GENERALIZED ROBE'S PROBLEM HAVING HETEROGENEOUS PRIMARY CONTAINING VISCOUS FLUID INSIDE THE OUTERMOST LAYER AND SPHERICAL SECONDARY WITH MODIFIED NEWTONIAN POTENTIAL

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***ABSTRACT**: The purpose of this paper is to study the motion properties of the third body which is moving inside the outer layer of the oblate heterogeneous body filled with the viscous fluid, under the influence of the oblate heterogeneous body and the radiating point mass which is producing the modified Newtonian potential. Wealso assume that the system is perturbed by the small perturbations in Coriolis and centrifugal forces. By using the evaluated system of equations of motion, we determine the locations of the collinear, non-collinear and out-of-plane equilibrium points. Finally, the stability examinations are done for these equilibrium points.*

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Key Words: Oblate heterogeneous body, viscous fluid, radiation force, perturbation, modified Newtonian potential.

1. INTRODUCTION

New kind of restricted 3-body problem was introduced by [14], where primary is taken as spherical shell which is filled with incompressible fluid and secondary is taken as point mass. These two bodies are moving in circular orbits around their common center of mass. And the third smallest body is moving inside the spherical shell in the fluid under the gravitational forces of the both massive bodies but not influencing them.

Afterward the Robe's problem was extended with various perturbations by many researchers. [15] studied the effect of the perturbations in Coriolis and centrifugal forces on the location of equilibrium point in the Robe's restricted problem of 3 bodies and they also supposed that the density of the infinitesimal body is equal to the density of the fluid filled in the primary. [13] investigated the Robe's problem by considering the first primary as Roche's ellipsoid and examined the linear stability of the equilibrium points. They also pointed out the connection between the buoyancy force and the Coriolis force. [8, 9, 10] studied the existence and stability of equilibrium points in the Robe's restricted 3-body problem in two cases for the movement of the primaries as elliptical and as circular. In the case of elliptical motion, they found only one equilibrium point while in the case of circular motion they found infinite number of equilibrium points. They also pointed out that the collinear equilibrium points are stable while in the classical case it is unstable and triangular equilibrium points are stable.

[16, 17, 18, 19, 20] investigated the locations and stability of the smallest body around equilibrium points in circular Robe's restricted 3-body problem under the supposition that both the primaries are oblate body. They found two equilibrium points in In-plane motion, one near the center of the first primary which is conditionally stable and second near the line joining the center of both the primaries, is unstable. They also found two equilibrium points in out-of-plane which are unstable. [3, 2, 4] studied the Robe's restricted problem by assuming that the fluid taken inside the first primary has viscous force and other primary is oblate in shape. They found two collinear equilibrium points, infinitely many circular equilibrium points and two out-of plane equilibrium points which all are unstable. [12] analyzed the equilibrium solutions and their linear stability of Robe's restricted problem when one primary is taken as spherical shell and another primary is considered as finite-straight segment. They observed that the collinear equilibrium points are conditionally stable while non-collinear and out-of-plane equilibrium points are unstable. [7] studied this problem by supposing the drag forces. [6] examined the stability of the triangular equilibrium points in the Robes restricted 3-body problem by supposing the effect of solar radiation pressure. Some more related studies are as follows: [21,22, 23, 24, 25].

We arranged the paper in five sections. The review of literature is given in section 1. In section 2, we determinedthe equations of motion. In section 3, we evaluated the existence of equilibrium points in three subsections 3.1, 3.2 as well as 3.3 and the stability examination is done in section 4. Finally, the conclusion is made in section 5.

