FRACTIONAL INEQUALITIES RELATED TO THE WEIGHTED CHEBYSHEV FUNCTIONAL TYPE

Mohamed Bezziou[1], Zoubir Dahmani[2], Alanzai T. Abdulrahman[3]
[1]UDBKM, University and laboratory LPAM, Mostaganem, Algeria
[2]Laboratory LPAM, Faculty of SEI, UMAB, University of Mostaganem, Algeria
[3]Mathematics Department, College of Science, University of Ha'il, Ha'il, KSA
m.bezziou@univ-dbkm.dz [1], zzdahmani@yahoo.fr [2], t.shyman@uoh.edu.sa [3]

ABSTRACT: *In this paper, we establish new weighted fractional inequalities of Chebyshev type, involving the* (k,s,h)*-Riemann-Liouville and*(k,h)*-Hadamard fractional integrals, with respect to another, increasing, positive and monotone function, with*$h' \in C^1(a,b)$*. In particular, some recent results are deduced.*

Keywords:(k,s,h)-Riemann-Liouville integral, (k,h)-Hadamard fractional operator,Chebyshev functional type, Fractional Factorial Design, Statistical Analysis.
AMS Subject Classification: 34A15, 35E10

1 INTRODUCTION

The integral inequalities are very important in many areas of science, especially in mathematics, physics, chemistry, biology. Many researchers have given a lot of attention to the generalization of fractional integral inequalities related to weighted Chebyshev functional. In fact, they established many results to Grüss and Chebyshev inequalities. For more details, we refer the reader to [1,5,6,7,8,9,11,13,14,15] and the references therein.

Let us now cite some recent works that have motivated the present paper. We begin by the paper [3], where the authors introduced two new fractional integral operators: the first one is the (k,s,h)- Riemann-Liouville fractional integral (for a function $f \in L^1([a,b])$) with respect to another monotone, increasing and positive function h with $h' \in C^1(a,b)$. It is given by

$$ {}_k^s J_{a,h}^{\alpha}\big(f(t)\big) = \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^t \left(h^{s+1}(t) - h^{s+1}(\tau)\right)^{\frac{\alpha}{k}-1} h^s(\tau) h'(\tau) f(\tau)\, d\tau, \qquad (1.1)$$

where $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}}\, dt,\ \alpha > 0, k > 0, s \in \mathbb{R} - \{-1\}$.

The second operator is the (k,h)-Hadamard fractional integral (of $f \in L^1([a,b])$ with respect to h). It is defined for $k > 0$ by

$$ {}_k I_{a,h}^{\alpha}\big(f(t)\big) = \frac{1}{k\Gamma_k(\alpha)} \int_a^t \left(\log \frac{h(t)}{h(\tau)}\right)^{\frac{\alpha}{k}-1} \frac{h'(\tau)}{h(\tau)} f(\tau)\, d\tau. \qquad (1.2)$$

Based on these two operators, we can state that:

Proposition 1.1[3] *We have*

$$ \lim_{s \to -1^+} {}_k^s J_{a,h}^{\alpha}\big(f(t)\big) = {}_k I_{a,h}^{\alpha}\big(f(t)\big) \quad . \qquad (1.3)$$

Now, by considering the weighted Chebyshev functional [6,7]:

$$ \begin{aligned} T(f,g,p,q) &:= \int_a^b p(x)dx \int_a^b q(x) f(x) g(x) dx + \int_a^b q(x) dx \\ &\times \int_a^b p(x) f(x) g(x) dx - \int_a^b p(x) f(x) dx \int_a^b q(x) g(x) dx \qquad (1.4) \\ &- \int_a^b q(x) f(x) dx \int_a^b p(x) g(x) dx, \end{aligned}$$

where f and g are two real-valued integrable functions and p, q are two positive integrable functions on a finite $[a,b]$.

Very recently [5] Z. Dahmani and M. Doubbi Bounoua proved the result:

Theorem1.1 *Let* $\phi : [a,b] \to \mathrm{R}$ *be an absolutely continuous function,* $p : [a,b] \to \mathrm{R}$ *an integrable function and* $(\phi')^2 \in L^1[a,b]$. *Then, for all* $\alpha > 0$, *and* $x \in]a,b]$, *we have:*

$$ \frac{J_a^\alpha(p\phi^2)(x)}{J_a^\alpha p(x)} - \left(\frac{1}{J_a^\alpha p(x)} I_a^\alpha(p\phi)(x)\right)^2 \le \frac{1}{(J_a^\alpha p(x))^2} \int_a^x \tilde{P}_x(t) \left(\phi'(x)\right)^2 dt \qquad (1.5)$$

with

$$\tilde{P}_x(t)=\frac{1}{\Gamma(\alpha)}\left[J_a^\alpha\big(xp(x)\big)\int_a^t (x-y)^{\alpha-1}p(y)\,dy-J_a^\alpha\big(p(x)\big)\int_a^t (x-y)^{\alpha-1}yp(y)\,dy\right]. \quad (1.6)$$

The main purpose of this paper is to establish some new inequalities for (1.4) by using the $(k,s,h)-$ Riemann-Liouville and (k,h)-Hadamard fractional operators with respect to another monotone, increasing and positive function h. We generalize some very recent results related to weighted Chebyshev inequalities [5].

2 MAIN RESULTS

We begin by proving the following identity:

Lemma2.1 *Let* $f,g\ :\ [a,b]\to \mathbb{R}$ *be two integrable functions an* $f'\in L^1[a,b]$, *assume that* h *is a increasing and positive function on* $(a,b]$, $h\in C^1(a,b)$ *and* $p,q\ :[a,b]\to \mathbb{R}^+$ *are positive integrable functions. Then for all* $\alpha>0, k>0, s\in \mathbb{R}-\{-1\}$ *and* $a<x\le b$ *, we have*

$$\left[{}_k^sJ_{a,h}^\alpha p(x)\right]\left[{}_k^sJ_{a,h}^\alpha (qfg)(x)\right]+\left[{}_k^sJ_{a,h}^\alpha q(x)\right]\left[{}_k^sJ_{a,h}^\alpha (pfg)(x)\right]$$
$$-\left[{}_k^sJ_{a,h}^\alpha (pf)(x)\right]\left[{}_k^sJ_{a,h}^\alpha (qg)(x)\right]-\left[{}_k^sJ_{a,h}^\alpha (qf)(x)\right]\left[{}_k^sJ_{a;h}^\alpha (pg)(x)\right]$$
$$=\int_a^x f'(t)\left(\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t\left(h^{s+1}(x)-h^{s+1}(y)\right)^{\frac{\alpha}{k}-1}h^s(y)h'(y)\,p(y)F_x(y)dy\right)dt \quad (2.1)$$
$$+\int_a^x f'(t)\left(\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t\left(h^{s+1}(x)-h^{s+1}(y)\right)^{\frac{\alpha}{k}-1}h^s(y)h'(y)q(y)G_x(y)dy\right)dt.$$

where,

$$F_x(y)={}_k^sJ_{a,h}^\alpha (qf)(x)-f(y)\ {}_k^sJ_{a,h}^\alpha q(x), \quad (2.2)$$

and

$$G_x(y)={}_k^sJ_{a,h}^\alpha (pf)(x)-f(y)\ {}_k^sJ_{a,h}^\alpha p(x). \quad (2.3)$$

