SOLVABILITY FOR A SYSTEM OF FRACTIONAL STURM-LIOUVILLE LANGEVIN EQUATIONS

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ABSTRACT: *In this paper, we study a new problem of coupled systems of Sturm-Liouville and langevin fractional differential equations. By means of the contraction mapping principle and O'Regan theorem, we present some existence result for the studied problem. Then, some Ulam type stabilities are analyzed. At the end, an illustrative example is discussed.*

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1- INTRODUCTION:

Fractional differential equations and systems arise in variety engineering and scientific disciplines. For some recent development on this theory, we refer the reader to [1, 2, 4, 5, 9, 10, 11, 12, 14, 23, 25, 27, 28, 29 and 30]. In a series of articles dating from $1836-1837$ Sturm and Liouville created a whole new subject in mathematical analysis. The theory, (later known as Sturm-Liouville theory) plays an important role in different many areas of science, for example, engineering, mathematics, physics, chemistry, etc... For more details, we refer the interested reader to [6, 17, 22, 26 and 31]. A standard form of the Sturm-Liouville differential equation is given as:

$$-\frac{d}{dt}[g(t)\frac{dy}{dt}]+h(t)y=\lambda f(t)y,\quad \lambda>0,\quad t\in[\alpha,\beta],$$

with separated boundary conditions of the form

$$\begin{cases}\alpha_1 y(\alpha)+\alpha_2 y'(\alpha)=0,\quad (\alpha_1)^2+(\alpha_2)^2>0,\\ \beta_1 y(\beta)+\beta_2 y'(\beta)=0,\quad (\beta_1)^2+(\beta_2)^2>0.\end{cases}$$

where the functions g and h are continuous over $[\alpha,\beta]$ such that $g(t)>0$, and

$f:[\alpha,\beta]\times \mathbb{R}\to\mathbb{R}^+$ is a continuous function.

The Langevin equation (first formulated by Langevin in 1908) is found to be an effective tool to describe the evolution of physical phenomena in fluctuating environments, in [19], Langevin introduced the classical Langevin equation as follows:

$$\frac{d}{dt}[\frac{dy}{dt}+\lambda h(t)]=f(t)y,\quad t\in[\alpha,\beta],\quad \lambda>0.$$

With various boundary conditions, the above problem has been studied by many authors, see for instance the works [3, 7, 19, 20, 21]. In addition, the Ulam stability of fractional differential equations can be considered as a new way for the researchers. Truthfully, we can inspect from it several topics in nonlinear analysis problems. Moreover, the analysis on stability of fractional order differential equations is more complex than that of classical differential equations, Ulam type stability problems has been attracted by many researchers, see [13].

In [17], Kiataramkul et al. have investigated the following existence of fractional order differential equation:

$$\begin{cases}D^\beta[p(t)D^\alpha+r(t)]x(t)=g(t,x(t)),\quad 1<t<T,\\ x(1)=-x(T),\quad D^\alpha x(1)=-D^\alpha x(T),\end{cases}$$

where D^ρ denotes the Caputo-type Hadamard derivative of order $\rho\in\{\alpha,\beta\}$, $p:[1,T]\to\mathbb{R}$ is continuous and $f\in C([1,T]\times\mathbb{R},\mathbb{R})$.

In [22], the autohrs have considered the following systems of fractional differential equations with anti-periodic boundary conditions:

$$\begin{cases}D^{\alpha_2}[p(t)D^{\alpha_1}+r(t)]x(t)=f(t,x(t),y(t)),\quad 0<t<T,\\ D^{\beta_2}[q(t)D^{\beta_1}+s(t)]x(t)=g(t,x(t),y(t)),\quad 0<t<T,\\ x(0)=-x(T),D^{\alpha_1}x(0)=-D^{\alpha_1}x(T),\\ y(0)=-y(T),D^{\beta_1}x(0)=-D^{\beta_1}x(T),\end{cases}$$

where D^δ is the Caputo fractional derivative of orders $\delta\in\{\alpha_1,\alpha_2,\beta_1,\beta_2\}$, $f,g\in C([0,T]\times\mathbb{R}^2,\mathbb{R})$ and $p,q\in C([0,T],\mathbb{R}-\{0\})$.

In [8], the authors have studied another boundary value problem of system of generalized Sturm-Liouville and Langevin Hadamard fractional differential equations. The existence and uniqueness of solutions have been proved via Banach contraction principle fixed point theorem, also, the

Ulam-Hyers stability has also been addressed for the proposed problem.

Recently in [24], the authors have proposed an approach to the fractional version of the Sturm-Liouville and Langevin problems.

In this paper, we are concerned with the following coupled system of Langevin tpye:

$$\begin{cases} D^{\alpha_2}[g_1(t)D^{\alpha_1}+h_1(t)]x_1(t)= f_1(t,x_1(t),x_2(t),D^{\delta_1}x_1(t),D^{\delta_2}x_2(t)), & 0<t<1, \\ D^{\beta_2}[g_2(t)D^{\beta_1}+h_2(t)]x_2(t)= f_2(t,x_1(t),x_2(t),D^{\delta_1}x_1(t),D^{\delta_2}x_2(t)), & 0<t<1, \end{cases} \cdots (1)$$

with the boundary conditions:

$$\begin{cases} x_1(0)+x_1(1)=0, & D^{\alpha_1}x_1(0)+D^{\alpha_1}x_1(1)=0, \\ x_2(0)+x_2(1)=0, & D^{\alpha_2}x_2(0)+D^{\alpha_2}x_2(1)=0, \end{cases} \cdots (2)$$

where D^{γ} is the Caputo fractianal derivative of orders $\gamma\in\{\alpha_1,\alpha_2,\beta_1,\beta_2,\delta_1,\delta_2\}$ with $0<\alpha_1,\alpha_2,\beta_1,\beta_2,\delta_1,\delta_2<1$ with $\delta_1<\alpha_1,\ \delta_2<\alpha_2$ and $f_1,\ f_2\in C([0,1]\times \mathbb{R}^4,\mathbb{R}),\ \ g_1,g_2\in C([0,1],\mathbb{R}-\{0\})$ with $|g_1(t)|>1, |g_2(t)|>1,\ h_1,h_1\in C([0,1],\mathbb{R}).$

Note that system (1)-(2) is a generalization of Sturm-Liouville and Langevin fractional differential systems.

We arrange the rest of the paper as follows. Section 2 contains an auxiliary result that plays a key role in analyzing the given problem. The main results for the problem (1)-(2) are discussed in section 3. We give an existence and uniqueness result with the dael of ORegan's theorem. In Section 4, we investigate some types of Ulam stability for this fractional system. Finally, the paper was appended examples which illustrate the applicability of the results in Section 5.

2- Basic Definitions and Relevant Lemmas

In this section, we introduce some fractional calculus notations and definitions that will be used in this paper, for more details, see [14, 15].

Definition 1: *The Riemann-Liouville fractional integral of order* $\alpha\ge 0$ *for a continuous function* $f\ :[a,b]\to\mathbb{R}$ *is defined as:*

$$J_a^{\alpha}f(t)= \begin{cases} \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}f(\tau)d\tau, & \alpha>0, \quad a<t\le b, \quad f(t), \quad \alpha=0, \\ f(t), & \alpha=0, \quad a<t\le b, \end{cases}$$

where $\Gamma(\alpha):=\int_0^{\infty}e^{-u}u^{\alpha-1}du.$

Note that for $\alpha>0,\beta>0$, we have:

$$J_a^{\alpha}J_a^{\beta}f(t)=J_a^{\alpha+\beta}f(t).$$

Definition 2: *The Caputo fractional derivative of order* $\alpha>0$ *of a function* $f\in C^n([a,b],\mathbb{R})$ *is given by:*

$$D^{\alpha}f(t) = J^{n-\alpha}D^nf(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t(t-s)^{n-\alpha-1}f^{(n)}(s)ds,$$

Where $n=[\alpha]+1$ *and* $[\alpha]$ *denotes the integer part of positive real number* α *, and* $\Gamma(.)$ *is the Gamma function.*

Definition 3: *The Mittag-Leffler function is an entire function defined by the series*

$$E_{\alpha}(z)=\sum_{k=0}^{\infty}\frac{(z)^k}{\Gamma(k\alpha+1)},\ \alpha>0,$$

where $\Gamma(.)$ *is the Gamma function.*

The following lemmas give some properties of fractional calculus theory, see [16, 18].