2. EQUATIONS OF MOTION

Assuming that there are two masses, m_1 as primary is an oblate heterogeneous body with N−layers having different densities ρ_N, and m_2 as secondary is a radiating point mass which is producing the modified Newtonian potential. These two bodies are moving in circular orbits around their common center of mass which is taken as origin O. The second primary m_2 is moving around the first primary m_1 in circular orbit also. There is a third infinitesimal body of mass m which is moving inside the outer layer (i.e. N^{th} – layer) of the primary m_1, this layer is filled with incompressible homogeneous viscous fluid. Therefore at the time of motion, body m has the following forces:

a. The gravitational force exerted by the fluid and buoyancy force in the N^th^ –layer i.e.

$$F_B = \frac{4\pi}{3}\rho_N mG(\frac{\rho_N}{\rho}-1)r_1$$

The gravitational force due to (N − 1) layers of an oblate heterogeneous primary m_1 i.e. (see [5])

$$F_1 = \frac{Gm_1^* m}{r_1^3} r_1 + \frac{3G}{2r_1^5}[h_1 - \frac{5h_3 z^2}{r_1^2}]\, r_1$$

The modified gravitational force due to the second primary m_2 i.e. (see [1])

$$F_2 = \frac{Gm_2 m(r_2^2-\varepsilon)}{(r_2^2+\varepsilon)^2 r_2} r_2$$

The force due to radiation pressure from the second primary i.e. (see [11])

$$F_2 - F_p = F_2(1-\frac{F_p}{F_2}) = F_2(1-p) = qF_2,$$

The viscous force i.e. $F_v = m(v_x, v_y, v_z) = m(-\alpha_0\dot{x}, -\alpha_0\dot{y}, -\alpha_0\dot{z}),$ with viscous constant α_0.

Hence the total force on m will be

$$F = \frac{Gm_1^* m}{r_1^3} r_1 + \frac{3G}{2r_1^5}[h_1 - \frac{5h_3 z^2}{r_1^2}]\, r_1 + \frac{Gm_2 mq(r_2^2-\varepsilon)}{(r_2^2+\varepsilon)^2 r_2} r_2 + Kmr_1 + F_v,$$

where

$$m_1^* = m_1 - \frac{4\pi}{3}(a_N b_N c_N - a_{N-1}b_{N-1}c_{N-1})\rho_N = m_1 - V_N\rho_N = m_1 - m_N,$$

m_N = the mass of the N^th^-layer,

p = Radiating force/Gravitational force,

$$K = \frac{4\pi}{3}\rho_N G(\frac{\rho_N}{\rho}-1),$$

$$h_1 = \frac{4\pi}{3}\sum_{i=1}^{N-1}\frac{1}{5}(\rho_i - \rho_{i+1})a_i b_i c_i(2a_i^2 - b_i^2 - c_i^2),$$

$$h_3 = \frac{4\pi}{3}\sum_{i=1}^{N-1}\frac{1}{5}(\rho_i - \rho_{i+1})a_i b_i c_i(a_i^2 - c_i^2).$$

For the non-dimensional units, we have $m_1 + m_2 = 1$, $G = 1$ and the separation distance between the primary and secondary is unity, and also $\mu = m_2/(m_1 + m_2)$. Hence $m_1 = 1-\mu$ and $m_1^* = 1-\mu-\mu_N$ with $\mu_N = m_N/(m_1+m_2)$. Therefore the equations of motion of the small body in the cartesian coordinate will be as follows:

$$\begin{aligned} \ddot{x} - 2n\alpha\dot{y} &= \cap_x + v_x, \\ \ddot{y} + 2n\alpha\dot{x} &= \cap_y + v_y, \\ \ddot{z} &= \cap_z + v_z, \end{aligned} \tag{1}$$

With, n is the mean motion of the system and given by Eq. (18). J_i and k are the dimensionless quantities of h_i and K respectively.

$$\cap_x = n^2\beta x + k(x+\mu) - \frac{\mu q(r_2^2-\varepsilon)(x+\mu-1)}{(r_2^2+\varepsilon)^2 r_2} - \frac{(1-\mu-\mu_N)(x+\mu)}{r_1^3} - \frac{3(x+\mu)}{2r_1^5}(J_1 - \frac{5J_3 z^2}{r_1^2}), \tag{2}$$

$$\cap_y = (n^2\beta + k - \frac{\mu q(r_2^2-\varepsilon)}{(r_2^2+\varepsilon)^2 r_2} - \frac{(1-\mu-\mu_N)}{r_1^3} - \frac{3}{2r_1^5}(J_1 - \frac{5J_3 z^2}{r_1^2}))y, \tag{3}$$

$$\cap_z = (k - \frac{\mu q(r_2^2-\varepsilon)}{(r_2^2+\varepsilon)^2 r_2} - \frac{(1-\mu-\mu_N)}{r_1^3} - \frac{3}{2r_1^5}(J_1 + 2J_3 - \frac{5J_3 z^2}{r_1^2}))z, \tag{4}$$

$$\begin{aligned} r_1^2 &= (x+\mu)^2 + y^2 + z^2, \\ r_2^2 &= (x+\mu-1)^2 + y^2 + z^2. \end{aligned} \tag{5}$$