Proof. Using integrating by parts and replacing the quantities (2.2) and (2.3) in (2.1), we get,

$$\int_a^x f'(t)\left(\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t\left(h^{s+1}(x)-h^{s+1}(y)\right)^{\frac{\alpha}{k}-1}h^s(y)h'(y)\,p(y)F_x(y)dy\right)dt$$
$$+\int_a^x f'(t)\left(\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t\left(h^{s+1}(x)-h^{s+1}(y)\right)^{\frac{\alpha}{k}-1}h^s(y)h'(y)q(y)G_x(y)dy\right)dt$$
$$=f(t)\left(\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^x\left(h^{s+1}(x)-h^{s+1}(y)\right)^{\frac{\alpha}{k}-1}h^s(y)h'(y)\,p(y)\right.$$
$$\times\left({}_k^sJ_{a,h}^\alpha (qf)(x)-f(y)\ {}_k^sJ_{a,h}^\alpha q(x)\right)dy\Big)$$
$$+\left[{}_k^sJ_{a,h}^\alpha q(x)\right]\left[{}_k^sJ_{a,h}^\alpha (pfg)(x)\right]-\left[{}_k^sJ_{a,h}^\alpha (qf)(x)\right]\left[{}_k^sJ_{a;h}^\alpha (pg)(x)\right]$$
$$+f(t)\left(\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^x\left(h^{s+1}(x)-h^{s+1}(y)\right)^{\frac{\alpha}{k}-1}h^s(y)h'(y)q(y)\right.$$
$$\times\left({}_k^sJ_{a,h}^\alpha (pf)(x)-f(y)\ {}_k^sJ_{a,h}^\alpha p(x)\right)dy\Big)$$
$$+\left[{}_k^sJ_{a,h}^\alpha p(x)\right]\left[{}_k^sJ_{a,h}^\alpha (qfg)(x)\right]-\left[{}_k^sJ_{a,h}^\alpha (pf)(x)\right]\left[{}_k^sJ_{a,h}^\alpha (qg)(x)\right].$$

Hence, the identity (2.1) is proved.

Fast of all, we give the following quantity, for all $a<x\le b$

$$A(x)$$
$$= \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\Bigg(\Bigg[\ {}_k^s J_{a,h}^{\alpha}[xq(x)]\int_a^t \left(h^{s+1}(x)-h^{s+1}(y)\right)^{\frac{\alpha}{k}-1} h^s(y)h'(y)p(y)dy$$
$$-{}_k^s J_{a,h}^{\alpha} q(x)\int_a^t \left(h^{s+1}(x)-h^{s+1}(y)\right)^{\frac{\alpha}{k}-1} h^s(y)h'(y)yp(y)dy\Bigg] \quad (2.10)$$
$$+\Bigg[\ {}_k^s J_{a,h}^{\alpha}[xp(x)]\int_a^t \left(h^{s+1}(x)-h^{s+1}(y)\right)^{\frac{\alpha}{k}-1} h^s(y)h'(y)q(y)dy$$
$$-{}_k^s J_{a,h}^{\alpha} p(x)\int_a^t \left(h^{s+1}(x)-h^{s+1}(y)\right)^{\frac{\alpha}{k}-1} h^s(y)h'(y)yq(y)dy\Bigg]\Bigg).$$

where $k>0$ and $s \in \mathrm{R}-\{-1\}$.

Based on the above lemma, we prove the following theorem.

Theorem2.1 *Let* $\varphi : [a,b] \to \mathrm{R}$ *be an absolutely continuous function on* $[a,b]$ *and assume that* h *is a monotone, increasing and positive function on* $(a,b]$, $h \in C^1(a,b)$ *and* $p,q : [a,b] \to \mathrm{R}^+$ *are positive integrable functions. If* $(\varphi')^2 \in L^1[a,b]$ *, then for all* $\alpha > 0, a < x \le b$ *, we have*

$$\left[{}_k^s J_{a,h}^{\alpha} p(x)\right]\left[{}_k^s J_{a,h}^{\alpha}\left(q\varphi^2\right)(x)\right]+\left[{}_k^s J_{a,h}^{\alpha} q(x)\right]\left[{}_k^s J_{a,h}^{\alpha}\left(p\varphi^2\right)(x)\right]$$
$$-2\left[{}_k^s J_{a,h}^{\alpha}(p\varphi)(x)\right]\left[{}_k^s J_{a,h}^{\alpha}(q\varphi)(x)\right] \quad (2.11)$$
$$\le \int_a^x A(x)[\varphi'(t)]^2\,dt.$$

Where $k>0, s \in \mathrm{R}-\{-1\}$ *and* $A(x)$ *is given by (2.10).*

Proof. We have

$$\left[{}_k^s J_{a,h}^{\alpha} p(x)\right]\left[{}_k^s J_{a,h}^{\alpha}\left(q\varphi^2\right)(x)\right]+\left[{}_k^s J_{a,h}^{\alpha} q(x)\right]\left[{}_k^s J_{a,h}^{\alpha}\left(p\varphi^2\right)(x)\right]$$
$$-2\left[{}_k^s J_{a,h}^{\alpha}(p\varphi)(x)\right]\left[{}_k^s J_{a,h}^{\alpha}(q\varphi)(x)\right]$$
$$= \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^x\int_a^x \left(h^{s+1}(x)-h^{s+1}(t)\right)^{\frac{\alpha}{k}-1} h^s(t)h'(t)p(t)q(\tau)$$
$$\times\left(h^{s+1}(x)-h^{s+1}(\tau)\right)^{\frac{\alpha}{k}-1} h^s(\tau)h'(\tau)[\varphi(t)-\varphi(\tau)]^2\,dt\,d\tau.$$

