AN APPROXIMATE SOLUTION FOR CASIMIR OSCILLATOR EQUATION USING LAPLACE DECOMPOSITION METHOD

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ABSTRACT: *In quantum field theory, unavoidable roles are having on the operation of micromechanical system. One of the quantum phenomena that have these roles is the Casimir effect [1]. In present paper, we study a simple linear harmonic micro spring that is under the influence of the Casimir pressure/force. It is behaving as an anharmonic-nonlinear Casimir oscillator. Since the equation of motion of this nonlinear-micromechanical Casimir oscillator has no precise solvable solution and turning points of the system have no fixed points we consider the Laplace decomposition method for obtaining series solutions of nonlinear oscillator differential equations. The equations are Laplace transformed and the nonlinear Casimir effect terms are represented by Adomian's polynomials. The results illustrate that Laplace decomposition method is an appropriate method in solving the highly nonlinear equations.*

Keywords: Casimir effect, nonlinear oscillator, Laplace transform, Adomian's polynomials, Laplace decomposition method.

I- INTRODUCTION

The Casimir effect, obtained by Casimir 1948, emerge from quantum fluctuations in the electromagnetic field [2,3]. As a result of the existence of these fluctuations, two uncharged parallel conducting plates, closely spaced, must experience an attractive force when the gap between opposing surface is less than a micrometer wide [4]. The magnitude of the attractive Casimir force per unit area A

$$F_c = \frac{\pi^2}{240}\ \frac{\hbar\, c}{x^4}$$

where c the speed of the light, ħ is Planck constant $h/2\pi$ and x is the plate separation.

The significance of the Casimir effect in Nano systems has been considered since about three decades ago. Because of the strong attractive Casimir force at small scales, moving parts of micromechanical system may stick to each other; this quantum phenomenon is called stiction, which is a troublemaker effect in micromechanical systems and can make them unstable [5].

Several an approximate methods, such as homotopy analysis method [6,7], Adomian decomposition method [8,9], homotopy perturbation method [10-12], differential transform method [13] , variational iteration method [14-16], Laplace decomposition method [17,18] and the homotopy perturbation transform method [19] were applied and introduced to get an approximate solutions. In this paper, we study a simple model of an oscillating system (micro spring) under the act of the Casimir force; then, we try to estimate and approximate the solution of this anharmonic nonlinear system using the Laplace decomposition method.

II- NONLINEAR CASIMIR OSCILLATOR MODEL

The geometry of the nonlinear casimir oscillator is shown in Fig.(1) where we consider a simple configuration consisting of a spring obeying Hook's law connected between two parallel plates. One of them is fixed and the other is moving. In this figure, x_0 denotes the normalized equilibrium position of the movable plate in the absence of the Casimir pressure, which represents the length of the spring with an elastic constant (k).

The moving plate is under the influence of two forces, one is the well-known restoring force of the spring and the other one is the effect of the Casimir pressure (force). The differential equation of motion for a nonlinear Casimir oscillator in one dimension, just perpendicular to the area of the plate and parallel to the spring length, is

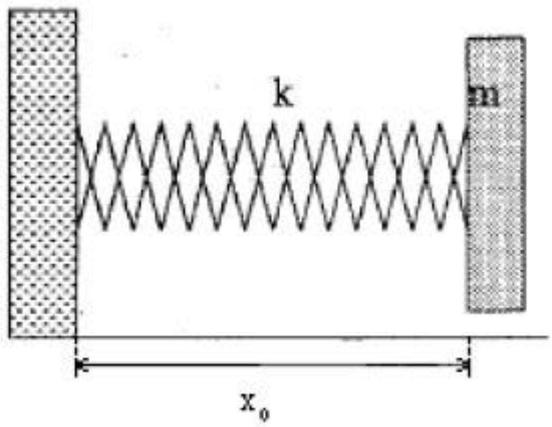

Figure (1) : A micro spring connected between two parallel conducting plates

$$\rho\ \frac{d^2x(t)}{dt^2} + \frac{\gamma}{A}\frac{dx}{dt} + \frac{k}{A}(x - x_0) = \frac{-\pi^2\hbar c}{240x^4} \qquad (1)$$