3 Determination of Equilibrium points

For the equilibrium points we have to put zero to all the derivative with respect to time in the system (1), hence

$$n^2\beta x + k(x+\mu) - \frac{\mu q(r_2^2-\varepsilon)(x+\mu-1)}{(r_2^2+\varepsilon)^2 r_2} - \frac{(1-\mu-\mu_N)(x+\mu)}{r_1^3} - \frac{3(x+\mu)}{2r_1^5}(J_1 - \frac{5J_3 z^2}{r_1^2}) = 0, \tag{6}$$

$$(n^2\beta + k - \frac{\mu q(r_2^2-\varepsilon)}{(r_2^2+\varepsilon)^2 r_2} - \frac{(1-\mu-\mu_N)}{r_1^3} - \frac{3}{2r_1^5}(J_1 - \frac{5J_3 z^2}{r_1^2}))y = 0, \tag{7}$$

$$(k - \frac{\mu q(r_2^2-\varepsilon)}{(r_2^2+\varepsilon)^2 r_2} - \frac{(1-\mu-\mu_N)}{r_1^3} - \frac{3}{2r_1^5}(J_1 + 2J_3 - \frac{5J_3 z^2}{r_1^2}))z = 0. \tag{8}$$

After solving equations (6), (7) and (8), we can find the locations of equilibrium points in three cases.

3.1 Locations of collinear equilibrium points

Collinear equilibrium points can be obtained from equation

(6) by taking $x \neq 0, y = 0, z = 0$.

$$n^2\beta x + k(x+\mu) - \frac{\mu q((x+\mu-1)^2-\varepsilon)(x+\mu-1)}{((x+\mu-1)^2+\varepsilon)^2\left|x+\mu-1\right|}$$
$$-\frac{(1-\mu-\mu_N)(x+\mu)}{\left|x+\mu\right|^3} - \frac{3(x+\mu)J_1}{2\left|x+\mu\right|^5} = 0, \qquad (9)$$

When the first primary is a spherical shell and also m_2 is moving around m_1 as well as m is moving inside m_1 and also there are no perturbations, i.e. $r_2 = -(x+\mu-1)$, then Eq. (9) reduces to

$$x + k(x+\mu) + \frac{\mu}{(x+\mu-1)^2} = 0, \qquad (10)$$

and hence

$$(x+\mu)[(1+k)x^2 + (2k\mu - 2k + \mu - 2)x + (1 + k - 2k\mu + \mu^2 k)] = 0. \qquad (11)$$

The real solutions of Eq. (11) inside the spherical shell are

$x_1 = -\mu,$ and

$$x_2 = \frac{2-\mu+2k-2k\mu+\sqrt{\mu(\mu-4k-4)}}{2(1+k)}. \qquad (12)$$

Now when the first primary is heterogeneous body with N-layers then there are two cases for r_1.

First case: $r_1 = (x + \mu)$.

Then Eq. (9) reduces to

$$n^2\beta x + k(x+\mu) + \frac{\mu q((x+\mu-1)^2-\varepsilon)}{((x+\mu-1)^2+\varepsilon)^2}$$
$$-\frac{(1-\mu-\mu_N)}{(x+\mu)^2} - \frac{3J_1}{2(x+\mu)^4} = 0, \qquad (13)$$