Applying Cauchy-Schwarz to: $[\varphi(t)-\varphi(\tau)]^2 = \left(\int_\tau^t \varphi'(r)dr\right)^2$, it yields that

$${}_k^s J_{a,h}^{\alpha} p(x)\,{}_k^s J_{a,h}^{\alpha}\left(q\varphi^2\right)(x) + {}_k^s J_{a,h}^{\alpha} q(x)\,{}_k^s J_{a,h}^{\alpha}\left(p\varphi^2\right)(x)$$
$$-2\,{}_k^s J_{a,h}^{\alpha}(p\varphi)(x)\,{}_k^s J_{a,h}^{\alpha}(q\varphi)(x)$$
$$\le \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^x\int_a^x \left(h^{s+1}(x)-h^{s+1}(t)\right)^{\frac{\alpha}{k}-1} h^s(t)h'(t)(t-\tau)$$
$$\times p(t)q(\tau)\left[\int_a^t [\varphi'(r)]^2 dr - \int_a^\tau [\varphi'(r)]^2 dr\right]dt\,d\tau \quad (2.12)$$
$$\le \left[{}_k^s J_{a,h}^{\alpha} q(x)\right]\left[{}_k^s J_{a,h}^{\alpha}\left(xp(x)\int_a^x [\varphi'(r)]^2 dr\right)\right]$$
$$+\left[{}_k^s J_{a,h}^{\alpha} p(x)\right]\left[{}_k^s J_{a,h}^{\alpha}\left(xq(x)\int_a^x [\varphi'(r)]^2 dr\right)\right]$$
$$-\left[{}_k^s J_{a,h}^{\alpha} xp(x)\right]\left[{}_k^s J_{a,h}^{\alpha}\left(q(x)\int_a^x [\varphi'(r)]^2 dr\right)\right]$$
$$-\left[{}_k^s J_{a,h}^{\alpha} xq(x)\right]\left[{}_k^s J_{a,h}^{\alpha}\left(p(x)\int_a^x [\varphi'(r)]^2 dr\right)\right].$$

Thanks to the above lemma, we have the following identity:

$$\left[{}_k^s J_{a,h}^{\alpha} p(x)\right]\left[{}_k^s J_{a,h}^{\alpha}\left(x(qg)(x)\right)\right]+\left[{}_k^s J_{a,h}^{\alpha} q(x)\right]\left[{}_k^s J_{a,h}^{\alpha}\left(x(pg)(x)\right)\right]$$
$$-\left[{}_k^s J_{a,h}^{\alpha}\left(x(p)(x)\right)\right]\left[{}_k^s J_{a,h}^{\alpha}(qg)(x)\right]-\left[{}_k^s J_{a,h}^{\alpha}\left(x(q)(x)\right)\right]\left[{}_k^s J_{a;h}^{\alpha}(pg)(x)\right]$$
$$=\int_a^x f'(t)\left(\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t\left(h^{s+1}(x)-h^{s+1}(y)\right)^{\frac{\alpha}{k}-1}h^s(y)h'(y)p(y)M_x(y)dy\right)dt \qquad (2.13)$$
$$+\int_a^x f'(t)\left(\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t\left(h^{s+1}(x)-h^{s+1}(y)\right)^{\frac{\alpha}{k}-1}h^s(y)h'(y)q(y)N_x(y)dy\right)dt,$$

where

$$M_x(y)={}_k^s J_{a,h}^{\alpha}[xq(x)]-y\,{}_k^s J_{a,h}^{\alpha}q(x), \qquad (2.14)$$

and

$$N_x(y)={}_k^s J_{a,h}^{\alpha}[xp(x)]-y\,{}_k^s J_{a,h}^{\alpha}p(x). \qquad (2.15)$$

Replacing (2.14) and (2.15) in (2.13), we get

$$\left[{}_k^sJ_{a,h}^{\alpha}p(x)\right]\left[{}_k^sJ_{a,h}^{\alpha}xqg(x)\right]+\left[{}_k^sJ_{a,h}^{\alpha}q(x)\right]\left[{}_k^sJ_{a,h}^{\alpha}xpg(x)\right]$$
$$-\left[{}_k^sJ_{a,h}^{\alpha}xp(x)\right]\left[{}_k^sJ_{a,h}^{\alpha}qg(x)\right]-\left[{}_k^sJ_{a,h}^{\alpha}xq(x)\right]\left[{}_k^sJ_{a,h}^{\alpha}pg(x)\right]$$
$$=\int_a^x f'(x)\left(\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t\left(h^{s+1}(x)-h^{s+1}(y)\right)^{\frac{\alpha}{k}-1}h^s(y)h'(y)p(y)\times\left({}_k^sJ_{a,h}^{\alpha}xq(x)-y\,{}_k^sJ_{a,h}^{\alpha}q(x)\right)dy\right)dt$$
$$+\int_a^x f'(x)\left(\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t\left(h^{s+1}(x)-h^{s+1}(y)\right)^{\frac{\alpha}{k}-1}h^s(y)h'(y)p(y)\times\left({}_k^sJ_{a,h}^{\alpha}xp(x)-y\,{}_k^sJ_{a,h}^{\alpha}p(x)\right)dy\right)dt$$
$$=\left(\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\right)^2\int_a^x\left\{\begin{array}{l}{}_k^sJ_{a,h}^{\alpha}xq(x)\int_a^t\left(h^{s+1}(x)-h^{s+1}(y)\right)^{\frac{\alpha}{k}-1}h^s(y)h'(y)p(y)dy\\ -{}_k^sJ_{a,h}^{\alpha}q(x)\int_a^t\left(h^{s+1}(x)-h^{s+1}(y)\right)^{\frac{\alpha}{k}-1}h^s(y)h'(y)yp(y)dy\\ +{}_k^sJ_{a,h}^{\alpha}xp(x)\int_a^t\left(h^{s+1}(x)-h^{s+1}(y)\right)^{\frac{\alpha}{k}-1}h^s(y)h'(y)q(y)dy\\ -{}_k^sJ_{a,h}^{\alpha}p(x)\int_a^t\left(h^{s+1}(x)-h^{s+1}(y)\right)^{\frac{\alpha}{k}-1}h^s(y)h'(y)yq(y)dy\end{array}\right\}g'(t)dt$$

(2.16)

Finally, by (2.12) and (2.16), we get (2.11).

Corollary2.1 *Let* $\varphi:[a,b]\to\mathrm{R}$ *be an absolutely continuous function and assume that* h *is a monotone, increasing and positive function on* (a,b) , $h\in C^1(a,b)$ *and* q *is a positive integrable function on* $[a,b]$ *. If* $(\varphi')^2\in L^1[a,b]$ *. Then for all* $\alpha>0, a<x\le b$ *, we have*

$$\frac{(h(b)-h(a))^{\alpha}}{\Gamma(\alpha+1)} J_{a,h}^{\alpha}\left(q\varphi^{2}\right)(b)+\left[J_{a,h}^{\alpha}q(b)\right]\left[J_{a,h}^{\alpha}\left(\varphi^{2}\right)(b)\right]$$
$$-2\left[J_{a,h}^{\alpha}(\varphi)(b)\right]\left[J_{a,h}^{\alpha}(q\varphi)(b)\right]$$
$$\leq \int_a^b \frac{1}{\Gamma(\alpha)}\left\{\left[J_{a,h}^{\alpha}[xq(x)]\int_a^t (h(b)-h(y))^{\alpha-1}h'(y)p(y)dy\right.\right.$$
$$\left.-J_{a,h}^{\alpha}q(x)\int_a^t (h(b)-h(y))^{\alpha-1}h'(y)yp(y)dy\right] \quad (2.17)$$
$$+\left[J_{a,h}^{\alpha}[b]\int_a^t (h(b)-h(y))^{\alpha-1}h'(y)q(y)dy-\frac{(h(b)-h(a))^{\alpha}}{\Gamma(\alpha+1)}\right.$$
$$\left.\left.\times\int_a^t (h(b)-h(y))^{\alpha-1}h'(y)yq(y)dy\right]\right\}[\varphi'(t)]^2\,dt,$$

Proof. Applying Theorem 2.1 with $x=b, k=1, s=0$ and $p(x)=1$, we get (2.17).

