CONFORMAL MINKOWSKI SPACETIME AND THE COSMOLOGICAL CONSTANT

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ABSTRACT: *This paper introduces the use of Noether symmetry equation for Lagrangian of conformal plane symmetric static Minkowski spacetime to find all those Minkowski spacetimes which admit the conformal factor. We also discuss the cosmological constant in these spacetimes.*

Keywords: Static Conformal Mikowski Spacetime, Cosmological Constant, Einstein Field Equations.

1 INTRODUCTION

Symmetries play an important role in different areas of research, including differential equations and general relativity [4, 6, 7, 10, 12]. The Einstein field equations(EFE) [9],

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}+\Lambda g_{\mu\nu}=\kappa T_{\mu\nu}. \tag{1}$$

are the building blocks of the theory of general relativity. These are Highly non-linear second order partial differential equations and it is not easy to obtain exact solutions of these equations. Symmetries help a lot in finding solutions of these equations. These solutions (spacetimes) have been classified by using different spacetime symmetries. Among different spacetime symmetries isometries or the Killing vectors (KVs) have the importance that they help in understanding the geometric properties of spaces also corresponding to each isometries there is some conserved quantity. They are also subset of the Noether symmetry (NS) [1-3] i.e.

$$KVs\subseteq NS.$$

This relation shows that the KVs do not lead to all the conserved quantities or the first integrals. Therefore, it looks reasonable to look for Noether symmetries from the Lagrangian of spacetimes. Instead of taking the Lagrangian from the most general form of the spacetime and solving a set of partial differential equations involving unknown metric coefficients one may adopt an easy approach and directly look for the NS from Lagrangian of all known spacetimes obtained through classification by KVs. However, we have adopted here the longer route, so that we also have a counter check on the spacetimes obtained through the classification by the KVs. Using the Lagrangian of plane and spherical symmetric static spacetimes, a complete list of Noether symmetries have been obtained and new first integral have been found. Some new solutions have also been found in the case of spherical symmetry. Here, we want to use the technique of Noether symmetries and find all those Minkowski spacetimes which admit the conformal factor [8]. We take the following form of the conformal Minkowski spacetime [11]

$$ds^2=e^{\mu(x)}(dt^2-dx^2-dy^2-dz^2), \tag{2}$$

For spacetime (2), the Einstein field equations (1) read

$$\begin{aligned}&\mu_{xx}(x)+\frac{1}{4}\mu_x^2(x)+\Lambda e^{\mu(x)}=kT_{00}=-kT_{22}=-kT_{33},\\&-\frac{3}{4}\mu_x^2(x)-\Lambda e^{\mu(x)}=kT_{11}.\end{aligned} \tag{3}$$

We will find all those classes of the spacetime given in equation (2) which admit the conformal factor $e^{\mu(x)}$ for some particular values of the function $\mu(x)$. The cosmological constant Λ is also discuss for the obtained spacetimes.

2 The Noether Symmetry Governing Equation

A symmetry $\mathbf{X}$ is the Noether symmetry if it satisfies the following equation,

$$\mathbf{X}^1L+(D\xi)L=D\mathbf{A}, \tag{4}$$

where

$$\mathbf{X}^1=\mathbf{X}+\eta^0_{,s}\frac{\partial}{\partial t}+\eta^1_{,s}\frac{\partial}{\partial x}+\eta^2_{,s}\frac{\partial}{\partial y}+\eta^3_{,s}\frac{\partial}{\partial z}, \tag{5}$$

is the first order prolongation of

$$\mathbf{X}=\xi\frac{\partial}{\partial s}+\eta^0\frac{\partial}{\partial t}+\eta^1\frac{\partial}{\partial x}+\eta^2\frac{\partial}{\partial y}+\eta^3\frac{\partial}{\partial z}, \tag{6}$$

D is differential operator define as

$$D=\frac{\partial}{\partial s}+\dot{t}\frac{\partial}{\partial t}+\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}+\dot{z}\frac{\partial}{\partial z}, \tag{7}$$

and $\mathbf{A}$ is a gauge function. ξ, η^i and $\mathbf{A}$ are functions of $s,t,x,y,z,$ and $\eta^i_{,s}$ are functions of $s,t,x,y,z,\dot{t},\dot{r},\dot{y},\dot{z}$, where ``." denotes differentiation with respect to s. From differential geometry we know that for the general cylindrically symmetric static space-times given by eq. (2), the usual Lagrangian is

$$L=e^{\mu(x)}(\dot{t}^2-\dot{r}^2-\dot{y}^2-\dot{z}^2), \tag{8}$$

Using eq.(8) in eq.(4) we get the following system of 19 partial differential equations (PDEs)

$$\begin{aligned}&\xi_t=0,\quad \xi_x=0,\quad \xi_y=0,\quad \xi_z=0,\quad A_s=0,\\&2e^{\mu(x)}\eta^0_s=A_t,\quad -2e^{\mu(x)}\eta^1_s=A_x,\\&-2b^2e^{\mu(x)}\eta^2_s=A_y,\quad -2e^{\mu(x)}\eta^3_s=A_z,\\&\mu_x(x)\eta^1+2\eta^2_y-\xi_s=0,\quad \eta^2_z+\eta^3_y=0,\\&\eta^1_y+\eta^2_x=0,\quad \eta^1_z+\eta^3_x=0,\\&\eta^0_x-\eta^1_t=0,\quad \eta^0_y-\eta^2_t=0,\\&\eta^0_z-\eta^3_t=0,\quad \mu_x(x)\eta^1+2\eta^0_t-\xi_s=0,\\&\mu_x(x)\eta^1+2\eta^3_z-\xi_s=0,\quad \mu_x(x)\eta^1+2\eta^1_x-\xi_s=0.\end{aligned}$$

