# ESTIMATION FOR PARAMETER ON GAMMA DISTRIBUTION BY MAXIMUM A POSTERIOR APPROACH

**Tammarat K.[1] ,Noppakun T.[2,*]**

[1]*Rajamangala University of Technology Isan, Faculty of Sciences and Liberal Arts,,Department of Applied Mathematics and Statistics Nakhon Ratchasima, Thailand, 30000*

[2]*Kalasin University, Faculty of Sciences and Health Technology, Department of Science and Mathematics Kalasin, Thailand, 46000*

[*]For correspondence; E-mail: nop_stat@hotmail.com

***ABSTRACT:*** *The objective of this research is to estimate the true parameter of gamma distribution by using Maximum a Posterior approach, Bayes approach, Maximum Likelihood approach which are Maximum a Posterior estimator, Bayes estimator and Maximum Likelihood estimator, respectively. These estimators are compared with Mean Square Error for estimation of the true parameter when the data drown from a gamma distribution. In the case of the small sample size, the Maximum a Posterior estimator is quite well as compared with Bayes estimator and Maximum Likelihood estimator base on Mean Square Error. In another case of the sample size appear that Maximum a Posterior estimator and Bayes estimators are less Mean Square Error than Maximum Likelihood estimators. Our results suggest that Maximum a Posterior estimator for estimation of the true parameter of gamma distribution because Mean Square Error of Maximum a Posterior estimator is quite well when it is compared with Bayes estimator and Maximum Likelihood estimator.*

**Keywords:** Maximum a Posterior, Bayes, Maximum Likelihood, Mean Square Error

## 1. INTRODUCTION

Estimation parameter $\theta$ is one of the statistical inferences that infer to population. Estimation $\theta$ presents two ways as point estimator and interval estimator by using the sample drown from the population. A point estimator is focusing on a single value referred to population, and a point estimation is called an estimator $\theta$. The popular approaches of estimation for the parameter $\theta$ are the Maximum Likelihood (ML) approach and Bayes (BAY) approach which are called ML estimator and BAY estimator, respectively. ML estimator and BAY estimator are widely employed in the estimation of the parameter, for example, Jae [1] used an ML estimator on a tutorial exposition and [2] implement an ML estimator on stochastic volatility models by using Monte Carlo simulations. Moreover, Nilanjan et al. [3] proposed an alternative ML estimator of using information from external big datasets while building refined regression models based on an individual analytic study. Kirsty, Rebecca and Julian [4] used an inverse gamma prior distribution of variance in Bayesian meta-analysis which led to more accurate estimates as well as Ameera and Khawla [5] conducted with a BAY estimator for a variance by using inverse gamma prior on the one-way repeated measurements model as a mixed model.

An indicator of the accuracy of parameter estimation is mean square error (MSE) which is a measure of the quality of an estimator. The MSE is defended as

$$\mathrm{MSE}(\hat{\theta}) = E\left(\hat{\theta}-\theta\right)^2 \qquad (1)$$

In this research is focused on the gamma distribution. The gamma distribution is presented in the fields of engineering, science, and business. For example, Roding et. al [6] estimated the mean self-diffusion coefficient with the gamma distribution model as well as Ramman et al. [7] studied the estimated parameter of the gamma distribution for modeling lifetime data. In addition, Gregory, Jeol, and Chris [8] used the gamma distribution to represent monthly rainfall in Africa for drought monitoring applications. The probability density functions of gamma distribution with the two parameters as $\alpha > 0$ and $\lambda > 0$ is

$$f\left(x;\alpha,\lambda\right) = \frac{1}{\Gamma(\alpha)}\lambda^{\alpha}x^{\alpha-1}e^{-\lambda x},\ x > 0 \qquad (2)$$

and it denoted by $Gamma\left(\alpha,\lambda\right)$ where the mean of X and the variance of X are $E\left(X\right) = \frac{\alpha}{\lambda}$ $V\left(X\right) = \frac{\alpha}{\lambda^2}$ and, respectively.

The imported parameter of the gamma distribution is the true parameter $\lambda$ related to the parameter on Poisson distribution and exponential distribution. Poisson distribution has the parameter as $\lambda$ where it is the mean rate which is the number of successes that occur in a fixed interval of time. The exponential distribution is associated with gamma distribution in the average waiting time until successes for the first time.

