VARIOUS PERTURBATIONS CNSIDERED ON THE GENERALIZED CIRCULAR RESTRICTED THREE-BODY PROBLEM

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ABSTRACT*: The motion of variable mass test particle in the generalized restricted 3-body system under the effect of zonal harmonic coefficients up to fourth-order is investigated. For numerical investigations, we derive the equations of motion of the test particle which varies its mass according to Jeans law where we show the effective variation due to parameters in the locations of stationary points and the basins of convergence. Finally, the stability of the stationary points have been examined by Meshcherskii transformations.*

AMS Subject Classification: 70F05, 70F15.

Key Words: *Zonal Harmonic; Variable mass; Stationary points; Periodic orbits.*

1. INTRODUCTION

In general all celestial bodies are having irregular shapes and zonal harmonic coefficients of the celestial bodies have a great impact on the motion of the test particles in the restricted problems. Therefore it has an attraction for the researchers. This problem is studied by many scientists with various perturbations including zonal harmonic coefficients in the configuration of restricted problems such as restricted three-body problem, four-body problem, restricted five-body problem, restricted six-body problem etc. [1-14].

Many other effects and perturbations are studied by various researchers separately as follows: [15] numerically investigated the effect of the oblateness of the more massive primary on the location of five equilibrium pointsin the circular restricted 3-body problem. They noticed that the patterns of angular frequencies have interesting differences. [16, 17] investigated the effects of radiation pressure, oblateness, variable mass and asteroid beltsin the framework of the circular restricted three-body problem by finding Lagrangian function with the help of kinetic energy and potential energy. [18] investigated the effect of variable mass in the frame of circular restricted three-body problem by considering that sum of the masses of all bodies are always constant. They obtained the zero-velocity surfaces using Jacobi-quasi-integral. They also presented an example for their model as binary stars system with the conservation of mass transfer. [19] studied the existence, position and stability of the collinear equilibrium points in the frame of generalized Hill's problem by hypothesizing primary as radiating and secondary as oblate. They also plotted the basins ofattraction by using a fast and simple Newton-Raphson iterative method for several cases of parameters. [20] investigated the effect of oblateness on the periodic orbits as well as on the regions of quasi-periodic motion aroundboth the primaries in the circular restricted problem of three bodies by using the numerical technique of Poincaré esurfaces of section. They found some variational effects in the stability of orbits due to oblateness. [21] studied the motion of the variable mass body in the circular restricted 3-body system by taking into account to one of the primaries having photo-gravitational effect, using Jeans law [22] and Meshcharskii space-time transformations [23,. 24, 25] studied the effect of oblateness, radiation pressure of the primaries as well as the effect of variable mass of the infinitesimal body in the frame of restricted problems. [26, 27, 28, 29] investigated the effects of the perturbations and variable mass in the frame of the restricted three-body problem. They also studied the effects of these perturbations on the locations of equilibrium points, Poincaré e surfaces of section, regions of possible andforbidden motion, basins of attraction and examined the stability of these equilibrium points where most of the cases they found that these points are unstable. The paper is organized in various sections and subsections as: The literature review is given in section 1. The determination of equations of motion performed in section 2 while section 3 represents the numerical studies with subsections 3.1 and 3.2. The stability of stationary points is studied in section 4. Finally, the conclusion is drawn in section 5.

2. EQUATIONS OF MOTION

The potential between two oblate celestial bodies having masses m_1 and m_2 with separation distance d under the effect of zonal harmonic coefficients up to fourth-order (J_2^1, J_4^1, J_2^2 J_{22} and J_4^2) can be written as ([30])

$$V = -Gm_1m_2[\frac{1}{d} + \frac{A_1+B_1}{2d^3} - \frac{3(A_2+B_2)}{8d^5}], \qquad (1)$$

where the oblateness coefficients are $A_i = J_{2\times i}^1 R_1^{2\times i}$, $B_i = J_{2\times i}^2 R_2^{2\times i}$ and R_i is the mean radius of the both celestial bodies (i= 1,2).