Lemma 4: *For* $\alpha>0, n\in\mathbb{N}^*$ *, the general solution of the equation* $D^{\alpha}x(t)=0$ *is given by*

$$x(t)=c_0+c_1t+c_2t^2+\cdots+c_{n-1}t^{n-1},$$

where $c_i\in\mathbb{R}, i=0,1,\cdots,n-1, n=[\alpha]+1.$

Lemma 5: *For* $\alpha>0,\ n\in\mathbb{N}^*$

$$J^{\alpha}[D^{\alpha}x(t)]=x(t)+c_0+c_1t+c_2t^2+\ldots+c_{n-1}t^{n-1},$$

for some $c_i\in\mathbb{R}, i=0,1,\cdots,n-1, n=[\alpha]+1$.

Lemma 6: *Let* $q>p>0$ *and* $f\in L^1([a,b])$ *. Then*

$$D^pJ^q[f(t)]=J^{q-p}[f(t)],\ \ t\in[a,b].$$

The following lemma is crucial for our results:

Lemma 7: *Let* Ω *be an open subset of a closed and convex set* Y *in Banach space* X *. Assume that* $0\in\Omega$ *and* $\Psi(\overline{\Omega})$ *is bounded, where* $\Psi:\overline{\Omega}\to Y$ *is given by* $\Psi=\varphi_1+\varphi_2$ *, in which* $\varphi_1:\overline{\Omega}\to X$ *is completely* *contimuous and* $\varphi_2 : \overline{\Omega} \to X$ *is called nonlinear contraction (i.e. there exists a nonnegative nondecreasing function* $\omega : [0,\infty) \to [0,\infty)$ *such that* $\omega(x) \le x, \forall x > 0$ *and* $\|\varphi_2(x) - \varphi_2(y)\| \le \omega\|x - y\|, \ \forall x, y \in \Omega$ *. Then, either* Ψ *has a fixed point* $x \in \overline{\Omega}$ *or, there exists a point* $x \in \partial\Omega$ *and* $\lambda \in [0,1]$ *, with* $x = \lambda \Psi x$ *, where* $\partial\Omega$ *represent the boundary of* Ω *.*

We prove the following lemma too:

Lemma 8: *Let* $f_1, f_2 \in (C[0,1], \mathrm{R})$ *be tow given functions. Then the solution of the problem*

$$\begin{cases} D^{\alpha_2}\left[g_1(t)D^{\alpha_1} + h_1(t)\right]x_1(t) = f_1(t), & 0 < t < 1, \\ D^{\beta_2}\left[g_2(t)D^{\beta_1} + h_2(t)\right]x_2(t) = f_2(t), & 0 < t < 1, \end{cases}$$

is given by the following expression:

$$\begin{aligned} x_1(t) \ = \ & \int_0^t \frac{(t-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_1^{-1}(\tau) \int_0^\tau \frac{(\tau-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} f_1(s)dsd\tau - \int_0^t \frac{(t-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_1^{-1}(\tau)h_1(\tau)x_1(\tau)d\tau \\ - \ & \frac{1}{2}\int_0^1 \frac{(1-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_1^{-1}(\tau) \int_0^\tau \frac{(\tau-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} f_1(s)dsd\tau + \frac{1}{2}\int_0^1 \frac{(t-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_1^{-1}(\tau)h_1(\tau)x_1(\tau)d\tau \\ - \ & \mathbf{K}_1\left(\int_0^t \frac{(t-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_1^{-1}(\tau)d\tau + \frac{1}{2}\int_0^1 \frac{(1-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_1^{-1}(\tau)\right) \times \\ & \left[g_1^{-1}(1)\int_0^1 \frac{(1-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)} f_1(\tau)d\tau + \left(g_1^{-1}(0)h_1(0) - g_1^{-1}(1)h_1(1)\right)x_1(1)\right], \qquad \text{....} \quad (3) \end{aligned}$$

and

$$\begin{aligned} x_2(t) \ = \ & \int_0^t \frac{(t-\tau)^{\beta_1-1}}{\Gamma(\beta_1)} g_2^{-1}(\tau) \int_0^\tau \frac{(\tau-s)^{\beta_2-1}}{\Gamma(\beta_2)} f_2(s)dsd\tau - \int_0^t \frac{(t-\tau)^{\beta_1-1}}{\Gamma(\beta_1)} g_2^{-1}(\tau)h_2(\tau)x_2(\tau)d\tau \\ - \ & \frac{1}{2}\int_0^1 \frac{(1-\tau)^{\beta_1-1}}{\Gamma(\beta_1)} g_2^{-1}(\tau) \int_0^\tau \frac{(\tau-s)^{\beta_2-1}}{\Gamma(\beta_2)} f_2(s)dsd\tau + \frac{1}{2}\int_0^1 \frac{(t-\tau)^{\beta_1-1}}{\Gamma(\beta_1)} g_2^{-1}(\tau)h_2(\tau)x_2(\tau)d\tau \\ - \ & \mathbf{K}_2\left(\int_0^t \frac{(t-\tau)^{\beta_1-1}}{\Gamma(\beta_1)} g_2^{-1}(\tau)d\tau + \frac{1}{2}\int_0^1 \frac{(1-\tau)^{\beta_1-1}}{\Gamma(\beta_1)} g_2^{-1}(\tau)\right) \times \\ & \left[g_2^{-1}(1)\int_0^1 \frac{(1-\tau)^{\beta_2-1}}{\Gamma(\beta_2)} f_2(\tau)d\tau + \left(g_2^{-1}(0)h_2(0) - g_2^{-1}(1)h_2(1)\right)x_2(1)\right]. \qquad \text{....} \quad (4) \end{aligned}$$

where

$$\mathbf{K}_i := \frac{1}{g_i^{-1}(1) + g_i^{-1}(0)}, \quad i = 1, 2.$$

Proof: Let $c_i, d_i \in \mathrm{R}, i = 0;1$. Hence, it yields that

$$\begin{aligned} x_1(t) = & \int_0^t \frac{(t-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_1^{-1}(\tau) \int_0^\tau \frac{(\tau-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} f_1(s)dsd\tau - \int_0^t \frac{(t-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_1^{-1}(\tau)h_1(\tau)x_1(\tau)d\tau \\ & - c_0\int_0^t \frac{(t-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_1^{-1}(\tau)d\tau - c_1, \qquad \text{....} \quad (5) \end{aligned}$$

and

$$\begin{aligned} x_2(t) = & \int_0^t \frac{(t-\tau)^{\beta_1-1}}{\Gamma(\beta_1)} g_2^{-1}(\tau) \int_0^\tau \frac{(\tau-s)^{\beta_2-1}}{\Gamma(\beta_2)} f_2(s)dsd\tau - \int_0^t \frac{(t-\tau)^{\beta_1-1}}{\Gamma(\beta_1)} g_2^{-1}(\tau)h_2(\tau)x_2(\tau)d\tau \\ & - d_0\int_0^t \frac{(t-\tau)^{\beta_1-1}}{\Gamma(\beta_1)} g_2^{-1}(\tau)d\tau - d_1. \qquad \text{....} \quad (6) \end{aligned}$$

Thanks to boundary conditions (2), we get

$$c_0 = \frac{1}{g_1^{-1}(0)+g_1^{-1}(1)}\left[g_1^{-1}(1)\int_0^1 \frac{(1-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)} f_1(\tau)d\tau + \left[g_1^{-1}(1)h_1(1)-g_1^{-1}(0)h_1(0) \mid x_1(1)\right]\right],$$

and

$$d_0 = \frac{1}{g_2^{-1}(0)+g_2^{-1}(1)}\left[g_2^{-1}(1)\int_0^1 \frac{(1-\tau)^{\beta_2-1}}{\Gamma(\beta_2)} f_1(\tau)d\tau + \left[g_2^{-1}(1)h_2(1)-g_1^{-1}(0)h_2(0)\right)x_2(1)\right].$$

On the other hand, we have

$$\begin{aligned} 2c_1 \quad = \quad & \frac{1}{2}\int_0^1 \frac{(1-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_1^{-1}(\tau)\int_0^\tau \frac{(\tau-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} f_1(s)dsd\tau - \frac{1}{2}\int_0^1 \frac{(t-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_1^{-1}(\tau)h_1(\tau)x_1(\tau)d\tau \\ - \quad & \frac{K_1}{2}\int_0^1 \frac{(1-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_1^{-1}(\tau)d\tau\left[g_1^{-1}(1)\int_0^1 \frac{(1-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)} f_1(\tau)d\tau + \left(g_1^{-1}(0)h_1(0)-g_1^{-1}(1)h_1(1)\right)x_1(1)\right], \end{aligned}$$

and

$$\begin{aligned} 2d_1 \quad = \quad & \frac{1}{2}\int_0^1 \frac{(1-\tau)^{\beta_1-1}}{\Gamma(\beta_1)} g_1^{-1}(\tau)\int_0^\tau \frac{(\tau-s)^{\beta_2-1}}{\Gamma(\beta_2)} f_2(s)dsd\tau - \frac{1}{2}\int_0^1 \frac{(t-\tau)^{\beta_1-1}}{\Gamma(\beta_1)} g_2^{-1}(\tau)h_2(\tau)x_2(\tau)d\tau \\ - \quad & \frac{K_2}{2}\int_0^1 \frac{(1-\tau)^{\beta_1-1}}{\Gamma(\beta_1)} g_2^{-1}(\tau)d\tau\left[g_1^{-1}(1)\int_0^1 \frac{(1-\tau)^{\beta_2-1}}{\Gamma(\beta_2)} f_2(\tau)d\tau + \left(g_2^{-1}(0)h_2(0)-g_2^{-1}(1)h_2(1)\right)x_2(1)\right]. \end{aligned}$$

Substituting the values in (5) and (6), we get (3) and (4), and this completes the proof.