where ρ represents the mass of the unit area (A) of the moving plate, and γ is the possible damping spring's coefficient and it is good approximation to neglect it ($\gamma = 0$) at micro (Nano) world scales ; we get

$$\rho\ \frac{d^2x(t)}{dt^2} + \frac{k}{A}(x - x_0) = \frac{-\pi^2\hbar c}{240x^4} \qquad (2)$$

rewrite the last equation, we get a simple form

$$\ddot{x} + \frac{k}{\rho A}(x - x_0) = \frac{-\pi^2\hbar c}{240\rho x^4} \qquad (3)$$

In the absence of the Casimir effect $\hbar \to 0$, equation (3) reduces to the familiar linear harmonic oscillator which has well-known solutions. However, in the presence of the Casimir effect, it is a nonlinear differential equation which we will be solved by Laplace decomposition method.

III- LAPLACE DECOMPOSITION METHOD

The Laplace transform is considered an elementary, but useful technique for solving linear ordinary differential equations that are widely used by scientists and engineers for obstruction linearized models. In fact, the Laplace transform is one of only a few techniques that can be applied to linear systems. Although the Laplace transform is great usefulness in solving linear problems, it is totally incapable of handling nonlinear differential equations because of the difficulties that are caused by the nonlinear terms. This paper considers the effectiveness of the Laplace decomposition method in solving nonlinear Casimir oscillator differential equations. The equation will be transformed using Laplace algorithm, and the nonlinear will be represented by the Adomian's polynomials.

Many papers introduced this method to solve a various nonlinear partial differential equations. Khuri [20] used this method for the approximate solution of a class of a nonlinear ordinary differential equations. Handibag and Karande [21] applied this method for the solution of the linear and nonlinear heat equation. Elgazery [22] exploited this method to solve Falkner-Skan equation. The Laplace decomposition method was employed in [23] to get approximate analytical solutions of the linear and the nonlinear fractional diffusion-wave equations.

To illustrate the idea of the Laplace decomposition method, we consider the nonlinear ordinary differential equation in general form as:

$$Gu = g \quad \Rrightarrow \quad Lu + Ru + Nu = g \qquad (4)$$

where G represents a nonlinear differential operator. The method consists of decomposing the linear part of G in L+R , where L is an operator that have inverse L^{-1} , and R is the remaining part . denote the nonlinear term by N. the first step in Laplace decomposition method is the applying of Laplace transform on both sides of equation (4), we get

$$L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu \qquad (5)$$

The second step represents the key of this technique, the nonlinear term N in equation (4) is decomposed into particular series of polynomials as follows:

$$Nu = \sum_{n=0}^{\infty} A_n \quad ; A_n \equiv \text{Adomian polynomials.} \qquad (6)$$

where

$$A_n(u_0, u_1, u_2, \dots, u_n) = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0} \qquad (7)$$

The first five Adomian polynomials [24] for the variable Nu=f(u) are given by:

$A_0 = f(u_0)$, $A_1 = u_1 f'(u_0)$

$A_2 = u_2 f'(u_0) + \frac{1}{2!}u_1^2 f''(u_0)$, $A_3 = y_3 f'(u_0) + u_1 u_2 f''(u_0) + \frac{1}{3!}u_1^3 f^{(3)}(u_0)$

$A_4 = u_4 f'(u_0) + \left(u_1 u_3 + \frac{1}{2!}u_2^2\right) f''(u_0) + \frac{1}{2!}u_1^2 u_2 f^{(3)}(u_0) + \frac{1}{4!}u_1^4 f^{(4)}(u_0)$ and so on ... (8)

In the next section, the Laplace decomposition method will be introduced to obtain an approximate solution for equation (3) and the $x(t)$ will be plotted. It is important point to mention the constant x_0 that is included in the problem will be in micro(Nano) scales and to be able to apply the simple Casimir force, the area (A) should be much larger than the second power of the separation distance $(A \gg x_0^2)$.