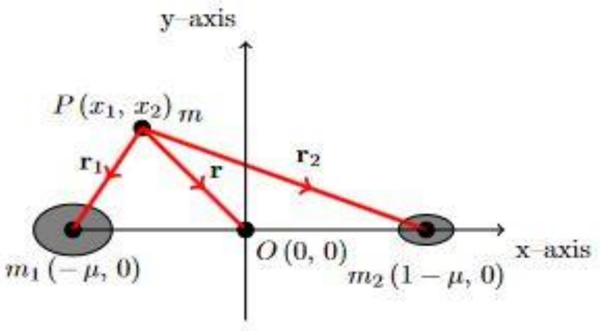


Figure-1: Geometric plan for generalized restricted problem.

Let m be the mass of the test particle which varies its mass according to Jeans law and moving under the influence of two heavy celestial bodies but not affecting them. Let xyz−be forming right-handed system where system is rotating through z-axis with mean motion n (see fig. 1). We also assume that the syderal coordinate system coincides with the synodic coordinate system. Let both the heavy celestial bodies be placed at the x-axis and revolving about their common center of mass which is taken as origin. Let α and β be the small perturbationsin the coriolis and centrifugal forces respectively and also qi be the small parameters for the radiation pressuresfrom both bodies m_i (i = 1,2). Following the procedure given by [31], we can write the equations of motion of the test particle with non-dimensional variables where the variation of mass of the smallest body originates from one point and have zero momenta as:

$$\frac{\dot m}{m}(\dot x_1 - n\alpha x_2) + (\ddot x_1 - 2n\alpha \dot x_2) = U_{x_1},$$
$$\frac{\dot m}{m}(\dot x_2 + n\alpha x_1) + (\ddot x_2 + 2n\alpha \dot x_1) = U_{x_2}, \qquad (2)$$

where,

$$U = \frac{n^2\beta}{2}(x_1^2 + x_2^2) + \frac{(1-\mu)q_1}{r_1} + \frac{(1-\mu)q_1A_1}{2r_1^3} + \frac{\mu\, q_2}{r_2} - \frac{3(1-\mu)q_1A_2}{8r_1^5} + \frac{\mu\, q_2 B_1}{2r_2^3} - \frac{3\mu\, q_2 B_2}{8r_2^5}, \qquad (3)$$

with,

$$n^2 = 1 + \frac{3(A_1+B_1)}{2} - \frac{15(A_2+B_2)}{8},$$

$$\alpha = 1+\alpha_1, \beta = 1+\beta_1, \alpha_1 << 1, \beta_1 << 1,$$

$$r_1^2 = (x_1+\mu)^2 + x_2^2,\ \ r_2^2 = (x_1+\mu-1)^2 + x_2^2.$$

To study the dynamical behaviour of the test particle, we must determine the equations of motion which preserve the dimension of the space and time, therefore we will use Jean's law [22] and Meshcherskii space-time transformations [23], which are as follow:

$$dt = d\tau, \qquad m = m_0 e^{-\delta_1 t},$$
$$(x_1, x_2) = \delta_2^{-1/2}(x_3, x_4),$$
$$(\dot x_1, \dot x_2) = \delta_2^{-1/2}\{(\dot x_3 + \frac{\delta_1}{2}x_3), (\dot x_4 + \frac{\delta_1}{2}x_4)\}, \qquad (4)$$
$$(\ddot x_1, \ddot x_2) = \delta_2^{-1/2}\{(\ddot x_3 + \delta_1\dot x_3 + \frac{\delta_1^2}{4}x_3), (\ddot x_4 + \delta_1\dot x_4 + \frac{\delta_1^2}{4}x_4)\},$$

Where δ_1 is the variation constant, $\delta_2 = m/m_0$ and m_0 is the mass of test particle at time t = 0.

