qd(T)/Qd(T),$$

then J(T) is a continuous function of T satisfying $0 \le J(T) \le 1$.

We note that there are operators for which $J(T) > \frac{1}{2}$, e.g. if T is a non-unitary isometry then J(T)=1.

In [3, Theorem 2.3] we prove that for any operator T in B(H), there exists a thin operator T_o such that

$$Qd(T) = \|T - T_o\| = d(T,[T]),$$

and also [3, Corollary 3] if T is an invertible operator in B(H), then

$$qd(T) \le \|\pi(T)\|/2 .$$

Thus we see that corresponding to every operator T in B(H) there exists an operator T_o such that $Qd(T) = \|T - T_o\|$. Let A(H) denote the class of all those operators T on H for which $T - T_o$ has finite-dimensional null space. We now study the class of non-quasidiagonal operators T belonging to A(H). One can see that the unilateral shift of multiplicity one belongs to A(H).

3 RESULTS AND DISCUSSIONS

We make use of the above mentioned results in the following

Theorem 1. Suppose that for an operator T in A(H), $J(T) > \frac{1}{2}$. Then

$$T = qd(T)V+S$$

With V a non-unitary isometry and $Qd(S) = Qd(T) - qd(T)$.

Proof. Let T_o be chosen as in [3, Theorem 2.3]. Then we have

$$Qd(T) = \|T - T_o\| = d(T,[T]).$$

Also

$$d(T,[T]) = d(T - T_o,[T]) \ge \|\pi(T) - \pi(T_o)\| = \|T - T_o\|.$$

Since T_o is thin, We obtain

$$qd(T - T_o) = qd(T) > Qd(T)/2 = \|\pi(T) - \pi(T_o)\|/2,$$

and therefore by [3, Corollary 3] T is not invertible. Also since $N(T - T_o)$ is finite-dimensional, without loss of generality, we can find a non-unitary isometry V such that $T - T_o = VA$ is the polar decomposition of $T - T_o$, where $A = ((T - T_o)^*(T - T_o))^{1/2}$.

If we set $S = T - qd(T)V$, then

$$T = qd(T)V + S .$$

Since Qd is a seminorm, we have

$$\begin{aligned} Qd(S) &= Qd(T - qd(T)V) \ge Qd(T) - Qd(T)V) \\ &= Qd(T) - qd(T)QD(V) \\ &= Qd(T) - qd(T) \end{aligned}$$

Now, for any compact operator K, we have

$$\begin{aligned} Qd(S) &= Qd(S - T_o) = Qd(V(A - qd(T))) \\ &= Qd(V(A - qd(T) + K)) = Q(V(A - qd(T)+)) \\ &\le \|V(A - qd(T) + K)\| \le \|A - qd(T) + K\|. \end{aligned}$$

Thus,

$$Qd(S) \le \|\pi(A) - qd(T)\|.$$

As

$$\begin{aligned} qd(T) &= qd(T_o + VA) = qd(VA) \le \|VA\| = \|T - T_o\| \\ &= \|\pi(T) - \pi(T_o)\| = \|\pi(A)\|. \end{aligned}$$

We have

$$Qd(S) \le \|\pi(A)\| - qd(T) = \|\pi(T) - \pi(T_o)\| - qd(T) = Qd(T) - qd(T)$$

.

The desired conclusion follows.

Corollary 2. For any operator T in A(H), we have $J(T) > \frac{1}{2}$ if and only if $T = qd(T)V + S$ with V a non-unitary isometry and $Qd(S) < Qd(T)/2$.

Proof. If $J(T) > \frac{1}{2}$. Then by Theorem 1, $T = qd(T)V + S$ with V a non-unitary isometry and

$$\begin{aligned} Qd(S) &= Qd(T) - qd(T) \\ &= Qd(T)(1 - qd(T)/Qd(T)) \\ &< Qd(T)/2. \end{aligned}$$

Conversely, if $T = qd(T)V + S$ with V a non-unitary isometry and $Qd(S) < Qd(T)/2$ then

$$\begin{aligned} Qd(T) &= Qd(qd(T)V + S) \\ &\le Qd(qd(T)V) + Qd(S) \\ &= qd(T) + Qd(S) \\ &< qd(T) + Qd(T)/2. \end{aligned}$$

This gives

$$Qd(T)/2 < qd(T).$$

Corollary 3. Let $T \in A(H)$. Then $\lim_{P \in P(H)} \|TP - PT\|$ exists if and only if $T = \lambda V + S$ with λ is a scalar, V a non-unitary isometry and S a thin operator.

Proof. If $\lim_{P \in P(H)} \|TP - PT\|$ exists, we have

$$qd(T) = Qd(T).$$

Since every thin operator is quasidiagonal, the case when $Qd(T) = 0$ is trivial. So assume that $Qd(T) \ne 0$.