3- Main results:

This section focuses on two results. The first is concerned with the existence of a solution of the problem considered. While in the second result, we will investigate the issue of at least one solution for (1)-(2).

First of all, let us consider the Banach space:

$$X_1 = \left\{x_1 \mid x_1 \in C([0,1],\mathrm{R}), D^{\delta_1}x_1 \in C([0,1],\mathrm{R})\right\},$$

and the norm

$$\|x_1\|_{X_1} = \|x_1\| + \left\|D^{\delta_1}x_1\right\| = \sup_{t\in[0,1]}|x_1(t)| + \sup_{t\in[0,1]}\left|D^{\delta_1}x_1(t)\right|.$$

Then $(X_1, \|x_1\|_{X_1})$ is a Banach space. Similarly we define the space

$$X_2 = \left\{x_2 \mid x_2 \in C([0,1],\mathrm{R}), D^{\delta_2}x_2 \in C([0,1],\mathrm{R})\right\},$$

and the norm

$$\|x_2\|_{X_2} = \|x_2\| + \left\|D^{\delta_2}x_2\right\| = \sup_{t\in[0,1]}|x_2(t)| + \sup_{t\in[0,1]}\left|D^{\delta_2}x_2(t)\right|.$$

Also $(X_2, \|x_2\|_{X_2})$ is a Banach space.

Certainly $(X_1 \times X_2, \|.\|_{X_1\times X_2})$ is a Banach space.

Equipped with norm $\|(x_1,x_2)\|_{X_1\times X_2} = \|x_1\|_{X_1} + \|x_2\|_{X_2}$ for $(x_1,x_2)\in X_1\times X_2$.

In view of Lemma 8, me define an operator $\Psi : X_1\times X_2 \to X_1\times X_2$ by

$$\Psi(x_1,x_2)(t) := \left(\Psi_1(x_1,x_2)(t), \Psi_2(x_1,x_2)(t)\right).$$

As follows:

$$
\begin{aligned}
\Psi_1(x_1,x_2)(t) \;=\;& \int_0^t \frac{(t-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_1^{-1}(\tau) \int_0^\tau \frac{(\tau-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} f_1\left(s,x_1(s),x_2(s),D^{\delta_1}x_1(s),D^{\delta_2}x_2(s)\right) ds\, d\tau \\
-\;& \frac{1}{2}\int_0^1 \frac{(1-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_1^{-1}(\tau) \int_0^\tau \frac{(\tau-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} f_1\left(s,x_1(s),x_2(s),D^{\delta_1}x_1(s),D^{\delta_2}x_2(s)\right) ds\, d\tau \\
+\;& \frac{1}{2}\int_0^1 \frac{(t-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_1^{-1}(\tau) h_1(\tau) x_1(\tau) d\tau - \int_0^t \frac{(t-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_1^{-1}(\tau) h_1(\tau) x_1(\tau) d\tau \\
-\;& \mathbf{K}_1\left(\int_0^t \frac{(t-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_1^{-1}(\tau) d\tau + \frac{1}{2}\int_0^1 \frac{(1-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_1^{-1}(\tau)\right) \times \\
& \left[g_1^{-1}(1)\int_0^1 \frac{(1-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)} f_1\left(\tau,x_1(\tau),x_2(\tau),D^{\delta_1}x_1(\tau),D^{\delta_2}x_2(\tau)\right) d\tau \right. \\
+\;& \left. \left(g_1^{-1}(0)h_1(0) - g_1^{-1}(1)h_1(1)\right) x_1(1) \right], \qquad \ldots\ (7)
\end{aligned}
$$

and

$$
\begin{aligned}
\Psi_2(x_1,x_2)(t) \;=\;& \int_0^t \frac{(t-\tau)^{\beta_1-1}}{\Gamma(\beta_1)} g_2^{-1}(\tau) \int_0^\tau \frac{(\tau-s)^{\beta_2-1}}{\Gamma(\beta_2)} f_2\left(s,x_1(s),x_2(s),D^{\delta_1}x_1(s),D^{\delta_2}x_2(s)\right) ds\, d\tau \\
-\;& \frac{1}{2}\int_0^1 \frac{(1-\tau)^{\beta_1-1}}{\Gamma(\beta_1)} g_2^{-1}(\tau) \int_0^\tau \frac{(\tau-s)^{\beta_2-1}}{\Gamma(\beta_2)} f_2\left(s,x_1(s),x_2(s),D^{\delta_1}x_1(s),D^{\delta_2}x_2(s)\right) ds\, d\tau \\
+\;& \frac{1}{2}\int_0^1 \frac{(t-\tau)^{\beta_1-1}}{\Gamma(\beta_1)} g_2^{-1}(\tau) h_2(\tau) x_2(\tau) d\tau - \int_0^t \frac{(t-\tau)^{\beta_1-1}}{\Gamma(\beta_1)} g_2^{-1}(\tau) h_2(\tau) x_2(\tau) d\tau \\
-\;& \mathbf{K}_2\left(\int_0^t \frac{(t-\tau)^{\beta_1-1}}{\Gamma(\beta_1)} g_2^{-1}(\tau) d\tau + \frac{1}{2}\int_0^1 \frac{(1-\tau)^{\beta_1-1}}{\Gamma(\beta_1)} g_2^{-1}(\tau)\right) \times \\
& \left[g_2^{-1}(1)\int_0^1 \frac{(1-\tau)^{\beta_2-1}}{\Gamma(\beta_2)} f_2\left(\tau,x_1(\tau),x_2(\tau),D^{\delta_1}x_1(\tau),D^{\delta_2}x_2(\tau)\right) d\tau \right. \\
+\;& \left. \left(g_2^{-1}(0)h_2(0) - g_2^{-1}(1)h_2(1)\right) x_2(1) \right] \qquad \ldots\ (8)
\end{aligned}
$$

For calculation convenience, we introduce the quantities:

$$A_1 = \frac{3}{2} sup_{t\in[0,1]} |g_1^{-1}(t)| \left(\frac{1}{\Gamma(\alpha_1+\alpha_2+1)} + \frac{|k_1||g_1^{-1}(1)|}{\Gamma(\alpha_1+1)\,\Gamma(\alpha_2+1)}\right),$$

$$A_2 = \frac{3}{2} sup_{t\in[0,1]} |g_2^{-1}(t)| \left(\frac{1}{\Gamma(\beta_1+\beta_2+1)} + \frac{|k_2||g_2^{-1}(1)|}{\Gamma(\beta_1+1)\,\Gamma(\beta_2+1)}\right),$$

$$B_1 = \frac{3}{2} sup_{t\in[0,1]} |g_1^{-1}(t)| \left(\frac{|k_1|\left(g_1^{-1}(0)h_1(0) - g_1^{-1}(1)h_1(1)\right)}{\Gamma(\alpha_1+1)} + \frac{\left|sup_{t\in[0,1]} h_1(t)\right|}{\Gamma(\alpha_1+1)}\right),$$

$$B_2 = \frac{3}{2} sup_{t\in[0,1]} |g_2^{-1}(t)| \left(\frac{|k_2|(g_2^{-1}(0)h_2(0) - g_2^{-1}(1)h_2(1))}{\Gamma(\beta_1+1)} + \frac{\left|sup_{t\in[0,1]} h_2(t)\right|}{\Gamma(\beta_1+1)}\right),$$