From Eqs. (3) and (4), we get

$$\frac{\partial\Omega}{\partial x_3} = \ddot x_3 - 2n\alpha\dot x_4,$$
$$\frac{\partial\Omega}{\partial x_4} = \ddot x_4 + 2n\alpha\dot x_3, \qquad (5)$$

where,

$$\Omega = (\frac{n^2\beta}{2} + \frac{\delta_1^2}{8})(x_3^2 + x_4^2) + \delta_2^{3/2}(\frac{(1-\mu)q_1}{\rho_1} + \frac{(1-\mu)q_1A_1\delta_2}{2\rho_1^3} + \frac{\mu\, q_2}{\rho_2} - \frac{3(1-\mu)q_1A_2\delta_2^2}{8\rho_1^5} + \frac{\mu\, q_2B_1\delta_2}{2\rho_2^3} - \frac{3\mu\, q_2B_2\delta_2^2}{8\rho_2^5}), \qquad (6)$$

$$\rho_1 = \sqrt{(x_3 + \mu\delta_2^{1/2})^2 + x_4^2},$$

$$\rho_2 = \sqrt{(x_3 - (1-\mu)\delta_2^{1/2})^2 + x_4^2}.$$

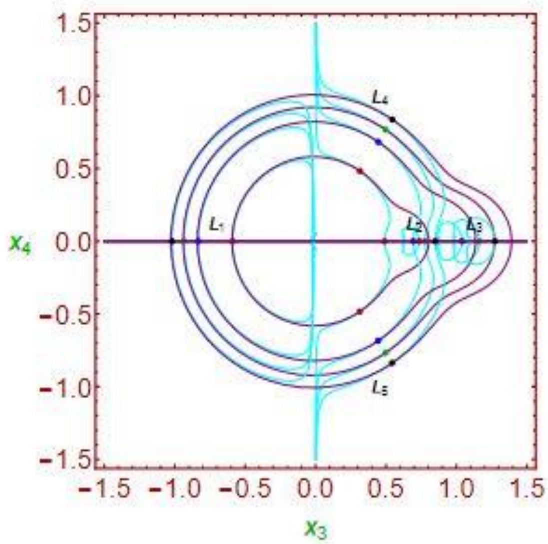
3 Numerical Studies

3.1 Location of stationary points

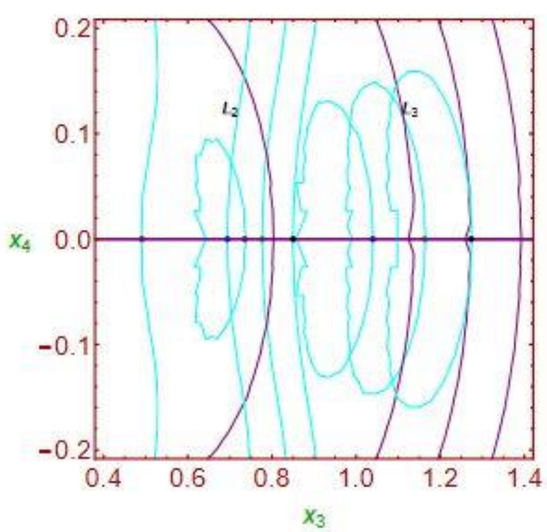
The locations of stationary points can be found by solving the following system,

$$\frac{\partial\Omega}{\partial x_3} = 0, \quad \frac{\partial\Omega}{\partial x_4} = 0. \qquad (7)$$

After numerically solving the system (7) for the parameters used with the help of well-known software, Mathematica, we get here five stationary points. Like classical case i.e. without perturbations([32]), three collinear (L_1, L_2, L_3) and two triangular stationary points (L_4, L_5). Figure 2(a) presents the graph for the various values of variation parameters δ_1 and δ_2 (= 0.4 (red), 0.8 (blue), 1(green), 1.2 (black)). From this figure, we observed that as increase the value of δ_2, the stationary points are moving away from origin. The locations of stationary points near L_2 and L_3 can be clearly seen in figure 2(b) which is zoomed part of figure 2(a).



(a) Locations for various values of δ_2 (= 0.4 (red), 0.8 (blue), 1 (green), 1.2 (black)



(b) Zoomed part of figure 2(a) near L_2 and L_3

Figure-2: Locations of stationary points in x_3-x_4-plane.