Estimation of the true parameter $\lambda$ of gamma distribution with the ML approach and BAY approach as follows.

The ML estimator is well-known that it is a general approach for parameter estimation. This estimator is to research the maximize of likelihood function given the parameter $\lambda$ and $\alpha$. Let the data $\mathbf{x} = \left(x_1, x_2, ..., x_n\right)$ drown from the gamma distribution of population, and the likelihood function of the gamma distribution $L\left(\alpha,\lambda|\mathbf{x}\right)$ is

$$L\left(\alpha,\lambda|\mathbf{x}\right) = \prod_{i=1}^{n} f\left(x_i|\alpha,\lambda\right)$$

$$= \left(\frac{\lambda^{\alpha}}{\Gamma(\alpha)}\right)^{n}\prod_{i=1}^{n} x^{\alpha-1}e^{-\lambda\sum_{i=1}^{n}x_i} \qquad (3)$$

After that, the log-likelihood function of the gamma distribution is

$$\ln L\left(\alpha,\lambda|\mathbf{x}\right) = \ln\prod_{i=1}^{n} f\left(x_i|\alpha,\lambda\right)$$

$$= n\alpha\ln(\lambda) - n\ln(\Gamma(\alpha)) + (\alpha-1)\sum_{i=1}^{n}\ln x_i - \lambda\sum_{i=1}^{n}x_i \qquad (4)$$

Next, the partial derivative of the log-likelihood function in (3) with respect to $\lambda$ is

$$\frac{\partial \ln L(\alpha,\lambda|\mathbf{x})}{\partial\lambda} = \frac{\partial}{\partial\lambda}\left(n\alpha\ln(\lambda) - n\ln(\Gamma(\alpha)) + (\alpha-1)\sum_{i=1}^{n}\ln x_i - \lambda\sum_{i=1}^{n}x_i\right)$$

$$= \frac{n\alpha}{\lambda} - \sum_{i=1}^{n}x_i \qquad (5)$$

Finally, setting in (4) is equal to zero given by

$$\frac{n\alpha}{\lambda} - \sum_{i=1}^{n}x_i = 0$$

$$\lambda = \frac{\alpha}{\bar{x}} \qquad (6)$$

Hence, the ML estimator is $\hat{\lambda}_{MLE} = \frac{\alpha}{\bar{x}}$.

The BAY estimator employed both the evidence contained in the data and the accumulated prior distribution of the true parameter $\lambda$. The conjugate prior distribution of the true parameter $\lambda$ is the gamma distribution with the parameters a and b hence the density function of $\lambda$ is

$$p(\lambda;a,b) = \frac{b^a}{\Gamma(a)}\lambda^{a-1}e^{-b\lambda}, \ \lambda > 0. \qquad (7)$$

And the likelihood function is

$$L(\alpha,\lambda|\mathbf{x}) = \left(\frac{\lambda^{\alpha}}{\Gamma(\alpha)}\right)^n \prod_{i=1}^{n} x^{\alpha-1} e^{-\lambda\sum_{i=1}^{n}x_i} \qquad (8)$$

Hence, the posterior distribution is

$$p(\lambda|\mathbf{x},a,b,\alpha) \propto \frac{b^a}{(\Gamma(\alpha))^n\Gamma(a)}\prod_{i=1}^{n}x^{\alpha-1}\lambda^{n\alpha+a-1}e^{-\lambda\left(\sum_{i=1}^{n}x_i+b\right)} \qquad (9)$$

The posterior distribution has distributed in a class of gamma distribution as $\lambda|\mathbf{x} \sim gamma(n\alpha+a, n\bar{x}+b)$.

The estimator by using Bayes approach is

$$\hat{\lambda}_{Bayes} = E(\lambda|\mathbf{x}) = \int \lambda p(\lambda|\mathbf{x})d\lambda = \frac{n\alpha+a}{n\bar{x}+b} \qquad (10)$$

Moreover, an interesting approach to estimating the true parameter $\lambda$ associated with the BAY approach is the Maximum a Posterior (MAP) approach, and it is used a process of BAY approach and ML approach. This estimator from the MAP approach is called the MAP estimator. The step of the MAP estimator is as follows. From the posterior distribution in (9), the maximum value of the true parameter $\lambda$ is determined as

$$\hat{\lambda}_{MAP} = \arg\max_{\lambda} \ln p(\lambda|\mathbf{x},a,b,\alpha). \qquad (11)$$

In this research, we suggested the MAP approach for parameter estimation is an alternative of estimation for the true parameter $\lambda$ on the gamma distribution. The MAP estimator is compared with ML estimator and BAY estimator based on MSE which is a measure of the quality of an estimator. This research is organized as follows. In section 2, the MAP estimator with gamma prior distribution is described. In section 3, simulation studies and the results were showed and section 4 the conclusion were presented.