Assume that $x_1 + R_1$, $|R_1| << 1$ and $x_2 + R_2$, $|R_2| << 1$ are the two real roots of Eq. (13). Putting these values of solutions in Eq. (13) and rejecting the second and higher powers of R_1, R_2 and J_1, we get

$$R_1 = \frac{1+\varepsilon}{2} \text{ and } R_2 = \frac{R_{2N}}{R_{2D}}, \qquad (14)$$

where,

$$R_{2N} = (2A_1^2 A_2 + J_1 - 2A_1^4(k + n^2 x_2 \beta))(A_3^2+\varepsilon)^2 - 2A_1^4 q(A_3^2-\varepsilon)\mu,$$
$$R_{2D} = 2(-6A_3 J_1(A_3^2+\varepsilon) + 4A_1^3 k(1 + A_1(-3+2A_1) + \varepsilon)(A_3^2+\varepsilon) - 2A_1 A_2(A_3^2+\varepsilon)(1 + x_2(-4 + 3x_2) + \varepsilon - 4\mu + 6x_2\mu + 3\mu^2) + A_1^3 n^2 \beta \times (A_3^2+\varepsilon)(4A_1 A_3 x_2 + (A_3^2+\varepsilon)(5x_2+\mu)) + 2A_1^3 q(2 + A_1(-5+3A_1) - 2\varepsilon)\mu),$$
$$A_1 = x_2 + \mu,\; A_2 = 1 - \mu - \mu_N,\; A_3 = A_1 - 1.$$

According to the values of R_1 and R_2, we get $(-\mu + \frac{1+\varepsilon}{2}, 0, 0)$ and $(x_2 + R_2, 0, 0)$ two equilibrium points.

Second case: $r_1 = -(x + \mu)$.

Then Eq. (9) reduces to

$$n^2\beta x + k(x+\mu) + \frac{\mu q((x+\mu-1)^2-\varepsilon)}{((x+\mu-1)^2+\varepsilon)^2}$$
$$+\frac{(1-\mu-\mu_N)}{(x+\mu)^2} + \frac{3J_1}{2(x+\mu)^4} = 0, \qquad (15)$$

Assume that $x_1 + R_3$, $|R_3| << 1$ and $x_2 + R_4$, $|R_4| << 1$ are the two real roots of Eq. (15). Putting these values of solutions in Eq. (15) and rejecting the second and higher powers of R_3, R_4 and J_1, we get

$$R_3 = \frac{1+\varepsilon}{2} \text{ and } R_4 = \frac{R_{4N}}{R_{4D}}, \qquad (16)$$

According to the values of R_3 and R_4, we get $(-\mu + \frac{1+\varepsilon}{2}, 0, 0)$ and $(x_2 + R_4, 0, 0)$ two equilibrium points

. Thus, the collinear equilibrium points are $(-\mu + \frac{1+\varepsilon}{2}, 0, 0)$, $(x_2 + R_2, 0, 0)$ and $(x_2 + R_4, 0, 0)$, provided these points lying inside the outer layer of oblate heterogeneous body.

3.2 Locations of non-collinear stationary points

Non-collinear stationary points can be obtained from equations (6) and (7) by taking z = 0, we get

$$(x-1+\mu)^2 + y^2 = 1 - \varepsilon = \ell^2. \qquad (17)$$

Which is the equation of circle with center at the center of the secondary and radius $\ell = \sqrt{1-\varepsilon}$. Provided

$$n^2 = \frac{q(1-\frac{3}{2}\varepsilon)}{\beta}. \qquad (18)$$

The general coordinate of the stationary point is $(1-\mu+\ell\,\mathrm{Cos}\,\alpha, \ell\,\mathrm{Sin}\,\alpha, 0)$, where α is a parameter. These points will be the coordinates of the stationary points when they will be in the outer layer of the oblate heterogeneous body.

3.3 Locations of out-of-plane stationary points

When the shape of the first primary is spherical then from equations (6) and (8) by taking y = 0, we get

$$x = -k,\; r_2 = \left(\frac{\mu}{k}\right)^{1/3}. \qquad (19)$$

Which confirms the result of [16]. Now when the shape of the first primary is heterogeneous with N-layers, then let

$$x = -k + \delta_1,\; r_2 = \beta_1 + \delta_2,\; \beta_1 = \left(\frac{\mu}{k}\right)^{1/3} (let),$$
$$r_1^2 = B_1^2 + 2\beta_1\delta_2 + 2\delta_1,\; z^2 = B_2 + 2\beta_1\delta_2 - 2\beta_2\delta_1,$$
$$\beta_2 = \mu - k - 1,\; (let),\quad \beta_3 = \beta_1^{-5}(let), \qquad (20)$$

$$B_1^2 = \beta_1^2 + 1 + 2\beta_2, \quad B_2 = \beta_1^2 - \beta_2^2, \; B_3 = B_1^{-1}, (let).$$