$$M_1 = sup_{t\in[0,1]} |g_1^{-1}(t)| \left(\frac{1}{\Gamma(\alpha_1+\alpha_2+1-\delta_1)} + \frac{|k_1||g_1^{-1}(1)|}{\Gamma(\alpha_1+1-\delta_1)\,\Gamma(\alpha_2+1)}\right),$$

$$M_2 = sup_{t\in[0,1]} |g_1^{-1}(t)| \left(\frac{1}{\Gamma(\alpha_1+\alpha_2+1-\delta_1)} + \frac{|k_1||g_1^{-1}(1)|}{\Gamma(\alpha_1+1-\delta_1)\,\Gamma(\alpha_2+1)}\right)$$

$$N_1 = sup_{t\in[0,1]}|g_1^{-1}(t)|\left(\frac{|k_1|(g_1^{-1}(0)h_1(0)-g_1^{-1}(1)h_1(1))}{\Gamma(\alpha_1+1-\delta_1)}+\frac{|sup_{t\in[0,1]}h_1(t)|}{\Gamma(\alpha_1+1-\delta_1)}\right),$$

$$N_2 = sup_{t\in[0,1]}|g_2^{-1}(t)|\left(\frac{|k_2|(g_2^{-1}(0)h_2(0)-g_2^{-1}(1)h_2(1))}{\Gamma(\beta_1+1-\delta_2)}+\frac{|sup_{t\in[0,1]}h_2(t)|}{\Gamma(\beta_1+1-\delta_2)}\right).$$

In the following, we need the assumptions:

(H_1): There exist non negative constants $l_{i,j}$, $i=1,2,$ and $j=1\ldots4,$ such that for each $t\in[0,1]$ and for all $u_j,v_j\in R,\ j=1\ldots4,$ we have

$$|f_i(t,u_1,u_2,u_3,u_4)-f_i(t,v_1,v_2,v_3,v_4)|\le\sum_{j=1}^{4}l_{i,j}|u_j-v_j|,\quad L_i=\max(l_{i,j}),\quad i=1,2.$$

(H_2): The functions $f_i\ :[0,1]\times\mathbb{R}^4\to\mathbb{R}$ are a jointly continuous functions.

(H_3): There exist nonnegative functions $\Phi_1,\Phi_2\in C([0,1],\mathbb{R})$, and nondecreasing functions $\phi_j,\ \phi_j\ :[0,\infty)\to[0,\infty)$, $j=1?4$, for each $t\in[0,1]$ and for all $x_j\in\mathbb{R}$ such that

$$|f_i(t,x_1,x_2,x_3,x_4)|\le\Phi_i(t)\left[\sum_{j=1}^{4}\phi_i(\|x_j\|)\right],\quad i=1,2.$$

(H_4): There exists $\chi_1,\chi_2\in C^1([0,1],\mathbb{R})$ and there exists $\varepsilon_1,\varepsilon_2>0$. For each $t\in[0,1]$

$|\chi_1|\le\varepsilon_1E_{\alpha_2}$,

and

$|\chi_2|\le\varepsilon_2E_{\beta_2}$.

For simplicity, we define

$$F_{x,i}(t)=f_i(t,x_1(t),x_2(t),D^{\delta_1}x_1(t),D^{\delta_2}x_2(t)),\quad i=1,2.$$

And

$$\Delta\ :=\sum_{i=1}^{2}L_i\left(A_i+M_i\right)+B_i+N_i.$$

3.1- Existence of Unique Solutions:

Our main result is given by the following theorem:

Theorem *9: Assume that (H1) holds and suppose that* $0<\Delta<1$ *. Then the problem (1)-(2) has a unique solution on* $[0,1]$.

Proof: The procedure is performed in two steps:

Step 1: We show that the operator Ψ is contractive. So, we take $x_i,y_i\in X_i,\ i=1,2$. Then, for each $t\in[0,1]$, we have

$$\begin{aligned}|\Psi_1(x_1,x_2)(t)-\Psi_1(y_1,y_2)(t)| \le\ & \int_0^t\frac{(t-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}|g_1^{-1}(\tau)|\int_0^\tau\frac{(\tau-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}|F_{x,1}(s)-F_{y,1}(s)|\,ds\,d\tau\\
&+\frac{1}{2}\int_0^1\frac{(1-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}|g_1^{-1}(\tau)|\int_0^\tau\frac{(\tau-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}|F_{x,1}(s)-F_{y,1}(s)|\,ds\,d\tau\\
&+\frac{1}{2}\int_0^1\frac{(t-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}|g_1^{-1}(\tau)||h_1(\tau)||x_1(\tau)-y_1(\tau)|\,d\tau\\
&+\int_0^t\frac{(t-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}|g_1^{-1}(\tau)||h_1(\tau)||x_1(\tau)-y_1(\tau)|\,d\tau\\
&+|\mathbf{K}_1|\left(\int_0^t\frac{(t-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}|g_1^{-1}(\tau)|\,d\tau+\frac{1}{2}\int_0^1\frac{(1-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}|g_1^{-1}(\tau)|\right)\times\\
&\quad\left[|g_1^{-1}(1)|\int_0^1\frac{(1-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)}|F_{x,1}(\tau)-F_{y,1}(\tau)|\,d\tau\right.\\
&\quad\left.+\left(g_1^{-1}(0)h_1(0)-g_1^{-1}(1)h_1(1)\right)|x_1(1)-y_1(1)|\right],\end{aligned}$$

and

$$
\begin{aligned}
|\Psi_2(x_1,x_2)(t)-\Psi_2(y_1,y_2)(t)| \leq & \int_0^t \frac{(t-\tau)^{\beta_1-1}}{\Gamma(\beta_1)}|g_2^{-1}(\tau)|\int_0^\tau \frac{(\tau-s)^{\beta_2-1}}{\Gamma(\beta_2)}|F_{x,2}(s)-F_{y,2}(s)|dsd\tau \\
+ & \frac{1}{2}\int_0^1 \frac{(1-\tau)^{\beta_1-1}}{\Gamma(\beta_1)}|g_2^{-1}(\tau)|\int_0^\tau \frac{(\tau-s)^{\beta_2-1}}{\Gamma(\beta_2)}|F_{x,2}(s)-F_{y,2}(s)|dsd\tau \\
+ & \frac{1}{2}\int_0^1 \frac{(t-\tau)^{\beta_1-1}}{\Gamma(\beta_1)} g_2^{-1}(\tau)h_2(\tau)|x_2(\tau)-y_2(\tau)|d\tau \\
+ & \int_0^t \frac{(t-\tau)^{\beta_1-1}}{\Gamma(\beta_1)}|g_2^{-1}(\tau)||h_2(\tau)||x_2(\tau)-y_2(\tau)|d\tau \\
+ & |\mathbf{K}_2|\left(\int_0^t \frac{(t-\tau)^{\beta_1-1}}{\Gamma(\beta_1)}|g_2^{-1}(\tau)|d\tau+\frac{1}{2}\int_0^1 \frac{(1-\tau)^{\beta_1-1}}{\Gamma(\beta_1)}|g_2^{-1}(\tau)|\right)\times \\
& \left[g_2^{-1}(1)\int_0^1 \frac{(1-\tau)^{\beta_2-1}}{\Gamma(\beta_2)}|F_{x,2}(\tau)-F_{y,2}(\tau)|d\tau\right. \\
+ & \left.\left(g_2^{-1}(0)h_2(0)-g_2^{-1}(1)h_2(1)\right)|x_2(1)-y_2(1)|\right].
\end{aligned}
$$

Using (H1), we obtain:

$$
\begin{aligned}
\left\|\Psi_1(x_1,x_2)-\Psi_1(y_1,y_2)\right\| \leq & \frac{3}{2}\sup_{t\in[0,1]}\left|g_1^{-1}(t)\right| \times \\
& \left[L_1\left(\|x_1-y_1\|_{X_1}+\|x_2-y_2\|_{X_2}\right)\left(\frac{1}{\Gamma(\alpha_1+\alpha_2+1)}+\frac{|\mathbf{K}_1|\left|g_1^{-1}(1)\right|}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\right)\right. \\
+ & \left.\|x_1-y_1\|_\infty\left(\frac{|\mathbf{K}_1|\left(g_1^{-1}(0)h_1(0)-g_1^{-1}(1)h_1(1)\right)}{\Gamma(\alpha_1+1)}+\frac{|\sup_{t\in[0,1]} h_1(t)|}{\Gamma(\alpha_1+1)}\right)\right],
\end{aligned}
$$

and

$$
\begin{aligned}
\left\|\Psi_2(x_1,x_2)-\Psi_2(y_1,y_2)\right\|_\infty \leq & \frac{3}{2}\sup_{t\in[0,1]}\left|g_2^{-1}(t)\right| \times \\
& \left[L_2\left(\|x_1-y_1\|_{X_1}+\|x_2-y_2\|_{X_2}\right)\left(\frac{1}{\Gamma(\beta_1+\beta_2+1)}+\frac{|\mathbf{K}_2|\left|g_2^{-1}(1)\right|}{\Gamma(\beta_1+1)\Gamma(\beta_2+1)}\right)\right. \\
+ & \left.\|x_1-y_1\|_\infty\left(\frac{|\mathbf{K}_2|\left(g_2^{-1}(0)h_2(0)-g_2^{-1}(1)h_2(1)\right)}{\Gamma(\beta_1+1)}+\frac{|\sup_{t\in[0,1]} h_2(t)|}{\Gamma(\beta_1+1)}\right)\right].
\end{aligned}
$$