## 2. MAP estimator with Gamma Prior Distribution

In this section, it presented the process of MAP estimator for estimating a rate parameter on gamma distribution as follows. Let $\lambda$ is distributed as gamma prior distribution with the scale parameter a > 0 and the rate parameter b > 0. Hence, the probability density function $\lambda$ is

$$f(\lambda;\mathrm{a},\mathrm{b}) = \frac{1}{\Gamma(a)}b^a\lambda^{a-1}e^{-b\lambda} \qquad (12)$$

The posterior distribution in (9) of $\lambda$ is

$$p(\lambda|\mathbf{x},a,b,\alpha) \propto \frac{b^a}{(\Gamma(\alpha))^n\Gamma(a)}\prod_{i=1}^{n}x^{\alpha-1}\lambda^{n\alpha+a-1}e^{-\lambda\left(\sum_{i=1}^{n}x_i+b\right)} \qquad (13)$$

The MAP estimator $\hat{\lambda}_{MAP}$ is the maximum of $\ln p(\lambda|\mathbf{x},a,b,\alpha)$ respect $\lambda$.

The maximum of $\ln p(\lambda|\mathbf{x},a,b,\alpha)$ respect to $\lambda$ is

$$\frac{\partial}{\partial\lambda}\ln p(\lambda|\mathbf{x},a,b,\alpha) = \frac{\partial}{\partial\lambda}(n\alpha+a)\ln\alpha - \frac{\partial}{\partial\lambda}\ln\lambda - \frac{\partial}{\partial\lambda}n\lambda\bar{x} - \frac{\partial}{\partial\lambda}b\lambda$$

$$= \frac{n\alpha+a}{\lambda} - \frac{1}{\lambda} - n\bar{x} - b. \qquad (14)$$

From in (14) find $\hat{\lambda}_{MAP}$, we set $\frac{n\alpha+a}{\alpha} - \frac{1}{\lambda} - n\bar{x} - b = 0$ that is

$$\frac{n\alpha+a}{\alpha} - \frac{1}{\lambda} - n\bar{x} - b = 0$$

$$\frac{n\alpha+a}{\lambda} - \frac{1}{\lambda} = n\bar{x} + b$$

$$\frac{1}{\lambda}(n\alpha+a-1) = n\bar{x}+b$$

$$\frac{1}{\lambda} = \frac{n\bar{x}+b}{(n\alpha+a-1)}$$

$$\lambda = \frac{n\alpha+a-1}{n\bar{x}+b}. \qquad (15)$$

The result in (15) is the MAP estimator $\hat{\lambda}_{MAP} = \frac{n\alpha+a-1}{n\bar{x}+b}$ with $a = 2$ and $b = 1$.

## 3. SIMULATION STUDY AND RESULTS

In this section, the estimated value of the rate parameter $\lambda$ on gamma distribution by ML approach, BAY approach, MAP approach is ML estimator, BAY estimator, MAP estimator, respectively. Further, the MSE of theses estimators is presented in Table I and II by Monte Carlo simulation. We performed a Monte Carlo simulation consisting of 100,000 iterations to compute the estimated value of ML estimator, BAY estimator, MAP estimator, and compute MSE of ML estimator, BAY estimator, MAP estimator. We denoted that $\lambda = 2, 4$ and sample sizes ranging from very small to moderate under $\alpha = 4,5,6,7,8$.