This system consists of seven unknowns, $\xi,\eta^i(i=0,1,2,3)$, μ, and A. Solutions of this system give the Lagrangian along with the Noether symmetries. Corresponding to these Lagrangian one may easily write spacetimes, which are the exact solutions of EFE. In the following sections different solutions of the system (9) are discussed.

3 Solution of the System (9)

In this section we discuss all the possible conformal Minkowski plane symmetric static spacetimes

Solution 1: The first solution of the system (9) is

$$ds^2 = e^{\mu(x)}(dt^2 - dx^2 - dy^2 - dz^2), \tag{10}$$

$$\begin{gathered} A(s,t,x,y,z) = C, \xi(s,t,x,y,z) = C_1, \eta^0(s,t,x,y,z) = C_3 y + zC_4 + C_5, \\ \eta^1(s,t,x,y,z) = 0, \eta^2(s,t,x,y,z) = C_3 t + C_6 z + C_7, \eta^3(s,t,x,y,z) = C_4 t - C_6 y + C_2, \end{gathered} \tag{11}$$

we see that in equation (11) there are seven parameters which implies that there are seven Noether symmetries, four translations, along s, t, y and z. Two boasts one along y axis and the other along z axis. One rotation in the yz plane. The Einstein field equation (15) are the same for spacetime given in equation (10) in which the cosmological constant Λ depends upon the values of function $\mu(x)$.

Solution 2: The second solution of (9) is

$$ds^2 = (\frac{x}{\alpha})^a (dt^2 - dx^2 - dy^2 - dz^2), \tag{12}$$

$$\begin{gathered} A(s,t,x,y,z) = C, \xi(s,t,x,y,z) = C_1 s + C_2, \eta^0(s,t,x,y,z) = \frac{C_1 t}{a+2} + C_4 y + C_5 z + C_6, \\ \eta^1(s,t,x,y,z) = \frac{C_1 x}{a+2}, \eta^2(s,t,x,y,z) = C_4 t + \frac{C_1 y}{a+2} + C_7 z + C_8, \\ \eta^3(s,t,x,y,z) = C_5 t - C_7 y + \frac{C_1 z}{a+2} + C_3 \end{gathered} \tag{13}$$

the Lagrangian of the spacetime given in equation (12) admit eight Noether symmetries which are given in equation (13). These symmetries have one additional symmetry along with symmetries discussed in solution 1. The additional Noether symmetry is the scaling symmetry. The Einstein field equations for spacetime (12) are

$$\begin{gathered} \frac{a(a-4)}{4x^2} + \Lambda(\frac{x}{\alpha})^a = kT_{00} = -kT_{22} = -kT_{33}, \\ -\frac{3a^2}{4x^2} - \Lambda(\frac{x}{\alpha})^a = kT_{11}. \end{gathered} \tag{14}$$

Rearranging the system (14) we have

$$\begin{gathered} \Lambda = (\frac{x}{\alpha})^{-a}(kT_{00} - \frac{a(a-4)}{4x^2}), \\ \Lambda = (\frac{x}{\alpha})^{-a}(kT_{11} + \frac{3a^2}{4x^2}). \end{gathered} \tag{15}$$

We discuss the value of Λ for $a < -2$ and $a > -2$ here. The value of Λ for $a = -2$ will discuss in solution 4. The value of the cosmological constant $\Lambda \to \infty$ as $x \to \infty$ for $a < -2$. While for $a > -2$ the value of $\Lambda \to 0$ as $x \to \infty$.

Solution 3: The third solution of (9) is as follows

$$ds^2 = e^{\frac{x}{\alpha}}(dt^2 - dx^2 - dy^2 - dz^2), \tag{16}$$

$$\begin{gathered} A(s,t,x,y,z) = -2\alpha^2 C_1 e^{\frac{x}{\alpha}} + C, \xi(s,t,x,y,z) = 1/2 C_1 s^2 + C_2 s + C_3, \\ \eta^0(s,t,x,y,z) = C_5 y + C_6 z + C_7, \eta^1(s,t,x,y,z) = (C_1 s + C_2)\alpha, \\ \eta^2(s,t,x,y,z) = C_5 t + C_8 z + C_9, \eta^3(s,t,x,y,z) = C_6 t - C_8 y + C_4. \end{gathered} \tag{17}$$