From Eqs. (6) and (8), we get

$$(n^2\beta + \frac{3J_3}{r_1^5})x + \frac{3J_3\mu}{r_1^5} + \frac{\mu q(r_2^2 - \varepsilon)}{(r_2^2 + \varepsilon)^2 r_2} = 0, \qquad (21)$$

Putting the valuesfrom Eq. (20) in Eqs. (21) and (8), correspondingly we get

$$\begin{aligned} &n^2\beta\beta_1\delta_1 - 3kq\delta_2 = C_1 k + C_2\mu, \\ &C_3\delta_1 + C_4\delta_2 = C_5, \end{aligned} \qquad (22)$$

Where

$$\begin{aligned} C_1 &= 3B_3^5 J_3\beta_1 + n^2\beta\beta_1 - q, \\ C_2 &= 3q\varepsilon\beta_3 - 3B_3^5 J_3\beta_1, \\ C_3 &= A_2 B_3^5, \\ C_4 &= 3q\mu\beta_1\beta_3 + B_3^5 A_2\beta_1, \\ C_5 &= -k + q\mu(\beta_1^2 - 3\varepsilon)\beta_3 + B_3^3 A_2 + \tfrac{3}{2}B_3^5(J_1 \\ &\quad + 2J_3) - \tfrac{15}{2}B_3^7 B_2 J_3. \end{aligned}$$

Solving equation (22) for δ_1 and δ_2 , we get

$$\delta_1 = \frac{C_1C_4k + C_2C_4\mu + 3kqC_5}{n^2\beta\beta_1C_4 + 3kqC_3},$$

$$\delta_2 = \frac{n^2\beta\beta_1C_5 - kC_1C_3 - \mu C_2C_3}{n^2\beta\beta_1C_4 + 3kqC_3}.$$

Therefore,

$$\begin{cases} x = -k + \delta_1, \\ z = \pm\sqrt{B_2 + 2\beta_1\delta_2 - 2\beta_2\delta_1} \end{cases}. \qquad (23)$$

The Eq. (23) represents the coordinates of out-of-plane provided these points are lie in the outer layer (N^{th}-layer) of the heterogeneous body.

4 Stability

We examine the stability of an equilibrium point (ξ_0, η_0, ζ_0) to observe the behaviour of small body's motion in its vicinity ($\xi_0 + \alpha_1$, $\eta_0 + \alpha_2$, $\zeta_0 + \alpha_3$), where (α_1, α_2, α_3) are small displacements from the equilibrium points. The variational equations for the system (1) can be written as

$$\begin{aligned} &\ddot{\alpha}_1 - 2n\alpha\dot{\alpha}_2 = (\textstyle\bigcap_{xx}^0 + v_{xx}^0)\alpha_1 + (\bigcap_{xy}^0 + v_{xy}^0)\alpha_2 \\ &+(\textstyle\bigcap_{xz}^0 + v_{xz}^0)\alpha_3 + v_{x\dot{x}}^0\dot{\alpha}_1 + v_{x\dot{y}}^0\dot{\alpha}_2 + v_{x\dot{z}}^0\dot{\alpha}_3, \\ &\ddot{\alpha}_2 + 2n\alpha\dot{\alpha}_1 = (\textstyle\bigcap_{yx}^0 + v_{yx}^0)\alpha_1 + (\bigcap_{yy}^0 + v_{yy}^0)\alpha_2 \\ &+(\textstyle\bigcap_{yz}^0 + v_{yz}^0)\alpha_3 + v_{y\dot{x}}^0\dot{\alpha}_1 + v_{y\dot{y}}^0\dot{\alpha}_2 + v_{y\dot{z}}^0\dot{\alpha}_3, \\ &\ddot{\alpha}_3 = (\textstyle\bigcap_{zx}^0 + v_{zx}^0)\alpha_1 + (\bigcap_{zy}^0 + v_{zy}^0)\alpha_2 \\ &+(\textstyle\bigcap_{zz}^0 + v_{zz}^0)\alpha_3 + v_{z\dot{x}}^0\dot{\alpha}_1 + v_{z\dot{y}}^0\dot{\alpha}_2 + v_{z\dot{z}}^0\dot{\alpha}_3. \end{aligned} \qquad (24)$$