Thus, it yields that

$$
\left\|\Psi_1(x_1,x_2)-\Psi_1(y_1,y_2)\right\|_\infty \leq \left(L_1A_1+B_1\right)\left(\|x_1-y_1\|_{X_1}+\|x_2-y_2\|_{X_2}\right).
$$

Also, we have

$$
\left\|\Psi_2(x_1,x_2)-\Psi_2(y_1,y_2)\right\|_\infty \leq \left(L_2A_2+B_2\right)\left(\|x_1-y_1\|_{X_1}+\|x_2-y_2\|_{X_2}\right).
$$

Step 2: To facilitate the proof, we calculate $D^{\delta_i}\Psi_i,\ i=1,2$. We have

$$\begin{aligned} D^{\delta_1}\Psi_1(x_1,x_2)(t) &= \int_0^t \frac{(t-\tau)^{\alpha_1-\delta_1-1}}{\Gamma(\alpha_1-\delta_1)} g_1^{-1}(\tau)\int_0^\tau \frac{(\tau-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}F_{x,1}(s)dsd\tau \\ &- \mathbf{K}_1\int_0^t \frac{(t-\tau)^{\alpha_1-\delta_1-1}}{\Gamma(\alpha_1-\delta_1)} g_1^{-1}(\tau)d\tau\Big[\Big(g_1^{-1}(0)h_1(0)-g_1^{-1}(1)h_1(1)\Big)x_1(1) \\ &+ g_1^{-1}(1)\int_0^1 \frac{(1-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)}F_{x,1}(\tau)d\tau\Big] - \int_0^t \frac{(t-\tau)^{\alpha_1-\delta_1-1}}{\Gamma(\alpha_1-\delta_1)} g_1^{-1}(\tau)h_1(\tau)x_1(\tau)d\tau, \end{aligned}$$

and

$$\begin{aligned} D^{\delta_2}\Psi_2(x_1,x_2)(t) &= \int_0^t \frac{(t-\tau)^{\beta_1-\delta_2-1}}{\Gamma(\beta_1-\delta_2)} g_2^{-1}(\tau)\int_0^\tau \frac{(\tau-s)^{\beta_2-1}}{\Gamma(\beta_2)}F_{x,2}(s)dsd\tau \\ &- \mathbf{K}_2\int_0^t \frac{(t-\tau)^{\beta_1-\delta_2-1}}{\Gamma(\beta_1-\delta_2)} g_2^{-1}(\tau)d\tau\Big[\Big(g_2^{-1}(0)h_2(0)-g_2^{-1}(1)h_2(1)\Big)x_2(1) \\ &+ g_2^{-1}(1)\int_0^1 \frac{(1-\tau)^{\beta_2-1}}{\Gamma(\beta_2)}F_{x,2}(\tau)d\tau\Big] - \int_0^t \frac{(t-\tau)^{\beta_1-\delta_2-1}}{\Gamma(\beta_1-\delta_2)} g_2^{-1}(\tau)h_2(\tau)x_2(\tau)d\tau. \end{aligned}$$

In the same manner of **step 1**, we can write:

$$\left\|D^{\delta_1}\Psi_1(x_1,x_2)-D^{\delta_1}\Psi_1(y_1,y_2)\right\|_\infty \le \Big(L_1M_1+N_1\Big)\Big(\|x_1-y_1\|_{X_1}+\|x_2-y_2\|_{X_2}\Big),$$

also

$$\left\|D^{\delta_2}\Psi_2(x_1,x_2)-D^{\delta_2}\Psi_2(y_1,y_2)\right\|_\infty \le \Big(L_2M_2+N_2\Big)\Big(\|x_1-y_1\|_{X_1}+\|x_2-y_2\|_{X_2}\Big).$$

Consequently

$$\|\Psi(x_1,x_2)-\Psi(y_1,y_2)\|_{X_1} \le \Big[L_1\Big(A_1+M_1\Big)+B_1+N_1\Big]\Big(\|x_1-y_1\|_{X_1}+\|x_2-y_2\|_{X_2}\Big),$$

and

$$\|\Psi(x_1,x_2)-\Psi(y_1,y_2)\|_{X_2} \le \Big[L_2\Big(A_2+M_2\Big)+B_2+N_2\Big]\Big(\|x_1-y_1\|_{X_1}+\|x_2-y_2\|_{X_2}\Big).$$

It yields then that

$$\|\Psi(x_1,x_2)-\Psi(y_1,y_2)\|_{X_1\times X_2} \le \Delta\Big(\|x_1-y_1\|_{X_1}+\|x_2-y_2\|_{X_2}\Big).$$

We deduce that Ψ is contractive. Thanks to Banach contraction principle, we conclude that has a unique fixed point which is the solution of (1)-(2).

3.2- At least one solution via ORegan's Theorem:

Our second main result is based on Lemma 7. We have:

Theorem *10:* *Assume that the hypotheses (H2) and (H3) hold. Then, the problem (1)-(2) has at least a solution on* $[0,1]$.

Proof: For the forthcoming theorems proof, we split both (7) and (8) as follows:

$$\begin{aligned} \varphi_{1,1}(x_1,x_2)(t) &= \int_0^t \frac{(t-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_1^{-1}(\tau)\int_0^\tau \frac{(\tau-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}F_{x,1}(s)dsd\tau \\ &- \frac{1}{2}\int_0^1 \frac{(1-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_1^{-1}(\tau)\int_0^\tau \frac{(\tau-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}F_{x,1}(s)dsd\tau \\ &- \mathbf{K}_1\left(\int_0^t \frac{(t-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_1^{-1}(\tau)d\tau + \frac{1}{2}\int_0^1 \frac{(1-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_1^{-1}(\tau)d\tau\right) \\ &\times g_1^{-1}(1)\int_0^1 \frac{(1-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)}F_{x,1}(\tau)d\tau, \end{aligned}$$

$$\varphi_{1,2}(x_1)(t) = \frac{1}{2}\int_0^1 \frac{(t-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_1^{-1}(\tau)h_1(\tau)x_1(\tau)d\tau - \int_0^t \frac{(t-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_1^{-1}(\tau)h_1(\tau)x_1(\tau)d\tau$$
$$- \mathbf{K}_1\left(\int_0^t \frac{(t-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_1^{-1}(\tau)d\tau + \frac{1}{2}\int_0^1 \frac{(1-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_1^{-1}(\tau)\right)\left(g_1^{-1}(0)h_1(0) - g_1^{-1}(1)h_1(1)\right)x_1(1),$$

and

$$\varphi_{2,1}(x_1,x_2)(t) = \int_0^t \frac{(t-\tau)^{\beta_1-1}}{\Gamma(\beta_1)} g_2^{-1}(\tau)\int_0^\tau \frac{(\tau-s)^{\beta_2-1}}{\Gamma(\beta_2)} F_{x,2}(s)dsd\tau$$
$$- \frac{1}{2}\int_0^1 \frac{(1-\tau)^{\beta_1-1}}{\Gamma(\beta_1)} g_2^{-1}(\tau)\int_0^\tau \frac{(\tau-s)^{\beta_2-1}}{\Gamma(\beta_2)} F_{x,2}(s)dsd\tau$$
$$- \mathbf{K}_2\left(\int_0^t \frac{(t-\tau)^{\beta_1-1}}{\Gamma(\beta_1)} g_2^{-1}(\tau)d\tau + \frac{1}{2}\int_0^1 \frac{(1-\tau)^{\beta_1-1}}{\Gamma(\beta_1)} g_2^{-1}(\tau)\right)$$
$$\times \quad g_2^{-1}(1)\int_0^1 \frac{(1-\tau)^{\beta_2-1}}{\Gamma(\beta_2)} F_{x,2}(\tau)d\tau,$$