If $J(T) \le \frac{1}{2}$, then we arrive at a contradiction. Hence we assume that $J(T) > \frac{1}{2}$. Thus by Theorem 1,

$$T = qd(T)V + S$$

With V a non-unitary isometry and $Qd(S) = Qd(T) - qd(T) = 0$.

Conversely, let $T = \lambda V + S$ with λ a scalar, V a non-unitary isometry and $Qd(S) = 0$. Then

$$qd(T) = qd(\lambda V + S) = \lambda qd(V) = \lambda .$$

Also,

$$Qd(T) = Qd(\lambda V + S) = \lambda Qd(V) = \lambda .$$

Hence

$$qd(T) = Qd(T) .$$

Theorem 4. Suppose an operator T in A(H) satisfies $J(T) > \frac{1}{2}$. Then there exists an open disk D of radius $2qd(T) - Qd(T)$ such that $\sigma_p(T^*) \supset D$.

Proof. As $J(T) > \frac{1}{2}$, by Theorem 1, T has the representation $T = qd(T)V + S$ with V a non-unitary isometry and $Qd(S) = Qd(T) - qd(T)$. BY [3, Theorem 2.3] there exists a thin operator T_o such that

$$Qd(S) = \|S - T_o\| .$$

There exists a scalar λ_o , such that $T_o - \lambda_o = K_o$ is a compact operator.

Let $D = \left\{ \lambda : \left| \lambda - \overline{\lambda_o} \right| < 2qd(T) - Qd(T) \right\}$.

Suppose now that there exists $\lambda \in D$ such that $\lambda \notin \sigma_P(T^*)$. We have

$$\left\| V(S^* - T_O^* + \overline{\lambda_o} - \lambda) \right\| \le \left\| S^* - T_O^* \right\| + \left\| \lambda - \overline{\lambda_o} \right\|$$
$$< Qd(T) - qd(T) + 2qd(T) - Qd(T)$$
$$= qd(T).$$

Thus,

$R = qd(T) + V(S^* - T_o^* + \overline{\lambda_o} - \lambda)$ is invertible, and hence $T^* - \lambda = K_O^* + V^* R = V^*(I + VK_O^* R^{-1}) = \{0\}$. Since $VK_O^* R^{-1}$ is compact, we get (see [8, no.85]) $I + VK_O^* R^{-1}$ is invertible.

Hence

$N(V^*) = \{0\}$. This is contradiction to the fact that V is a non-unitary. Thus $D \subset \sigma_p(T^*)$.

Theorem 5. For any operator T in A(H)

$$qd(T) + q(T^*) \le Qd(T) .$$

Proof. The case when $Qd(T) = 0$ is trivial and so assume that $Qd(T) \ne 0$. Let, if possible

$qd(T) + q(T^*) > Qd(T)$. This gives that $2qd(T) > Qd(T)$ and so $qd(T)/Qd(T) > \frac{1}{2}$. By Theorem 1, $T = qd(T)V + S$ with V a non-unitary isometry and $Qd(S) = Qd(T) - qd(T)$. Now

$$\begin{aligned} q(T^*) &= q(qd(T)V^* + S^*) \\ &\le qd(T)q(V^*) + Q(S^*) \\ &= qd(T).O + Qd(S^*) \\ &= Qd(S) \\ &= Qd(T) - qd(T). \end{aligned}$$

Thus $qd(T) + q(T^*) \le Qd(T)$, Which is a contradiction. The desired conclusion follows.

5. CONCLUSION

We have identified a sub-class of non-quasidiagonal operators and its relationship with the class of thin operators by measuring certain estimates for the modules of the quasi-diagonality.

REFERENCES

1. 1. C. Apostal, Quasitriangularity in Hilbert Space, Indiana Univ. Math. J., 22 No. 9, 817-825, 1973.
2. S. C. Arora and S. K.Sahdev, On Quasidiagonal Operators, SERDICA- Bulgaricae, Mathematicae publications 20, 298-305, 1994.
3. S. C. Arora and S. K.Sahdev, On Quasidiagonal Operatiors-II, J. Indian Math. Soc., 59, 1-8, 1993.
4. C. Foias and L. Zsido, Some results on non-quasitriangular operators, (preprint).
5. P. R. Halmos, Quasitriangular Operators, Acta Sci. Math. (Sxeged), 29, 283-293, 1968.
6. P. R. Halmos, Ten Problems in Hilbert Space, Bull. Amer. Math. Soc., 76, 887-933, 1970.
7. D. A. Herrero, Quasitrinagularity, Approximation of Hilbert Space Operators, Vol. I, Research Notes in Math., Pitman Advanced Publishing Program, 135-167, 1982.
8. F. Riesz and B. Sx-Nagy, Leconsd' analyse functionnelle, Budapest, 1968.