Where the superscript 0 denotes the value at the corresponding equilibrium point. Here, $v_{\xi\eta}^0 = 0,$ and

$$v_{\xi\zeta}^0 = \begin{cases} 0, & \zeta \neq \dot{\xi}, \\ -\alpha_0, & \zeta = \dot{\xi} \end{cases}, \text{ where } (\xi, \eta) = (x, y, z).$$

The characteristic polynomial of equation (24) can be written as

$$p(\lambda) = \lambda^6 + d_5\lambda^5 + d_4\lambda^4 + d_3\lambda^3 + d_2\lambda^2 + d_1\lambda + d_0, \qquad (25)$$

with

$$\begin{aligned} d_5 &= -3\alpha_0, \\ d_4 &= -4n^2\alpha^2 - 3\alpha_0^2 + \textstyle\bigcap_{xx}^0 + \bigcap_{yy}^0 + \bigcap_{zz}^0, \\ d_3 &= \alpha_0[-4n^2\alpha^2 - \alpha_0^2 + 2(\textstyle\bigcap_{xx}^0 + \bigcap_{yy}^0 + \bigcap_{zz}^0)], \\ d_2 &= \alpha_0^2(\textstyle\bigcap_{xx}^0 + \bigcap_{yy}^0 + \bigcap_{zz}^0) + (\bigcap_{xy}^0)^2 + (\bigcap_{xz}^0)^2 + (\bigcap_{yz}^0)^2 \\ &+4n^2\alpha^2\textstyle\bigcap_{zz}^0 - (\bigcap_{xx}^0\bigcap_{yy}^0 + \bigcap_{xx}^0\bigcap_{zz}^0 + \bigcap_{yy}^0\bigcap_{zz}^0), \\ d_1 &= \alpha_0[(\textstyle\bigcap_{xy}^0)^2 + (\bigcap_{xz}^0)^2 + (\bigcap_{yz}^0)^2 - (\bigcap_{xx}^0\bigcap_{yy}^0 \\ &+\textstyle\bigcap_{xx}^0\bigcap_{zz}^0 + \bigcap_{yy}^0\bigcap_{zz}^0)], \\ d_0 &= -[\textstyle\bigcap_{xx}^0(\bigcap_{yz}^0)^2 + \bigcap_{yy}^0(\bigcap_{xz}^0)^2 + \bigcap_{zz}^0(\bigcap_{xy}^0)^2] \\ &+2\textstyle\bigcap_{xy}^0\bigcap_{yz}^0\bigcap_{zx}^0 + 2\bigcap_{xx}^0\bigcap_{yy}^0\bigcap_{zz}^0. \end{aligned}$$

Now $p(\lambda) \to \infty \; as \; \lambda \to \infty$ and $p(0) = d_0$. Here the stability of the equilibrium points will depend on the value of d_0, i.e. if $d_0 < 0$ then there will be at least one positive root, so the equilibrium points will be unstable.

5 Conclusion

New kind of Robe's problem is investigated where one of the primaries as heterogeneous body with N-layers having different densities and second radiating primary is producing the modified Newtonian potential. Here the third infinitesimal body is moving inside the N th -layer of the first primary having viscous incompressible fluid. We have determined the equations of motion which is depend on the density parameters of the heterogeneous body and the modified parameter of the secondary. The location of equilibrium points are evaluated where we found three collinear equilibrium points, infinite numbers of circular equilibrium points and two out-of-plane equilibrium points provided these points lie inside the N th -layer of the heterogeneous body. In this case we found three-collinear equilibrium points which are depending on the taken parameters while in the classical case only two collinear equilibrium points are exist. Finally we have examined the stability of equilibrium points which all are unstable.

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