$$\varphi_{2,2}(x_2)(t) = \frac{1}{2}\int_0^1 \frac{(t-\tau)^{\beta_1-1}}{\Gamma(\beta_1)} g_2^{-1}(\tau)h_2(\tau)x_2(\tau)d\tau - \int_0^t \frac{(t-\tau)^{\beta_1-1}}{\Gamma(\beta_1)} g_2^{-1}(\tau)h_2(\tau)x_2(\tau)d\tau$$
$$- \mathbf{K}_2\left(\int_0^t \frac{(t-\tau)^{\beta_1-1}}{\Gamma(\beta_1)} g_2^{-1}(\tau)d\tau + \frac{1}{2}\int_0^1 \frac{(1-\tau)^{\beta_1-1}}{\Gamma(\beta_1)} g_2^{-1}(\tau)\right)\left(g_2^{-1}(0)h_2(0) - g_2^{-1}(1)h_2(1)\right)x_2(1),$$

where

$$\Psi(x_1,x_2)(t) = \varphi_1(x_1,x_2)(t) + \varphi_2(x_1,x_2)(t),$$

and

$$\varphi_1(x_1,x_2)(t) = \varphi_{1,1}(x_1,x_2)(t) + \varphi_{2,1}(x_1,x_2)(t),$$
$$\varphi_2(x_1,x_2)(t) = \varphi_{1,2}(x_1)(t) + \varphi_{2,2}(x_2)(t).$$

We build the demonstration in four steps:

Let us consider the following subset: $\Omega_r = \left[(x_1,x_2) \in X_1 \times X_2 \ : \|(x_1,x_2)\| \le r\right]$ with

$$r \ge \frac{\sum_{i=1}^{2}\|\Phi_i(t)\|\left(A_i + M_i\right)\left[\sum_{j=1}^{4}\phi_i(\|r\|)\right]}{1-\Upsilon}, \quad .... \quad (9)$$

where

$$\Upsilon := \max\left\{ B_1 + N_1,\ B_2 + N_2 \right\}.$$

Step 1 : Our first claim is to prove that Ψ is uniformly bounded on $\overline{\Omega}$.

Let $(x_1,x_2) \in \overline{\Omega}$. Then, by (H3), it follows that

$$\|\varphi_{1,1}(x_1,x_2)(t)\| \le \|\Phi_1(t)\|\left[\phi_1(\|x_1\|) + \phi_2(\|x_2\|) + \phi_3(\|D^{\delta_1}x_1\|) + \phi_4(\|D^{\delta_2}x_2\|)\right]A_1,$$

By taking the norm, we can state that

$$\|\varphi_{1,1}(x_1,x_2)(t)\| \le \|\Phi_1(t)\|\left[\phi_1(\|r\|) + \phi_2(\|r\|) + \phi_3(\|r\|) + \phi_4(\|r\|)\right]A_1.$$

Thus, we can write

$$\|\varphi_{2,1}(x_1,x_2)(t)\| \le \|\Phi_2(t)\|\left[\phi_1(\|r\|) + \phi_2(\|r\|) + \phi_3(\|r\|) + \phi_4(\|r\|)\right]A_2.$$

Moreover, we have:

$$\left\| D^{\delta_1}\varphi_{1,1}(x_1,x_2)(t) \right\| \le \left\| \Phi_1(t) \right\| \left[\sum_{j=1}^{4} \phi_i(\|r\|) \right] M_1,$$

and

$$\left\| D^{\delta_1}\varphi_{2,1}(x_1,x_2)(t) \right\| \le \left\| \Phi_2(t) \right\| \left[\sum_{j=1}^{4} \phi_i(\|r\|) \right] M_2.$$

Then, We deduce that:

$$\left\| \varphi_1(x_1,x_2)(t) \right\|_{X_1\times X_2} \le \sum_{i=1}^{2} \left\| \Phi_i(t) \right\| \left(A_i + M_i \right) \left[\sum_{j=1}^{4} \phi_i(\|r\|) \right]. \quad .... \quad (10)$$

Therefore, φ_1 is uniformly bounded. In the same way, we obtain:

$$\left\| \varphi_{2,1}(x_1)(t) \right\|_{X_1} \le (B_1 + N_1) \left\| x_1 \right\|,$$

and

$$\left\| \varphi_{2,2}(x_2)(t) \right\|_{X_2} \le (B_2 + N_2) \left\| x_2 \right\|.$$

Therefore,

$$\left\| \varphi_2(x_1,x_2)(t) \right\|_{X_1\times X_2} \le \Upsilon\, r.$$

Thus, φ_2 is bounded.

Consequently, Ψ is bounded, indeed

$$\left\| \Psi(x_1,x_2)(t) \right\|_{X_1\times X_2} \le \sum_{i=1}^{2} \left\| \Phi_i(t) \right\| \left(A_i + M_i \right) \left[\sum_{j=1}^{4} \phi_i(\|r\|) \right] + \Upsilon\, r \le r.$$

Step 2 : We shall show that φ_2 is a contraction mapping.

Let $x_i, y_i \in X_i,\ i = 1,2$. Then, for each $t \in [0,1]$, we have

$$\left\| \varphi_{1,2}(x_1)(t) - \varphi_{1,2}(y_1)(t) \right\|_{X_1} \le \left(B_1 + N_1 \right) \left\| x_1 - y_1 \right\|,$$

and

$$\left\| \varphi_{2,2}(x_2)(t) - \varphi_{2,2}(y_2)(t) \right\|_{X_2} \le \left(B_2 + N_2 \right) \left\| x_2 - y_2 \right\|.$$

From the above inequalities, we get

$$\left\| \varphi_2(x_1,x_2)(t) - \varphi_2(y_1,y_2)(t) \right\|_{X_1\times X_2} \le \Upsilon \left(\left\| x_1 - y_1 \right\| + \left\| x_2 - y_2 \right\| \right).$$

Then φ_2 is contractive.

Step 3: Next, we need to show that φ_1 is completely continuous.

3a : Since the functions $f_i, i = 1,2$ are continuous by (H2), hence the operator φ_1 is also continuous, so this proof is omitted.

3b: Via the inequality (10), φ_1 is uniformly bounded.

3c: We will show that φ_1 is equicontinuous.

Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$. This yields

$$
\begin{aligned}
|\varphi_{1,1}(x_1,x_2)(t_2)-\varphi_{1,1}(x_1,x_2)(t_1)| = & \Bigg| \int_0^{t_1} \frac{(t-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_1^{-1}(\tau) \int_0^{\tau} \frac{(\tau-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} F_{x,1}(s)\,ds\,d\tau \\
& - \int_0^{t_1} \frac{(t-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_1^{-1}(\tau) \int_0^{\tau} \frac{(\tau-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} F_{x,1}(s)\,ds\,d\tau \\
& - \mathbf{K}_1 \left(\int_0^{t_2} \frac{(t-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_1^{-1}(\tau)\,d\tau - \int_0^{t_1} \frac{(t-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_1^{-1}(\tau)\,d\tau \right) \\
& \times \left(g_1^{-1}(1) \int_0^1 \frac{(1-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)} F_{x,1}(\tau)\,d\tau \right) \Bigg|,
\end{aligned}
$$

$$
\begin{aligned}
|\varphi_{1,1}(x_1,x_2)(t_2)-\varphi_{1,1}(x_1,x_2)(t_1)| \le & \frac{\sup_{t\in[0,1]} |g_1^{-1}(t)| \|\Phi_1(t)\| \left[\sum_{j=1}^{4} \phi_1\left(\|x_j\|\right) \right] \left(t_2^{\alpha_1+\alpha_2}-t_1^{\alpha_1+\alpha_2}\right)}{\Gamma(\alpha_1+\alpha_2+1)} \\
& + \frac{\sup_{t\in[0,1]} |g_1^{-1}(t)| \|\Phi_1(t)\| \left[\sum_{j=1}^{4} \phi_1\left(\|x_j\|\right) \right] \mathbf{K}_1 |g_1^{-1}(1)| \left(t_2^{\alpha_1}-t_1^{\alpha_1}\right)}{\Gamma(\alpha_1+1)},
\end{aligned}
$$

and

$$
\begin{aligned}
|\varphi_{2,1}(x_1,x_2)(t_2)-\varphi_{2,1}(x_1,x_2)(t_1)| \le & \frac{\sup_{t\in[0,1]} |g_2^{-1}(t)| \|\Phi_2(t)\| \left[\sum_{j=1}^{4} \phi_2\left(\|x_j\|\right) \right] \left(t_2^{\beta_1+\beta_2}-t_1^{\beta_1+\beta_2}\right)}{\Gamma(\beta_1+\beta_2+1)} \\
& + \frac{\sup_{t\in[0,1]} |g_2^{-1}(t)| \|\Phi_2(t)\| \left[\sum_{j=1}^{4} \phi_2\left(\|x_j\|\right) \right] \mathbf{K}_2 |g_2^{-1}(1)| \left(t_2^{\beta_1}-t_1^{\beta_1}\right)}{\Gamma(\beta_1+1)}.
\end{aligned}
$$