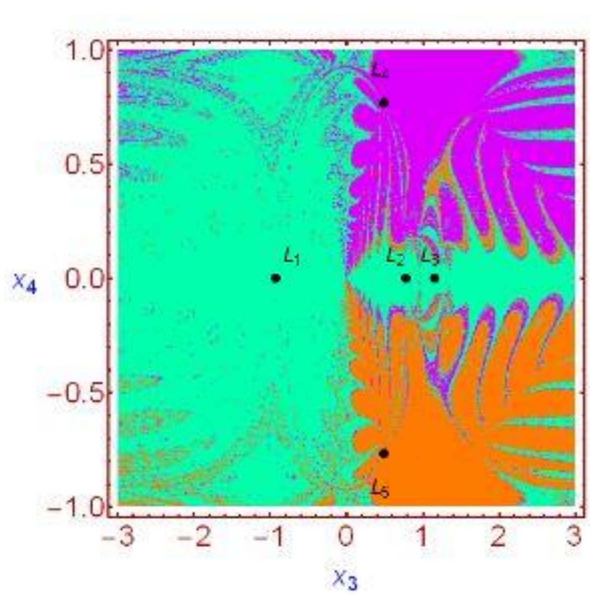
3.2 Basins of Convergence

Basins of convergence are one of the dynamical properties of the test particle where the initial point converges rapidly to one of the stationary points then that point will work as an attractor. We will use the very simple and well known N-R iterative method for the different values of variation parameter. The mathematical algorithm for this problem is as follows:

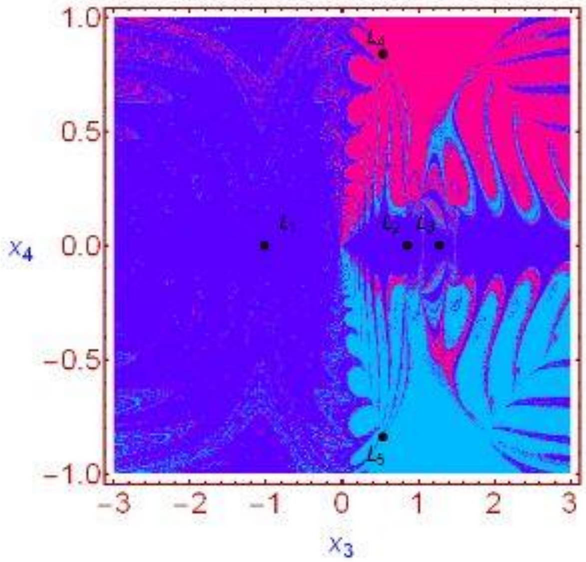
$$x_{3_{n+1}} = x_{3_n} - \left(\frac{\Omega_{x_3}\Omega_{x_4x_4} - \Omega_{x_4}\Omega_{x_3x_4}}{\Omega_{x_3x_3}\Omega_{x_4x_4} - \Omega_{x_3x_4}\Omega_{x_4x_3}}\right)_{(x_{3_n}, x_{4_n})}, \tag{8}$$

$$x_{4_{n+1}} = x_{4_n} - \left(\frac{\Omega_{x_4}\Omega_{x_3x_3} - \Omega_{x_3}\Omega_{x_4x_3}}{\Omega_{x_3x_3}\Omega_{x_4x_4} - \Omega_{x_3x_4}\Omega_{x_4x_3}}\right)_{(x_{3_n}, x_{4_n})}, \tag{9}$$

where x_{3n}, x_{4n} are the values of x_3 and x_4 coordinates of the n^{th} step of the iterative process. We have illustrated the basins of convergence in two cases firstly in the case of constant mass i. e. for $\delta_1 = 0$, $\delta_2 = 1$ (figure 3(a)) and secondly in the case when the mass is varying i.e. for $\delta_1 = 0.2$, $\delta_2 = 1.2$ (figure 3(a)). From figure 3(a), we observed that the attracting points L_1, L_2 and L_3 belong to light green color regions while L_4 and L_5 belong to purple and orange color regions respectively. Here all the regions extended to infinity. On the other hand figure 3(b), revealed that the attracting points L_1, L_2 and L_3 belong to blue color regions while L_4 and L_5 belong to red and light blue color regions respectively. Here also all the regions extended to infinity. We also observed that all the corresponding regions are expanding when we increase the value of the variation parameter δ_2. The black dots in both cases represent the location of attracting points



(a) For $\delta_1 = 0$, $\delta_2 = 1$



(b) For $\delta_1 = 0.2$, $\delta_2 = 1.2$

Figure-3: Basins of convergence in x_3-x_4-plane.