The spacetime given in equation (16) admit two additional Noether symmetrie along with symmetries given in solution 1. The 8^{th} symmetry is the mix symmetry of scaling and translation which corresponds to C_2 in equation (17) and 9^{th} symmetry is corresponds to the parameter C_1 in the above solution (17), which also carry a gauge function $-2\alpha^2 e^{\frac{x}{\alpha}}$. The Einstein field equations (15) for spacetime (16) are

$$\begin{gathered} \frac{1}{4\alpha^2} + \Lambda e^{\frac{x}{\alpha}} = kT_{00} = -kT_{22} = -kT_{33}, \\ -\frac{3}{4\alpha^2} - \Lambda e^{\frac{x}{\alpha}} = kT_{11}. \end{gathered} \tag{18}$$

Which can also be written as

$$\begin{gathered} \Lambda = e^{-\frac{x}{\alpha}}(kT_{00} - \frac{1}{4\alpha^2}), \\ \Lambda = e^{-\frac{x}{\alpha}}(kT_{11} + \frac{3}{4\alpha^2}). \end{gathered} \tag{19}$$

The system (19) give us one equation in three variables T_{00}, T_{11} and α for $x = 0$ which has many solutions. And the value of the cosmological constant Λ tends to zero as x tends to infinity.

Solution 4: The following is the fourth solution which has the maximum Noether symmetries for the plane symmetric static conformal Minkowski spacetime

$$ds^2 = (\frac{x}{\alpha})^{-2}(dt^2 - dx^2 - dy^2 - dz^2), \tag{20}$$

$$
\begin{aligned}
&A(s,t,x,y,z) = C, \xi(s,t,x,y,z) = C_1,\\
&\eta^0(s,t,x,y,z) = 1/2\left(t^2 + x^2 + y^2 + z^2\right)C_3 + \left(C_4 y + C_5 z + C_6\right)t + C_8 z + C_7 y + C_9,\\
&\eta^1(s,t,x,y,z) = \left(C_3 t + C_4 y + C_5 z + C_6\right)x,\\
&\eta^2(s,t,x,y,z) = 1/2\left(t^2 - x^2 + y^2 - z^2\right)C_4 + \left(C_3 t + C_5 z + C_6\right)y + C_7 t + C_{10} z + C_{11},\\
&\eta^3(s,t,x,y,z) = 1/2\left(t^2 - x^2 - y^2 + z^2\right)C_5 + \left(C_3 t + C_4 y + C_6\right)z + C_8 t - C_{10} y + C_2.
\end{aligned} \quad (21
$$

The solution (21) consist of eleven Noether symmetries in which ten are the isometries of the spacetime given in equation (20) which there are 11 conservation laws corresponding to different Noether symmetries. This is like the De Sitter spacetime which has also ten isometries. The Einstein field equations for spacetime given in equation (21) is

$$
\begin{aligned}
&\frac{3}{x^2} + \Lambda\left(\frac{x}{\alpha}\right)^{-2} = kT_{00},\\
&-\frac{3}{x^2} - \Lambda\left(\frac{x}{\alpha}\right)^{-2} = kT_{11},
\end{aligned} \quad (22)
$$

adding the equations we have $T_{00} = -T_{11}$ in that case the system reduce to equation

$$
\Lambda = \frac{x^2}{\alpha^2} kT_{00} - \frac{3}{\alpha^2}. \quad (23)
$$

At $x = 0$ we have $\Lambda = -\frac{3}{\alpha^2}$ which shows that Λ is negative for all values of α. And $\Lambda \to \infty$ as $x \to \pm\infty$. It mean that for $T_{00} = \frac{3}{kx^2}$, $\Lambda = 0$, for $T_{00} < \frac{3}{kx^2}$, $\Lambda < 0$ and for $T_{00} > \frac{3}{kx^2}$ $\Lambda > 0$. That is for the manifold describe by the metric given in equation (20) the cosmological constant is negative, positive and zero in different regions of the given spacetime. Which can be interpreted as it is attractive in some region and repulsive in some other region. There is a place somewhere in the spacetime where the value of the cosmological constant is zero.

4 CONCLUSION

In this paper we classify the static conformal Minkowski spacetimes according to Noether symmetries. It has been shown here that the static conformal Minkowski spacetimes admit, at least 7 Noether symmetries and at most 11 Noether symmetries. We obtain four classes of the static conformal Minkowski spacetimes. They are discussed as solution 1, 2, 3 and 4 respectively. The first solution has 7 Noether symmetries and 6 isommetries, the second solution has 8 Noether symmetries in which 6 are the isometries, one homothety and one pure Noether symmetry. In solution 3 we obtained 9 Noether symmetries, in which 6 are isometries of the corresponding spacetime and the remaining are purely Noether symmetries. In solution 4 we obtained 11 Noether Symmetries in which 10 are the isometries of the spacetime given in equation (20) and only one is purely Noether symmetry. This shows that the static conformal Minkowski admit either 6 isometries or 10 isometries. The cosmological constant Λ is discussed in each solution. In solution 4 the cosmological constant is negative for some region and positive other. Its value become very large for very large x. This mean that the cosmological constant change the sign when we go through the spacetime. Somewhere it is positive and somewhere its value is negative and there is certain region where the value of the cosmological constant is zero.

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