**Table 1. The estimated value of ML estimator, BAY estimator, MAP estimator, and MSE of ML estimator, BAY estimator, MAP estimator when $\lambda = 2$.**

| $\alpha$ | n | $\hat{\lambda}_{ML}$ | $\hat{\lambda}_{BAY}$ | $\hat{\lambda}_{MAP}$ | MSE ($\hat{\lambda}_{ML}$) | MSE ($\hat{\lambda}_{BAY}$) | MSE ($\hat{\lambda}_{MAP}$) |
|---|---|---|---|---|---|---|---|
| 4 | 5 | 2.1051 | 2.0852 | 1.9905 | 0.2572 | 0.1992 | 0.1750 |
| | 10 | 2.0506 | 2.0456 | 1.9969 | 0.1133 | 0.1011 | 0.0943 |
| | 15 | 2.0350 | 2.0327 | 2.0000 | 0.0722 | 0.0671 | 0.0639 |
| | 20 | 2.0256 | 2.0243 | 1.9996 | 0.0534 | 0.0506 | 0.0488 |
| | 30 | 2.0167 | 2.0162 | 1.9996 | 0.0346 | 0.0334 | 0.0326 |
| 5 | 5 | 2.0811 | 2.0685 | 1.9919 | 0.1946 | 0.1600 | 0.1441 |
| | 10 | 2.0405 | 2.0373 | 1.9981 | 0.0879 | 0.0804 | 0.0760 |
| | 15 | 2.0273 | 2.0258 | 1.9995 | 0.0572 | 0.0540 | 0.0520 |
| | 20 | 2.0194 | 2.0186 | 1.9988 | 0.0420 | 0.0403 | 0.0392 |
| | 30 | 2.0135 | 2.0131 | 1.9999 | 0.0276 | 0.0268 | 0.0263 |
| 6 | 5 | 2.0698 | 2.0608 | 1.9964 | 0.1594 | 0.1360 | 0.1241 |
| | 10 | 2.0330 | 2.0308 | 1.9981 | 0.0724 | 0.0673 | 0.0642 |
| | 15 | 2.0234 | 2.0224 | 2.0004 | 0.0476 | 0.0454 | 0.0439 |
| | 20 | 2.0175 | 2.0170 | 2.0004 | 0.0349 | 0.0337 | 0.0329 |
| | 30 | 2.0109 | 2.0107 | 1.9996 | 0.0229 | 0.0224 | 0.0220 |
| 7 | 5 | 2.0572 | 2.0508 | 1.9953 | 0.1314 | 0.1151 | 0.1065 |
| | 10 | 2.0291 | 2.0275 | 1.9993 | 0.0614 | 0.0577 | 0.0554 |
| | 15 | 2.0199 | 2.0192 | 2.0003 | 0.0403 | 0.0387 | 0.0376 |
| | 20 | 2.0135 | 2.0131 | 1.9989 | 0.0294 | 0.0285 | 0.0280 |
| | 30 | 2.0100 | 2.0098 | 2.0003 | 0.0196 | 0.0192 | 0.0190 |
| 8 | 5 | 2.0504 | 2.0455 | 1.9968 | 0.1131 | 0.1009 | 0.0942 |
| | 10 | 2.0258 | 2.0245 | 1.9999 | 0.0537 | 0.0509 | 0.0491 |
| | 15 | 2.0166 | 2.0160 | 1.9995 | 0.0348 | 0.0336 | 0.0328 |
| | 20 | 2.0135 | 2.0132 | 2.0008 | 0.0258 | 0.0251 | 0.0247 |
| | 30 | 2.0087 | 2.0085 | 2.0002 | 0.0170 | 0.0167 | 0.0165 |

**Table 2. The estimated value of ML estimator, BAY estimator, MAP estimator, and MSE of ML estimator, BAY estimator, MAP estimator when $\lambda = 4$.**