4. Stability of stationary points

The stability of stationary points is examined under the effect of perturbations in the generalized restricted 3-body problem for which we consider the motion near the stationary point (x_{30}, x_{40}) as ($x_{30} + x_{31}$, $x_{40} + x_{41}$), where (x_{31}, x_{41}) are small displacements. The system (5) can be reformulated in the phase space as:

$$\begin{aligned} \dot{x}_{31} &= x_{32}, \\ \dot{x}_{41} &= x_{42}, \\ \dot{x}_{32} &= 2n\alpha x_{42} + \Omega^0_{x_3x_3} x_{31} + \Omega^0_{x_3x_4} x_{41}, \\ \dot{x}_{42} &= -2n\alpha x_{32} + \Omega^0_{x_4x_3} x_{31} + \Omega^0_{x_4x_4} x_{41}, \end{aligned} \tag{10}$$

where the superscript 0 indicates the value of the second derivative of Ω from Eq. (6) at the corresponding stationary point (x_{30}, x_{40}). To preserve the dimension of space and time, we will use Meshcherskii-space-time inverse transformations and also following the procedure given by [21] we can write the characteristic equation for our model as

$$\lambda^4 + R_3\lambda^3 + R_2\lambda^2 + R_1\lambda + R_0 = 0, \tag{11}$$

where,

$$R_3 = -2\delta_1,$$

$$R_2 = 4n^2\alpha^2 + \frac{3}{2}\delta_1^2 - \Omega^0_{x_3x_3} - \Omega^0_{x_4x_4},$$

$$R_1 = -\delta_1(4n^2\alpha^2 + \frac{1}{2}\delta_1^2 - \Omega^0_{x_3x_3} - \Omega^0_{x_4x_4}),$$

$$R_0 = \frac{\delta_1^4}{16} + \frac{\delta_1^2}{4}(4n^2\alpha^2 - \Omega^0_{x_3x_3} - \Omega^0_{x_4x_4})$$

$$+ \Omega^0_{x_3x_3}\Omega^0_{x_4x_4} - \Omega^0_{x_3x_4}\Omega^0_{x_4x_3},$$

We have numerically evaluated the characteristic roots for each stationary point from Eq. (11) for the various values of the variation of mass parameter δ_2. From these values we observed that all the roots have either at-least one positive real root or a positive real part of the complex roots. Therefore all the stationary points are unstable.

5. CONCLUSION

The perturbed generalized restricted three-body problem is studied with the variation effect of mass for the testparticle through the Jeans and Meshcherskii law. The equations of motion determined here are clearly different from the equations of motion evaluated by [30, 32] with the mass variation factors. After this, we have studied the effect of these parameters on the various dynamical properties such as locations of stationary points and basins of convergence. We will explain here these effects one by one in the various subsections. In the locations of stationary points, we obtained that as increase the value of the variation parameters, the locations of stationary are moving away from the origin (figure 2). Further, in the next subsection, we have studied the most effective dynamical property which is the basins of convergence. Convergence means one of the initial value is approaching the attracting value. Here we observed that all the regions related to the attracting points are extending to infinity and also as increase the value of the variation parameter, the attracting regions are expanding (figure 3). In the last stability section, we have examined the stability of stationary points for the perturbed generalized restricted three-body problem through the Meshcherskii inverse transformations. We have evaluated the required characteristic roots for eachstationary point where we have found at least one positive real root or positive real part of the complex roots. Hence all the stationary points are unstable which are different from the result obtained in the classical case ([32], [30]). From these studies we revealed that the variation parameters have a great impact on the perturbed generalized restricted 3-body system.

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