Corollary2.2 *Assume that* $\varphi:[a,b]\to \mathrm{R}$ *is an absolutely continuous function. If* $(\varphi')^2\in L^1[a,b]$. *Then for all* $\alpha>0, a<x\leq b$, *we have*

$$\frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)}\left[J_a^{\alpha}\left(\varphi^2\right)(x)\right]-\left[J_a^{\alpha}(\varphi)(x)\right]^2$$
$$\leq \int_a^x \frac{1}{\Gamma(\alpha)}\left[J_a^{\alpha}[x]\int_a^t (x-y)^{\alpha-1}dy-\frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)}\int_a^t (x-y)^{\alpha-1}ydy\right][\varphi'(t)]^2\,dt. \quad (2.18)$$

Proof. Taking $k=1, s=0$ and $p(x)=q(x)=1, h(x)=x$ in Theorem 6, we obtain (2.18).

Corollary2.3 *Let* $\varphi:[a,b]\to \mathrm{R}$ *be an absolutely continuous function on* $[a,b]$ *and assume that* h *is a monotone, increasing and positive function on* $(a,b]$, $h\in C^1(a,b)$ *and* $p,q:[a,b]\to \mathrm{R}^+$ *are positive integrable functions. If* $(\varphi')^2\in L^1[a,b]$. *Then for all* $\alpha>0, a<x\leq b$ *and* $k>0$, *we have*

$$\left[{}_kI_{a,h}^{\alpha}p(x)\right]\left[{}_kI_{a,h}^{\alpha}\left(q\varphi^2\right)(x)\right]+\left[{}_kI_{a,h}^{\alpha}q(x)\right]\left[{}_kI_{a,h}^{\alpha}\left(p\varphi^2\right)(x)\right]$$
$$-2\left[{}_kI_{a,h}^{\alpha}(p\varphi)(x)\right]\left[{}_kI_{a,h}^{\alpha}(q\varphi)(x)\right]$$
$$\leq \int_a^x \frac{1}{k\Gamma_k(\alpha)}\left\{\left[{}_kI_{a,h}^{\alpha}[xq(x)]\int_a^t \left(\log\left(\frac{h(x)}{h(y)}\right)\right)^{\frac{\alpha}{k}-1}\frac{h'(y)}{h(y)}p(y)dy\right.\right.$$
$$\left.-{}_kI_{a,h}^{\alpha}q(x)\int_a^t \left(\log\left(\frac{h(x)}{h(y)}\right)\right)^{\frac{\alpha}{k}-1}\frac{h'(y)}{h(y)}yp(y)dy\right] \quad (2.19)$$
$$+\left[{}_kI_{a,h}^{\alpha}[xp(x)]\int_a^t \left(\log\left(\frac{h(x)}{h(y)}\right)\right)^{\frac{\alpha}{k}-1}\frac{h'(y)}{h(y)}q(y)dy\right.$$
$$\left.\left.-{}_kI_{a,h}^{\alpha}p(x)\int_a^t \left(\log\left(\frac{h(x)}{h(y)}\right)\right)^{\frac{\alpha}{k}-1}\frac{h'(y)}{h(y)}yq(y)dy\right]\right\}[\varphi'(t)]^2\,dt.$$

Proof. *Applying the* $\lim\limits_{s\to -1^+}$ *to the two sides of (2.11), we obtain*

$$\lim_{s\to-1^+}\left\{\left[{}_k^sJ_{a,h}^{\alpha}p(x)\right]\left[{}_k^sJ_{a,h}^{\alpha}\left(q\varphi^2\right)(x)\right]+\left[{}_k^sJ_{a,h}^{\alpha}q(x)\right]\left[{}_k^sJ_{a,h}^{\alpha}\left(p\varphi^2\right)(x)\right]\right.$$
$$\left.-2\left[{}_k^sJ_{a,h}^{\alpha}(p\varphi)(x)\right]\left[{}_k^sJ_{a,h}^{\alpha}(q\varphi)(x)\right]\right\}$$
$$\leq \lim_{s\to-1^+}\left\{\int_a^x \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\left[{}_k^sJ_{a,h}^{\alpha}[xq(x)]\int_a^t \left(h^{s+1}(x)-h^{s+1}(y)\right)^{\frac{\alpha}{k}-1}h^s(y)h'(y)p(y)dy\right.\right.$$
$$\left.-{}_k^sJ_{a,h}^{\alpha}q(x)\int_a^t \left(h^{s+1}(x)-h^{s+1}(y)\right)^{\frac{\alpha}{k}-1}h^s(y)h'(y)yp(y)dy\right][\varphi'(t)]^2\,dt \quad (2.20)$$
$$+\int_a^x \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\left[{}_k^sJ_{a,h}^{\alpha}[xp(x)]\int_a^t \left(h^{s+1}(x)-h^{s+1}(y)\right)^{\frac{\alpha}{k}-1}h^s(y)h'(y)q(y)dy\right.$$
$$\left.\left.-{}_k^sJ_{a,h}^{\alpha}p(x)\int_a^t \left(h^{s+1}(x)-h^{s+1}(y)\right)^{\frac{\alpha}{k}-1}h^s(y)h'(y)yq(y)dy\right][\varphi'(t)]^2\,dt\right\},$$