In a similar manner, we can find that

$$
\begin{aligned}
|D^{\delta_1}\varphi_{1,1}(x_1,x_2)(t_2)-D^{\delta_1}\varphi_{1,1}(x_1,x_2)(t_1)| \le & \sup_{t\in[0,1]} |g_1^{-1}(t)| \|\Phi_1(t)\| \left[\sum_{j=1}^{4} \phi_1\left(\|x_j\|\right) \right] \times \\
& \left(\frac{\left(t_2^{\alpha_1+\alpha_2}-t_1^{\alpha_1+\alpha_2}\right)}{\Gamma(\alpha_1+\alpha_2+1-\delta_1)} + \frac{\mathbf{K}_1 |g_1^{-1}(1)| \left(t_2^{\alpha_1}-t_1^{\alpha_1}\right)}{\Gamma(\alpha_1+1-\delta_1)} \right),
\end{aligned}
$$

and

$$
\begin{aligned}
|D^{\delta_2}\varphi_{2,1}(x_1,x_2)(t_2)-D^{\delta_2}\varphi_{2,1}(x_1,x_2)(t_2)| \le & \sup_{t\in[0,1]} |g_2^{-1}(t)| \|\Phi_2(t)\| \left[\sum_{j=1}^{4} \phi_2\left(\|x_j\|\right) \right] \times \\
& \frac{\left(t_2^{\beta_1+\beta_2}-t_1^{\beta_1+\beta_2}\right)}{\Gamma(\beta_1+\beta_2+1-\delta_2)} + \frac{\mathbf{K}_2 |g_2^{-1}(1)| \left(t_2^{\beta_1}-t_1^{\beta_1}\right)}{\Gamma(\beta_1+1-\delta_2)}.
\end{aligned}
$$

The right hand sides of (a), (b), (c) and (d) are independent of the pair (x_1, x_2) and tend to zero as $t_1 \to t_2$. Therefore, φ_1 is an equicontinuous operator.

As a consequence of the previous steps and thanks to Arzela-Ascoli theorem, we conclude that φ_1 is completely continuous.

Step 4: We suppose that there exists $\lambda \in [0,1]$ and there exists the pair $(x_1, x_2) \in \partial\Omega$ such that $(x_1, x_2) = \lambda\Psi(x_1, x_2)$, then $\|(x_1, x_2)\|_{X_1\times X_2} = r$. Thanks to **step 1**, we get

$$
r \le \sum_{i=1}^{2} \|\Phi_i(t)\| \left(A_i + M_i \right) \left[\sum_{j=1}^{4} \phi_i\left(\|r\|\right) \right] + \Upsilon r,
$$

thus we obtain

$$r \le \frac{\sum_{i=1}^{2}\|\Phi_i(t)\|\left(A_i+M_i\right)\left[\sum_{j=1}^{4}\phi_i(\|r\|)\right]}{1-\Upsilon}.$$

Which is clearly contradicted to (9) consequently, we have proved that the operators the operator Ψ has at least one fixed point. Therefore, the problem (1)-(2) has a solution on [0, 1] .

4- Ulam-Hyers-Mittag-Leffler stability:

In this section, we provide results regarding Ulam-Hyers-Mittag-Leffler stability for (1)-(2) .
We consider the following inequalities:

$$\begin{cases} \left| D^{\alpha_2}\left[g_1(t)D^{\alpha_1}+h_1(t)\right]x_1(t)-f_1\left(t,x_1(t),x_2(t),D^{\delta_1}x_1(t),D^{\delta_2}x_2(t)\right)\right| \le \varepsilon_1 E_{\alpha_2}(t^{\alpha_2}), & 0<t<1, \\ \left| D^{\beta_2}\left[g_2(t)D^{\beta_1}+h_2(t)\right]x_2(t)-f_2\left(t,x_1(t),x_2(t),D^{\delta_1}x_1(t),D^{\delta_2}x_2(t)\right)\right| \le \varepsilon_2 E_{\beta_2}(t^{\beta_2}), & 0<t<1, \end{cases}$$

where $E_{\alpha_2}(\cdot)$ and $E_{\beta_2}(\cdot)$ represent the Mittag-Lefler function defined by:

$$E_{\alpha_2}(.)=\sum_{k=0}^{\infty}\frac{(.)^k}{\Gamma\,(k\alpha_2+1)},$$

$$E_{\beta_2}(.)=\sum_{k=0}^{\infty}\frac{(.)^k}{\Gamma\,(k\beta_2+1)},$$

$$(.)\in Re(\alpha_2), Re(\beta_2)>0.$$

Definition 11: *The problem (1)-(2) is Ulam-Hyers-Mittag-Leffler stable with respect to* $E_\eta\left(t^\eta\right)$ *if there exists* $c_\eta>0$ *such that for each* $\varepsilon>0$ *and each solution* $(y_1,y_2)\in X_1\times X_2$ *of the inequalities (15), there exists a solution* $(x_1,x_2)\in X_1\times X_2$ *of the problem (1)-(2) with*

$$\|(y_1,y_2)-(x_1,x_2)\|_{X_1\times X_2}\le c_\eta \varepsilon E_\eta\left(t^\eta\right),\ \ t\in[0,1].$$

Next, Ulam-Hyers-Mittag-Leffler stable results will be provided.

Theorem 12: *Assume that the hypotheses of Theorem 9 are valid and (H4) holds. Then, (1)-(2) is generalized Ulam-Hyers-Mittag-Leffler stable.*

Proof: Let y_1, y_2 be solutions of (15), and we suppose that

$$\begin{aligned} y_1(t) \;=\; & \int_0^t \frac{(t-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_1^{-1}(\tau)\int_0^\tau \frac{(\tau-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} F_{y,1}(s)ds d\tau - \int_0^t \frac{(t-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_1^{-1}(\tau)h_1(\tau)x_1(\tau)d\tau \\ & - \frac{1}{2}\int_0^1 \frac{(1-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_1^{-1}(\tau)\int_0^\tau \frac{(\tau-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} F_{y,1}(s)ds d\tau + \frac{1}{2}\int_0^1 \frac{(t-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_1^{-1}(\tau)h_1(\tau)x_1(\tau)d\tau \\ & - \mathbf{K}_1\left(\int_0^t \frac{(t-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_1^{-1}(\tau)d\tau + \frac{1}{2}\int_0^1 \frac{(1-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_1^{-1}(\tau)\right)\times \\ & \left[g_1^{-1}(1)\int_0^1 \frac{(1-\tau)^{\alpha_2-1}}{\Gamma(\alpha_2)} F_{y,1}(\tau)d\tau + \left(g_1^{-1}(0)h_1(0)-g_1^{-1}(1)h_1(1)\right)x_1(1)\right] \\ & + \int_0^t \frac{(t-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_1^{-1}(\tau)\int_0^\tau \frac{(\tau-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} \chi_1(s)ds d\tau \;:=I_1, \end{aligned}$$

and

$$
\begin{aligned}
y_2(t) \;=\;& \int_0^t \frac{(t-\tau)^{\beta_1-1}}{\Gamma(\beta_1)} g_2^{-1}(\tau) \int_0^\tau \frac{(\tau-s)^{\beta_2-1}}{\Gamma(\beta_2)} F_{y,2}(s)\,ds\,d\tau - \int_0^t \frac{(t-\tau)^{\beta_1-1}}{\Gamma(\beta_1)} g_2^{-1}(\tau) h_2(\tau) x_2(\tau)\,d\tau \\
-\;& \frac{1}{2}\int_0^1 \frac{(1-\tau)^{\beta_1-1}}{\Gamma(\beta_1)} g_2^{-1}(\tau) \int_0^\tau \frac{(\tau-s)^{\beta_2-1}}{\Gamma(\beta_2)} F_{y,2}(s)\,ds\,d\tau + \frac{1}{2}\int_0^1 \frac{(t-\tau)^{\beta_1-1}}{\Gamma(\beta_1)} g_2^{-1}(\tau) h_2(\tau) x_2(\tau)\,d\tau \\
-\;& \mathtt{K}_2 \left(\int_0^t \frac{(t-\tau)^{\beta_1-1}}{\Gamma(\beta_1)} g_2^{-1}(\tau)\,d\tau + \frac{1}{2}\int_0^1 \frac{(1-\tau)^{\beta_1-1}}{\Gamma(\beta_1)} g_2^{-1}(\tau) \right) \times \\
& \left[g_2^{-1}(1) \int_0^1 \frac{(1-\tau)^{\beta_2-1}}{\Gamma(\beta_2)} F_{y,2}(\tau)\,d\tau + \left(g_2^{-1}(0) h_2(0) - g_2^{-1}(1) h_2(1) \right) x_2(1) \right] \\
+\;& \int_0^t \frac{(t-\tau)^{\beta_1-1}}{\Gamma(\beta_1)} g_1^{-1}(\tau) \int_0^\tau \frac{(\tau-s)^{\beta_2-1}}{\Gamma(\beta_2)} \chi_2(s)\,ds\,d\tau \;:= I_2.
\end{aligned}
$$