IV- METHOD OF SOLUTION AND DISCUSSION

To apply the Laplace decomposition method we consider equation (3) in a simple form with initial conditions given by:

$$\ddot{x} + \alpha x + \gamma x^{-4} = \beta \qquad \text{where } \alpha = \frac{k}{\rho A}, \beta = \frac{k}{\rho A} x_0 \text{ and } \gamma = \frac{\pi^2 \hbar c}{240\rho} \qquad (9)$$

with $x(0) = c$ and $\dot{x}(0) = 0$

according to the Laplace decomposition method, taking Laplace transform of both sides of equation (9):

$$s^2\mathcal{L}\{x(t)\} - sx(0) - x'(0) = -\alpha\mathcal{L}\{x(t)\} + \mathcal{L}\{\beta\} - \mathcal{L}\{\gamma x^{-4}\} \qquad (10)$$

The initial conditions imply

$$s^2\mathcal{L}\{x(t)\} = c\,s - \alpha\mathcal{L}\{x(t)\} + \mathcal{L}\{\beta\} - \gamma\mathcal{L}\{x^{-4}\} \qquad (11)$$

or

$$\mathcal{L}\{x(t)\} = \frac{c}{s} - \frac{\alpha}{s^2}\mathcal{L}\{x(t)\} + \frac{1}{s^2}\mathcal{L}\{\beta\} - \frac{\gamma}{s^2}\mathcal{L}\{x^{-4}\} \qquad (12)$$

Following the technique, if we assume an infinite series solution of the form

$$x(t) = \sum_{n=0}^{\infty} x_n(t) \qquad (13)$$

We obtain

$$\mathcal{L}\left\{\sum_{n=0}^{\infty} x(t)\right\} = \frac{c}{s} - \frac{\alpha}{s^2}\mathcal{L}\left\{\sum_{n=0}^{\infty} x(t)\right\} + \frac{1}{s^2}\mathcal{L}\{\beta\} - \frac{\gamma}{s^2}\mathcal{L}\left\{\sum_{n=0}^{\infty} A_n\right\} \qquad (14)$$

where the nonlinear operator $f(x) = x^{-4}$ is decomposed as in (6) in terms of the Adomian polynomials

$A_0 = {x_0}^{-4}$, $A_1 = -4x_1{x_0}^{-5}$, $A_2 = -4x_2{x_0}^{-5} + 10{x_1}^2{x_0}^{-6}$,

$A_3 = -4x_3{x_0}^{-5} + 20x_1x_2{x_0}^{-6} - 20{x_1}^3{x_0}^{-7}$ ... and so on .

Take the inverse Laplace transform of equation (14) .

$$\sum_{n=0}^{\infty} x(t) = c + \frac{\beta}{2}t^2 - \mathcal{L}^{-1}\left\{\frac{\alpha}{s^2}\mathcal{L}\left\{\sum_{n=0}^{\infty} x(t)\right\} - \mathcal{L}^{-1}\left\{\frac{\gamma}{s^2}\mathcal{L}\left\{\sum_{n=0}^{\infty} A_n\right\}\right\}\right\} \qquad (15)$$

Upon using the linearity of Laplace transform then matching both sides of (14), results in iteration scheme

$$x_0(t) = \mathcal{L}^{-1}\left\{\frac{c}{s}\right\} \quad , \quad x_1(t) = \mathcal{L}^{-1}\left\{\frac{\beta}{s^3}\right\}$$

$$x_2(t) = -\mathcal{L}^{-1}\left\{\frac{\alpha}{s^2}\mathcal{L}\{x_0(t)\}\right\} - \mathcal{L}^{-1}\left\{\frac{\gamma}{s^2}\mathcal{L}\{A_0\}\right\}$$