Thanks to (1.3), we get

$$\left[{}_kI^{\alpha}_{a,h}p(x)\right]\left[{}_kI^{\alpha}_{a,h}(q\varphi^2)(x)\right]+\left[{}_kI^{\alpha}_{a,h}q(x)\right]\left[{}_kI^{\alpha}_{a,h}(p\varphi^2)(x)\right]$$
$$-2\left[{}_kI^{\alpha}_{a,h}(p\varphi)(x)\right]\left[{}_kI^{\alpha}_{a,h}(q\varphi)(x)\right]$$
$$\leq\int_a^x\frac{1}{k\Gamma_k(\alpha)}\Bigg[\ {}_kI^{\alpha}_{a,h}[xq(x)]\int_a^t\lim_{s\to-1^+}\left(\frac{h^{s+1}(x)-h^{s+1}(y)}{s+1}\right)^{\frac{\alpha}{k}-1}h^s(y)h'(y)p(y)dy$$
$$-{}_kI^{\alpha}_{a,h}q(x)\int_a^t\lim_{s\to-1^+}\left(\frac{h^{s+1}(x)-h^{s+1}(y)}{s+1}\right)^{\frac{\alpha}{k}-1}h^s(y)h'(y)yp(y)dy\Bigg][\varphi'(t)]^2dt \quad (2.21)$$
$$+\int_a^x\frac{1}{k\Gamma_k(\alpha)}\Bigg[\ {}_kI^{\alpha}_{a,h}[xp(x)]\int_a^t\lim_{s\to-1^+}\left(\frac{h^{s+1}(x)-h^{s+1}(y)}{s+1}\right)^{\frac{\alpha}{k}-1}h^s(y)h'(y)q(y)dy$$
$$-{}^s_kJ^{\alpha}_{a,h}p(x)\int_a^t\lim_{s\to-1^+}\left(\frac{h^{s+1}(x)-h^{s+1}(y)}{s+1}\right)^{\frac{\alpha}{k}-1}h^s(y)h'(y)yq(y)dy\Bigg][\varphi'(t)]^2dt.$$

Now, we have

$$\lim_{s\to-1^+}\left(\frac{h^{s+1}(x)-h^{s+1}(y)}{s+1}\right)^{\frac{\alpha}{k}-1}h^s(y)=\left(\log\left(\frac{h(x)}{h(y)}\right)\right)^{\frac{\alpha}{k}-1}\frac{1}{h(y)}. \quad (2.22)$$

Replacing (2.22) in (2.21), we obtain (2.19).

Remark 2.1

(i) *Taking* $k=1, s=0$ *and* $p(x)=q(x), h(x)=x$ *in Theorem 2.1, we obtain Theorem 3.2 of [5].*

(ii) *Taking* $k=1$, $p(x)=q(x)$ *and* $h(x)=e^x$ *in Corollary 2.3, we obtain Theorem 3.2 of [5].*

Another main result that needs to be proved is the following theorem:

Theorem2.2 *Let* $f,g\ :\ [a,b]\to\mathrm{R}$ *be two integrable absolutely continuous functions on* $[a,b]$ *and assume thath is a monotone, increasing and positive function on* $(a,b]$, $h\in C^1(a,b)$ *and* $p,q\ :[a,b]\to\mathrm{R}^+$ *are integrable. If* $(f')^2,(g')^2\in L^1[a,b]$. *Then for all* $\alpha>0, a<x\leq b$, *we have*

$$\left[{}^s_kJ^{\alpha}_{a,h}p(x)\right]\left[{}^s_kJ^{\alpha}_{a,h}(qfg)(x)\right]+\left[{}^s_kJ^{\alpha}_{a,h}q(x)\right]\left[{}^s_kJ^{\alpha}_{a,h}(pfg)(x)\right]$$
$$-\left[{}^s_kJ^{\alpha}_{a,h}(pf)(x)\right]\left[{}^s_kJ^{\alpha}_{a,h}(qg)(x)\right]-\left[{}^s_kJ^{\alpha}_{a,h}(qf)(x)\right]\left[{}^s_kJ^{\alpha}_{a,h}(pg)(x)\right] \quad (2.23)$$
$$\leq\left(\int_a^xA(x)[f'(t)]^2dt\right)^{\frac{1}{2}}\left(\int_a^xA(x)[g'(t)]^2dt\right)^{\frac{1}{2}},$$

where $A(x)$ *is given by (2.10) and* $k>0,\ s\in\mathrm{R}-\{-1\}$.

Proof. Using the fractional Korkine identity, we get

$${}^s_kJ^{\alpha}_{a,h}p(x){}^s_kJ^{\alpha}_{a,h}(qfg)(x)+{}^s_kJ^{\alpha}_{a,h}q(x){}^s_kJ^{\alpha}_{a,h}(pfg)(x)$$
$$-{}^s_kJ^{\alpha}_{a,h}(pf)(x){}^s_kJ^{\alpha}_{a,h}(qg)(x)-\left({}^s_kJ^{\alpha}_{a,h}qf(x)\right)\left({}^s_kJ^{\alpha}_{a,h}pg(x)\right)$$
$$=\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^x\int_a^x\Bigg(\Bigg[\left(h^{s+1}(x)-h^{s+1}(r)\right)^{\frac{\alpha}{k}-1}h'(r)h^s(r)p(r)$$
$$\times\left(h^{s+1}(x)-h^{s+1}(\tau)\right)^{\frac{\alpha}{k}-1}h'(\tau)h^s(\tau)q(\tau)\Bigg]^{\frac{1}{2}}[f(r)-f(\tau)]\Bigg)drd\tau$$
$$\times\Bigg(\Bigg[\left(h^{s+1}(x)-h^{s+1}(r)\right)^{\frac{\alpha}{k}-1}h'(r)h^s(r)p(r)$$
$$\times\left(h^{s+1}(x)-h^{s+1}(\tau)\right)^{\frac{\alpha}{k}-1}h'(\tau)h^s(\tau)q(\tau)\Bigg]^{\frac{1}{2}}[g(r)-g(\tau)]\Bigg)drd\tau$$

Using Cauchy Schowarz inequality, we can prove that

$$
\begin{aligned}
&{}_k^s J_{a,h}^{\alpha} p(x)\, {}_k^s J_{a,h}^{\alpha}(qfg)(x) + {}_k^s J_{a,h}^{\alpha} q(x)\, {}_k^s J_{a,h}^{\alpha}(pfg)(x) \\
&-{}_k^s J_{a,h}^{\alpha}(pf)(x)\, {}_k^s J_{a,h}^{\alpha}(qg)(x) - \left({}_k^s J_{a,h}^{\alpha} qf(x)\right)\left({}_k^s J_{a,h}^{\alpha} pg(x)\right) \\
&\leq \left\{ \left(\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \right)^2 \int_a^x \int_a^x \left[\left(h^{s+1}(x) - h^{s+1}(r)\right)^{\frac{\alpha}{k}-1} \left(h^{s+1}(x) - h^{s+1}(\tau)\right)^{\frac{\alpha}{k}-1} \right. \right. \\
&\left. \left. \times h'(r) h^s(r) p(r) h'(\tau) h^s(\tau) q(\tau) \right] \left(f(r) - f(\tau) \right)^2 dr d\tau \right\}^{\frac{1}{2}} \\
&\times \left\{ \left(\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \right)^2 \int_a^x \int_a^x \left[\left(h^{s+1}(x) - h^{s+1}(r)\right)^{\frac{\alpha}{k}-1} \left(h^{s+1}(x) - h^{s+1}(\tau)\right)^{\frac{\alpha}{k}-1} \right. \right. \qquad (2.24) \\
&\left. \left. \times h'(r) h^s(r) p(r) h'(\tau) h^s(\tau) q(\tau) \right] \left(g(r) - g(\tau) \right)^2 dr d\tau \right\}^{\frac{1}{2}} \\
&\leq \left[{}_k^s J_{a,h}^{\alpha} q(x)\, {}_k^s J_{a,h}^{\alpha}(pf^2)(x) + {}_k^s J_{a,h}^{\alpha} p(x)\, {}_k^s J_{a,h}^{\alpha}(qf^2)(x) \right. \\
&\left. - 2\, {}_k^s J_{a,h}^{\alpha} qf(x)\, {}_k^s J_{a,h}^{\alpha}(pf)(x) \right]^{\frac{1}{2}} \\
&\times \left[{}_k^s J_{a,h}^{\alpha} q(x)\, {}_k^s J_{a,h}^{\alpha}(pg^2)(x) + {}_k^s J_{a,h}^{\alpha} p(x)\, {}_k^s J_{a,h}^{\alpha}(qg^2)(x) \right. \\
&\left. - 2\, {}_k^s J_{a,h}^{\alpha} qg(x)\, {}_k^s J_{a,h}^{\alpha}(pg)(x) \right]^{\frac{1}{2}}.
\end{aligned}
$$