Thanks to (H4), we have

$$
\begin{cases}
\left| y_1(t) - I_1 \right| \le \sup_{t\in[0,1]} | g_1^{-1}(t) | \dfrac{\varepsilon_1\, E_{\alpha_2}(t^{\alpha_2})}{\Gamma(\alpha_1+\alpha_2+1)}, \quad 0<t<1, \\
\left| y_2(t) - I_2 \right| \le \sup_{t\in[0,1]} | g_2^{-1}(t) | \dfrac{\varepsilon_2\, E_{\beta_2}(t^{\beta_2})}{\Gamma(\beta_1+\beta_2+1)}, \quad 0<t<1.
\end{cases} \qquad \text{....} \quad (16)
$$

By theorem 9, the problem (1)-(2) has a unique solution (x_1, x_2) . Then, using (16) and (H1), we get

$$
\|y_1 - x_1\|_\infty \le \left(L_1 A_1 + B_1 \right) \left(\|y_1 - x_1\|_{X_1} + \|y_2 - x_2\|_{X_2} \right) + \sup_{t\in[0,1]} | g_1^{-1}(t) | \frac{\varepsilon_1\, E_{\alpha_2}(t^{\alpha_2})}{\Gamma(\alpha_1+\alpha_2+1)},
$$

and

$$
\|y_2 - x_2\|_\infty \le \left(L_2 A_2 + B_2 \right) \left(\|y_1 - x_1\|_{X_1} + \|y_2 - x_2\|_{X_2} \right) + \sup_{t\in[0,1]} | g_2^{-1}(t) | \frac{\varepsilon_2\, E_{\beta_2}(t^{\beta_2})}{\Gamma(\beta_1+\beta_2+1)},
$$

also

$$
\|D^{\delta_1} y_1 - D^{\delta_1} x_1\|_\infty \le \left(L_1 M_1 + N_1 \right) \left(\|y_1 - x_1\|_{X_1} + \|y_2 - x_2\|_{X_2} \right) + \sup_{t\in[0,1]} | g_1^{-1}(t) | \frac{\varepsilon_1\, E_{\alpha_2}(t^{\alpha_2})}{\Gamma(\alpha_1+\alpha_2+1-\delta_1)}.
$$

Similarly, we get

$$
\|D^{\delta_2} y_2 - D^{\delta_2} x_2\|_\infty \le \left(L_2 M_2 + N_2 \right) \left(\|y_1 - x_1\|_{X_1} + \|y_2 - x_2\|_{X_2} \right) + \sup_{t\in[0,1]} | g_2^{-1}(t) | \frac{\varepsilon_2\, E_{\beta_2}(t^{\beta_2})}{\Gamma(\beta_1+\beta_2+1-\delta_2)}.
$$

Therefore,

$$
\|(y_1, y_2) - (x_1, x_2)\|_{X_1\times X_2} \le c_\eta \varepsilon E_\eta\left(t^\eta\right), \quad t\in[0,1],
$$

where,

$$
\varepsilon := \max\left\{ \varepsilon_1, \varepsilon_2 \right\},
$$

$$
E_\eta\left(t^\eta\right) := \max\left\{ E_{\alpha_2}(t^{\alpha_2}), E_{\beta_2}(t^{\beta_2}) \right\},
$$

$$
c_\eta := \frac{ \dfrac{\sup_{t\in[0,1]} | g_1^{-1}(t) |}{\Gamma(\alpha_1+\alpha_2+1)} + \dfrac{\sup_{t\in[0,1]} | g_2^{-1}(t) |}{\Gamma(\beta_1+\beta_2+1)} + \dfrac{\sup_{t\in[0,1]} | g_2^{-1}(t) |}{\Gamma(\alpha_1+\alpha_2+1-\delta_1)} + \dfrac{\sup_{t\in[0,1]} | g_2^{-1}(t) |}{\Gamma(\beta_1+\beta_2+1-\delta_2)} }{1-\Delta}.
$$

Thus, problem (1)-(2) is Ulam-Hyers-Mittag-Leffler stable.

5- Example:

We consider the following problem:

$$\begin{cases} D^{\alpha_2}[g_1(t)D^{\alpha_1}+h_1(t)]x_1(t)=f_1(t,x_1(t),x_2(t),D^{\delta_1}x_1(t),D^{\delta_2}x_2(t)), \quad 0<t<1, \\ D^{\beta_2}[g_2(t)D^{\beta_1}+h_2(t)]x_2(t)=f_2(t,x_1(t),x_2(t),D^{\delta_1}x_1(t),D^{\delta_2}x_2(t)), \quad 0<t<1, \\ x_1(0)+x_1(1)=0, \quad D^{\alpha_1}x_1(0)+D^{\alpha_1}x_1(1)=0, \\ x_2(0)+x_2(1)=0, \quad D^{\alpha_2}x_2(0)+D^{\alpha_2}x_2(1)=0. \end{cases} \quad \ldots \quad (17)$$

Here, we have

$$f_1(t,x_1(t),x_2(t),D^{\delta_1}x_1(t),D^{\delta_2}x_2(t)) = \frac{x_1(t)}{9x_1^2(t)+2}+\frac{x_2(t)}{x_2^2(t)(5+t)^2} + \frac{D_1^{\delta}x_1(t)\sin^2(\pi t)}{(4-t)^3}+\frac{D_2^{\delta}x_2(t)\cos^2(\pi t)}{(4+t)^2},$$

and

$$f_2(t,x_1(t),x_2(t),D^{\delta_1}x_1(t),D^{\delta_2}x_2(t)) = \frac{x_1(t)}{(x_1^2(t)+2)(6-t)^2}+\frac{x_2(t)}{(7x_2(t)+3)^2} + \frac{D_1^{\delta}x_1(t)\cos^2(2\pi t)}{19}+\frac{D_2^{\delta}x_2(t)\sin^2(2\pi t)}{(5-t)^3}.$$

Also

$$g_1(t)=t^{\frac{2}{7}}+1, \quad g_2(t)=t^{\frac{3}{2}}+2, \quad h_1(t)=\frac{t+2}{5(t+1)}, \quad h_2(t)=\frac{t+2}{4(t+3)}.$$

$$\alpha_1=0.9, \quad \alpha_2=0.8, \quad \beta_1=0.85, \quad \beta_2=0.75, \quad \delta_1=0.1, \quad \delta_2=0.1,$$

$$\mathbf{K}_1=1.2486, \quad \mathbf{K}_2=2.8235.$$

Clearly, for all $(x_1,y_1,z_1,w_1),(x_2,y_2,z_2,w_2)\in \mathsf{R}^4$, and $t\in[0,1]$, by Taylor's formula, we have:

$$|f_1(t,x_1,y_1,z_1,w_1)-f(t,x_2,y_2,z_2,w_2)| \le \frac{1}{9}|x_1-y_1|+\frac{1}{25}|x_1-y_1|+\frac{1}{4^3}|x_1-y_1|+\frac{1}{16}|x_1-y_1|,$$

$$|f_2(t,x_1,y_1,z_1,w_1)-f(t,x_2,y_2,z_2,w_2)| \le \frac{1}{36}|x_1-y_1|+\frac{1}{7^2}|x_1-y_1|+\frac{1}{19}|x_1-y_1|+\frac{1}{5^3}|x_1-y_1|,$$

consequently,

$$A_1=10.5314, \quad A_2=47.4814, \quad B_1=-0.3630, \quad B_2=-2.8809,$$

$$M_1=7.3701, \quad M_2=32.7561, \quad N_1=-0.1068, \quad N_2=-1.9081,$$

For $\Delta<0.9533$, it follows by Theorem 9 that problem (17) has a unique solution on $[0,1]$.

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