In general

$$x_{n+2}(t) = -\mathcal{L}^{-1}\left\{\frac{\alpha}{s^2}\mathcal{L}\{x_n(t)\}\right\} - \mathcal{L}^{-1}\left\{\frac{\gamma}{s^2}\mathcal{L}\{A_n\}\right\} \qquad (16)$$

So, the first few terms of the solution will be

$$x_0(t) = c\,, \qquad x_1(t) = \frac{\beta}{2}t^2$$

$$x_2(t) = -\mathcal{L}^{-1}\left\{\frac{\alpha}{s^2}\mathcal{L}\{c\}\right\} - \mathcal{L}^{-1}\left\{\frac{\gamma}{s^2}\mathcal{L}\{c^{-4}\}\right\} = -\alpha c\frac{t^2}{2!} - \gamma c^{-4}\frac{t^2}{2!}$$

$$x_3(t) = -\mathcal{L}^{-1}\left\{\frac{\alpha}{s^2}\mathcal{L}\left\{\frac{\beta}{2}t^2\right\}\right\} - \mathcal{L}^{-1}\left\{\frac{\gamma}{s^2}\mathcal{L}\left\{-4\frac{\beta}{2}t^2c^{-5}\right\}\right\}$$
$$= -\alpha\beta\frac{t^4}{4!} + 4\gamma\beta c^{-5}\frac{t^4}{4!}$$
$$x_4(t) = -\mathcal{L}^{-1}\left\{\frac{\alpha}{s^2}\mathcal{L}\left\{-\alpha c\frac{t^2}{2!} - \gamma c^{-4}\frac{t^2}{2!}\right\}\right\}$$
$$-\mathcal{L}^{-1}\left\{\frac{\gamma}{s^2}\mathcal{L}\left\{-4\left(-\alpha c\frac{t^2}{2!} - \gamma c^{-4}\frac{t^2}{2!}\right)c^{-5} + 10\left(\frac{\beta}{2}t^2\right)^2 c^{-6}\right\}\right\}$$
$$= \alpha^2 c\frac{t^4}{4!} - 3\gamma c^{-4}\frac{t^4}{4!} - 2\gamma^2 c^{-9}\frac{t^4}{4!} - (10)(3!)\beta^2\gamma c^{-6}\frac{t^6}{6!}$$

and so on …

Therefore the approximate solution is:

$$x(t) = x_0(t) + x_1(t) + x_2(t) + x_3(t) + x_4(t) + \cdots \quad (17)$$

or

$$x(t) = c + \frac{\beta}{2}t^2 - \alpha c\frac{t^2}{2!} - \gamma c^{-4}\frac{t^2}{2!} - \alpha\beta\frac{t^4}{4!} + 4\gamma\beta c^{-5}\frac{t^4}{4!} + \alpha^2 c\frac{t^4}{4!} - 3\gamma c^{-4}\frac{t^4}{4!} - 2\gamma^2 c^{-9}\frac{t^4}{4!} - (10)(3!)\beta^2\gamma c^{-6}\frac{t^6}{6!} + \cdots \quad (18)$$

V- CONCLUSION

In this paper we considered a model of nonlinear Casimir oscillator consisted of linear harmonic oscillator, which is under the influence of Casimir effect. The equation of the model was solved using a powerful and successive method which is known as Laplace decomposition method. We decomposed the nonlinear terms of the problem to the Adomian polynomials. The obtained approximate solution has a form of a series. Some figures were plotted to construct the effect of the Casimir force on the micromechanical system of the linear harmonic oscillator. It was noted that the tool found the solutions without any discretization or restrictive assumptions. The scheme described in this paper is expected to be further employed to solve most of the nonlinear problems in science.

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