According (2.24) with the Theorem 2.1, we obtain (2.23).

Now, we consider the quantity:

$$
\begin{aligned}
B(x) &= \lim_{s \to -1^+} A(x) \\
&= \frac{1}{k\Gamma_k(\alpha)} \left(\left[{}_k I_{a,h}^{\alpha}[xq(x)] \int_a^t \left(\log\left(\tfrac{h(x)}{h(y)}\right)\right)^{\frac{\alpha}{k}-1} \frac{h'(y)}{h(y)} p(y) dy \right. \right. \qquad (2.25) \\
&\quad \left. - {}_k I_{a,h}^{\alpha} q(x) \int_a^t \left(\log\left(\tfrac{h(x)}{h(y)}\right)\right)^{\frac{\alpha}{k}-1} \frac{h'(y)}{h(y)} y p(y) dy \right] \\
&\quad + \left[{}_k I_{a,h}^{\alpha}[xp(x)] \int_a^t \left(\log\left(\tfrac{h(x)}{h(y)}\right)\right)^{\frac{\alpha}{k}-1} \frac{h'(y)}{h(y)} q(y) dy \right. \\
&\quad \left. \left. - {}_k I_{a,h}^{\alpha} p(x) \int_a^t \left(\log\left(\tfrac{h(x)}{h(y)}\right)\right)^{\frac{\alpha}{k}-1} \frac{h'(y)}{h(y)} y q(y) dy \right] \right).
\end{aligned}
$$

Corollary2.4 *Let* $f, g : [a,b] \to \mathrm{R}$ *be two integrable and absolutely continuous functions on* $[a,b]$ *, assume that* h *is a monotone, increasing and positive function on* $(a,b]$, $h \in C^1(a,b)$ *and* $p, q : [a,b] \to \mathrm{R}^+$ *are integrable. If* $(f')^2, (g')^2 \in L^1[a,b]$. *Then for all* $\alpha > 0, a < x \leq b$ *and* $k > 0$*, we have*

$$
\begin{aligned}
&\left[{}_k I_{a,h}^{\alpha} p(x)\right]\left[{}_k I_{a,h}^{\alpha}(qfg)(x)\right] + \left[{}_k I_{a,h}^{\alpha} q(x)\right]\left[{}_k I_{a,h}^{\alpha}(pfg)(x)\right] \\
&-\left[{}_k I_{a,h}^{\alpha}(pf)(x)\right]\left[{}_k I_{a,h}^{\alpha}(qg)(x)\right] - \left[{}_k I_{a,h}^{\alpha}(qf)(x)\right]\left[{}_k I_{a,h}^{\alpha}(pg)(x)\right] \\
&\leq \left(\int_a^x B(x)[f'(t)]^2 dt \right)^{\frac{1}{2}} \left(\int_a^x B(x)[g'(t)]^2 dt \right)^{\frac{1}{2}},
\end{aligned}
\qquad (2.26)
$$

where $B(x)$ *is given by (2.25).*

Proof. By using (1.3) and (2.22) in (2.23), we get (2.26).

Remark 2.2

(1) *Taking* $k = 1, s = 0,$ $p(x) = q(x)$ *and* $h(x) = x$ *in Theorem 2.2, we obtain Theorem 3.6 of [5].*

(2) *Taking* $k = 1, s = 0,$ $p(x) = q(x)$ *and* $h(x) = e^x$ *in Corollary 2.4, we obtain Theorem 3.6 of [5].*

In the case of one nondecreasing function, we prove the following result:

Theorem2.3 *Let* $f : [a,b] \to \mathrm{R}$ *be an increasing absolutely continuous function on* $[a,b]$ *and assume that* h *is a monotone, increasing and positive function on* $(a,b]$ *,* $h \in C^1(a,b)$ *and* p,q *are positive integrable functions on* $[a,b]$ *. If* $g' \in L^\infty[a,b]$ *. Then for all* $\alpha > 0, a < x \le b$ *and* $k > 0$*, we have*

$$\Big| {}_k^s J_{a,h}^{\alpha} p(x) {}_k^s J_{a,h}^{\alpha} qfg(x) + {}_k^s J_{a,h}^{\alpha} q(x) {}_k^s J_{a,h}^{\alpha} pfg(x) - {}_k^s J_{a,h}^{\alpha} pf(x) {}_k^s J_{a,h}^{\alpha} qg(x) - {}_k^s J_{a,h}^{\alpha} qf(x) {}_k^s J_{a,h}^{\alpha} pg(x) \Big| \le \| g' \|_\infty \int_a^x A(x) f'(t) dt, \qquad (2.27)$$

where $A(x)$ *is defined by (2.10).*

Proof. We have,

$$\Big| {}_k^s J_{a,h}^{\alpha} p(x) {}_k^s J_{a,h}^{\alpha} qfg(x) + {}_k^s J_{a,h}^{\alpha} q(x) {}_k^s J_{a,h}^{\alpha} pfg(x) - {}_k^s J_{a,h}^{\alpha} pf(x) {}_k^s J_{a,h}^{\alpha} qg(x) - {}_k^s J_{a,h}^{\alpha} qf(x) {}_k^s J_{a,h}^{\alpha} pg(x) \Big|$$

$$= \left(\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \right)^2 \Big| \int_a^x \int_a^x \left(h^{s+1}(x) - h^{s+1}(r)\right)^{\frac{\alpha}{k}-1} \left(h^{s+1}(x) - h^{s+1}(\tau)\right)^{\frac{\alpha}{k}-1} \times h^s(r) h'(r) h^s(\tau) h'(\tau) p(r) q(\tau) [(f(r) - f(\tau))(g(r) - g(\tau))] dr d\tau \Big| \qquad (2.28)$$

$$\le \left(\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \right)^2 \int_a^x \int_a^x \left(h^{s+1}(x) - h^{s+1}(r)\right)^{\frac{\alpha}{k}-1} \left(h^{s+1}(x) - h^{s+1}(\tau)\right)^{\frac{\alpha}{k}-1} \times h^s(r) h'(r) h^s(\tau) h'(\tau) p(r) q(\tau) |(f(r) - f(\tau))(g(r) - g(\tau))| dr d\tau,$$

now, we have $g' \in L^\infty[a,b],$ so we can write

$$\Big| {}_k^s J_{a,h}^{\alpha} p(x) {}_k^s J_{a,h}^{\alpha} qfg(x) + {}_k^s J_{a,h}^{\alpha} q(x) {}_k^s J_{a,h}^{\alpha} pfg(x) - {}_k^s J_{a,h}^{\alpha} pf(x) {}_k^s J_{a,h}^{\alpha} qg(x) - {}_k^s J_{a,h}^{\alpha} qf(x) {}_k^s J_{a,h}^{\alpha} pg(x) \Big|$$

$$\le \| g' \|_\infty \left(\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \right)^2 \int_a^x \int_a^x \left(h^{s+1}(x) - h^{s+1}(r)\right)^{\frac{\alpha}{k}-1} \left(h^{s+1}(x) - h^{s+1}(\tau)\right)^{\frac{\alpha}{k}-1} \times h^s(r) h'(r) h^s(\tau) h'(\tau) p(r) q(\tau) |(f(r) - f(\tau))(r - \tau)| dr d\tau$$

As f is an increasing function, then we have

$$\Big| {}_k^s J_{a,h}^{\alpha} p(x) {}_k^s J_{a,h}^{\alpha} qfg(x) + {}_k^s J_{a,h}^{\alpha} q(x) {}_k^s J_{a,h}^{\alpha} pfg(x) - {}_k^s J_{a,h}^{\alpha} pf(x) {}_k^s J_{a,h}^{\alpha} qg(x) - {}_k^s J_{a,h}^{\alpha} qf(x) {}_k^s J_{a,h}^{\alpha} pg(x) \Big|$$

$$\le \| g' \|_\infty \left(\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \right)^2 \int_a^x \int_a^x \left(h^{s+1}(x) - h^{s+1}(r)\right)^{\frac{\alpha}{k}-1} \left(h^{s+1}(x) - h^{s+1}(\tau)\right)^{\frac{\alpha}{k}-1} \times h^s(r) h'(r) h^s(\tau) h'(\tau) p(r) q(\tau) (r - \tau)(f(r) - f(\tau)) dr d\tau$$

$$\le \| g' \|_\infty \Big\{ \left[{}_k^s J_{a,h}^{\alpha} p(x)\right]\left[{}_k^s J_{a,h}^{\alpha} (x(qf)(x))\right] + \left[{}_k^s J_{a,h}^{\alpha} q(x)\right]\left[{}_k^s J_{a,h}^{\alpha} (x(pf)(x))\right] - \left[{}_k^s J_{a,h}^{\alpha} (x(p)(x))\right]\left[{}_k^s J_{a,h}^{\alpha} (qf)(x)\right] - \left[{}_k^s J_{a,h}^{\alpha} (x(q)(x))\right]\left[{}_k^s J_{a;h}^{\alpha} (pf)(x)\right] \Big\}.$$

Applying (2.16), we get (2.27).

Corollary2.5 *Let* $f : [a,b] \to \mathrm{R}$ *be an increasing absolutely continuous function on* $[a,b]$ *and assume that* h *is a monotone, increasing and positive function on* $(a,b]$ *,* $h \in C^1(a,b)$ *and suppose tha* p,q *are positive integrable functions on* $[a,b]$ *. If* $g' \in L^\infty[a,b]$ *. Then for all* $\alpha > 0, a < x \le b$ *and* $k > 0$*, we have*

$$\Big| {}_kI_{a,h}^{\alpha}p(x)\,{}_kI_{a,h}^{\alpha}qfg(x)+{}_kI_{a,h}^{\alpha}q(x)\,{}_kI_{a,h}^{\alpha}pfg(x)$$
$$-{}_kI_{a,h}^{\alpha}pf(x)\,{}_kI_{a,h}^{\alpha}qg(x)-{}_kI_{a,h}^{\alpha}qf(x)\,{}_kI_{a,h}^{\alpha}pg(x)\Big| \qquad (2.29)$$
$$\le \| g'\|_\infty \int_a^x B(x)\, f'(t)dt,$$

where $B(x)$ *is defined by (2.25).*

Proof. Replacing (1.3) and (2.22) in (2.277), we get (44).

Remark 2.3

(1) *Taking* $k=1, s=0,\ p(x)=q(x)$ *and* $h(x)=x$ *in Theorem 2.3, we obtain Theorem 3.9 of [5].*

(2) *Taking* $k=1, s=0,\ p(x)=q(x)$ *and* $h(x)=e^x$ *in Corollary 2.5, we obtain Theorem 3.9 of [5].*

Theorem2.4 *Let* $f,g:[a,b]\to \mathrm{R}$ *be two increasing absolutely continuous functions on* $[a,b]$ *and assume that* h *is a monotone, increasing and positive function on* $(a,b]$, $h\in C^1(a,b)$ *and* p,q *are positive integrable functions on* $[a,b]$ *. If* $f',g'\in L^\infty[a,b]$ *. Then for all* $\alpha>0, a<x\le b$ *and* $k>0$*, we have*

$$\Big| {}_k^sJ_{a,h}^{\alpha}p(x)\,{}_k^sJ_{a,h}^{\alpha}qfg(x)+{}_k^sJ_{a,h}^{\alpha}q(x)\,{}_k^sJ_{a,h}^{\alpha}pfg(x)$$
$$-{}_k^sJ_{a,h}^{\alpha}pf(x)\,{}_k^sJ_{a,h}^{\alpha}qg(x)-{}_k^sJ_{a,h}^{\alpha}qf(x)\,{}_k^sJ_{a,h}^{\alpha}pg(x)\Big| \qquad (2.30)$$
$$\le \| f'\|_\infty \| g'\|_\infty \int_a^x A(x)dt,$$

where $A(x)$ *is defined by (2.10).*

Proof. We have,

$$\Big| {}_k^sJ_{a,h}^{\alpha}p(x)\,{}_k^sJ_{a,h}^{\alpha}qfg(x)+{}_k^sJ_{a,h}^{\alpha}q(x)\,{}_k^sJ_{a,h}^{\alpha}pfg(x)$$
$$-{}_k^sJ_{a,h}^{\alpha}pf(x)\,{}_k^sJ_{a,h}^{\alpha}qg(x)-{}_k^sJ_{a,h}^{\alpha}qf(x)\,{}_k^sJ_{a,h}^{\alpha}pg(x)\Big|$$
$$=\left(\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\right)^2 \Big|\int_a^x\int_a^x \left(h^{s+1}(x)-h^{s+1}(r)\right)^{\frac{\alpha}{k}-1}\left(h^{s+1}(x)-h^{s+1}(\tau)\right)^{\frac{\alpha}{k}-1} \qquad (2.31)$$
$$\times h^s(r)h'(r)h^s(\tau)h'(\tau)p(r)q(\tau)\left[(f(r)-f(\tau))(g(r)-g(\tau))\right]drd\tau\Big|$$
$$\le\left(\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\right)^2 \int_a^x\int_a^x \left(h^{s+1}(x)-h^{s+1}(r)\right)^{\frac{\alpha}{k}-1}\left(h^{s+1}(x)-h^{s+1}(\tau)\right)^{\frac{\alpha}{k}-1}$$
$$\times h^s(r)h'(r)h^s(\tau)h'(\tau)p(r)q(\tau)\left|(f(r)-f(\tau))(g(r)-g(\tau))\right|drd\tau,$$

we have $f',g'\in L^\infty[a,b],$ then can write

$$\Big| {}_k^sJ_{a,h}^{\alpha}p(x)\,{}_k^sJ_{a,h}^{\alpha}qfg(x)+{}_k^sJ_{a,h}^{\alpha}q(x)\,{}_k^sJ_{a,h}^{\alpha}pfg(x)$$
$$-{}_k^sJ_{a,h}^{\alpha}pf(x)\,{}_k^sJ_{a,h}^{\alpha}qg(x)-{}_k^sJ_{a,h}^{\alpha}qf(x)\,{}_k^sJ_{a,h}^{\alpha}pg(x)\Big|$$
$$\le \| f'\|_\infty \| g'\|_\infty \left(\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\right)^2 \int_a^x\int_a^x \left(h^{s+1}(x)-h^{s+1}(r)\right)^{\frac{\alpha}{k}-1}\left(h^{s+1}(x)-h^{s+1}(\tau)\right)^{\frac{\alpha}{k}-1}$$
$$\times h^s(r)h'(r)h^s(\tau)h'(\tau)p(r)q(\tau)(r-\tau)^2drd\tau$$
$$\le \| f'\|_\infty \| g'\|_\infty \Big\{\left[{}_k^sJ_{a,h}^{\alpha}p(x)\right]\left[{}_k^sJ_{a,h}^{\alpha}\left(x^2q(x)\right)\right]+\left[{}_k^sJ_{a,h}^{\alpha}q(x)\right]\left[{}_k^sJ_{a,h}^{\alpha}\left(x^2p(x)\right)\right]$$
$$-2\left[{}_k^sJ_{a,h}^{\alpha}\left(xp(x)\right)\right]\left[{}_k^sJ_{a,h}^{\alpha}\left(xq(x)\right)\right]\Big\}.$$

Hence by (2.16), we get (2.30).

Corollary2.6 *Let* $f,g:[a,b]\to \mathrm{R}$ *be two increasing absolutely continuous function on* $[a,b]$ *and assume that* h *is a* *monotone, increasing and positive function on* $(a,b]$, $h \in C^1(a,b)$ *and suppose that* p,q *are positive integrable functions on* $[a,b]$. *If* $f', g' \in L^\infty[a,b]$. *Then for all* $\alpha > 0, a < x \le b$ *and* $k > 0$, *we have*

$$\Big| {}_kI^{\alpha}_{a,h}p(x)\, {}_kI^{\alpha}_{a,h}qfg(x) + {}_kI^{\alpha}_{a,h}q(x)\, {}_kI^{\alpha}_{a,h}pfg(x) - {}_kI^{\alpha}_{a,h}pf(x)\, {}_kI^{\alpha}_{a,h}qg(x) - {}_kI^{\alpha}_{a,h}qf(x)\, {}_kI^{\alpha}_{a,h}pg(x)\Big| \qquad (2.32)$$

$$\le \| f' \|_\infty \| g' \|_\infty \int_a^x B(x)dt,$$

where $B(x)$ *is defined by (2.25).*

Proof. Replacing (1.3) and (2.22) in (2.30), we get (2.32).

REFERENCES

1. M. Bezziou, Z. Dahmani, A. Khameli: *Some Weighted Inequalities of Chebyshev Type Via RL-Approach.* Mathematica Math Jour. 60 (83), No.1, 12-20, (2018).
2. M. Bezziou, Z. Dahmani, A. Khameli: On *some double weighted frctional integral Inequalities.*Saragevo Journal of Mathematics. 15 (1), 23-36, (2019).
3. M. Bezziou, Z. Dahmani and M.Z. Sarikaya: *New operators for fractional integration theory with some applications*, JME. 12 (1), (2018).
4. P.L. Chebyshev: *Sur les expressions approximatives des integralesdefinis par les autres prises entre les memes limite.*Proc. Math. Soc. Charkov, 2, 93-98, (1882).
5. Z. Dahmani, M. D. Bounoua: *Further Results on Chebyshev and Steffensen Inequalities.* Kyungpook Mathematical Journal 58 (1), (2018).
6. Z. Dahmani: *New inequalities in fractional integrals.* International Journal of Nonlinear Sciences, 9(4), 493-497, (2010).
7. Z. Dahmani: *About some integral inequalities using Riemann-Liouville integrals.* General Mathematics, 20(4), 63-69, (2012).
8. Z. Dahmani, O. Mechouar, S. Brahami: *Certain inequalities related to the Chebyshev's functional involving Riemann-Liouville operator.* Bulletin of Mathematical Analysis and Applications, 3(4), 38-44, (2011).
9. H.J. Godwin: *On Generalizations of Chebyshev Inequality.* Journal of the American Statistical Association, 5(27), 923-945, (1955).
10. D.S. Mitrinovic, J. E. Pecaroc, A.M. Fink: *Classical and New Inequalities in Analysis*. Kluwer, Dordrecht, 1993.
11. A.M. Ostrowski: *On an integral inequality*. Aequationes Math, 4 (2), 358-373, (1970).
12. C.P. Niculescu: *An extension of Chebyshev's inequality and its connection with Jensen's inequality*. Journal of Inequalities and Applications, 6 (3), 451-462, (2001).
13. B.G. Pachpatte: *A note on Chebyshev-Grüss type inequalities for differential functions.*Tamsui Oxford Journal of Mathematical Sciences, 22 (1), 29-36, (2006).
14. S.G. Samko, A.A. Kilbas, O. I. Marichev: *Fractional Integrals and Derivatives Theory and Application.*Gordan and Breach Science, New York, (1993).
15. M.Z. Sarikatya, H. Yaldiz: *New generalization fractional inequalities of Ostrowski-Gruss type.*Lobachevskii Journal of Mathematics, 34(4), 326-331, (2013).