| $\alpha$ | n | $\hat{\lambda}_{ML}$ | $\hat{\lambda}_{BAY}$ | $\hat{\lambda}_{MAP}$ | MSE ($\hat{\lambda}_{ML}$) | MSE ($\hat{\lambda}_{BAY}$) | MSE ($\hat{\lambda}_{MAP}$) |
|---|---|---|---|---|---|---|---|
| 4 | 5 | 4.2098 | 3.7960 | 3.6235 | 1.0297 | 0.5622 | 0.5161 |
| | 10 | 4.1038 | 3.8994 | 3.8066 | 0.4556 | 0.3360 | 0.3480 |
| | 15 | 4.0697 | 3.9341 | 3.8707 | 0.2927 | 0.2387 | 0.2436 |
| | 20 | 4.0521 | 3.9508 | 3.9026 | 0.2139 | 0.1837 | 0.1863 |
| | 30 | 4.0324 | 3.9653 | 3.9328 | 0.1397 | 0.1265 | 0.1278 |
| 5 | 5 | 4.1703 | 3.8398 | 3.6976 | 0.7886 | 0.4837 | 0.4162 |
| | 10 | 4.0825 | 3.9197 | 3.8444 | 0.3490 | 0.2736 | 0.2812 |
| | 15 | 4.0533 | 3.9454 | 3.8942 | 0.2267 | 0.1931 | 0.1964 |
| | 20 | 4.0395 | 3.9588 | 3.9200 | 0.1679 | 0.1490 | 0.1508 |
| | 30 | 4.0277 | 3.9740 | 3.9479 | 0.1103 | 0.1017 | 0.1024 |
| 6 | 5 | 4.1325 | 3.8600 | 3.7394 | 0.6202 | 0.4168 | 0.4107 |
| | 10 | 4.0697 | 3.9342 | 3.8707 | 0.2907 | 0.2372 | 0.2421 |
| | 15 | 4.0445 | 3.9547 | 3.9117 | 0.1895 | 0.1658 | 0.1680 |
| | 20 | 4.0319 | 3.9648 | 3.9323 | 0.1392 | 0.1261 | 0.1274 |
| | 30 | 4.0201 | 3.9755 | 3.9537 | 0.0907 | 0.0850 | 0.0856 |
| 7 | 5 | 4.1203 | 3.8860 | 3.7810 | 0.5288 | 0.3730 | 0.3608 |
| | 10 | 4.0574 | 3.9417 | 3.8869 | 0.2437 | 0.2052 | 0.2090 |
| | 15 | 4.0395 | 3.9626 | 3.9256 | 0.1593 | 0.1419 | 0.1434 |
| | 20 | 4.0264 | 3.9690 | 3.9410 | 0.1185 | 0.1090 | 0.1100 |
| | 30 | 4.0196 | 3.9813 | 3.9625 | 0.0781 | 0.0737 | 0.0741 |
| 8 | 5 | 4.1029 | 3.8986 | 3.8058 | 0.4539 | 0.3351 | 0.3273 |
| | 10 | 4.0525 | 3.9512 | 3.9030 | 0.2141 | 0.1837 | 0.1864 |
| | 15 | 4.0345 | 3.9672 | 3.9347 | 0.1399 | 0.1264 | 0.1275 |
| | 20 | 4.0264 | 3.9760 | 3.9515 | 0.1038 | 0.0962 | 0.0968 |
| | 30 | 4.0185 | 3.9850 | 3.9685 | 0.0683 | 0.0649 | 0.0651 |

According to Table 1, by observing the estimated value of the true parameter $(\lambda = 2)$ and MSE of ML estimator, the BAY estimator and MAP estimator are showed as follows. The results indicated that for $\alpha = 4,5,6,7,8$ and n = 5, MSE of BAY estimator, ML estimator, and MAP estimator are more than 0.1 but a case for MAP estimator less than 0.1 as n = 5 and $\alpha = 8$. Again by observing in case n = 5, the MAP estimator is quite well as compared with the BAY estimator and ML estimator under MSE. When n = 10, 15, 20, 30, the results indicated that the estimated value of ML estimator, BAY estimator, and MAP estimator is not different but the MSE of BAY estimator and MAP estimator appeared that were smaller than the MSE of ML estimator.

Table 2 showed the estimated value of the rate parameter $(\lambda = 4)$ and MSE of ML estimator, BAY estimator, and MAP estimator. The results stated that the estimation of the rate parameter of ML estimator, BAY estimator, and MAP estimator were good performances for the estimated value of the rate parameter $(\lambda = 4)$. For MSE of ML estimator, BAY estimator, and MAP estimator, when n = 5, MSE of BAY estimator and MAP estimator are less than ML estimator under $\alpha = 4,5,6,7,8$. In cases n = 10, 15, 20 and $\alpha = 4,5,6,7,8$, it can be observed that MSE of three estimators as ML estimator, BAY estimator, and MAP estimator were similar. Meanwhile, the estimated value of the rate parameter $(\lambda = 4)$ of ML estimator, BAY estimator, and MAP estimator was not different.

## 4. CONCLUSIONS

In this research, we are interested in estimating the true parameter $\lambda$ of the gamma distribution base on MSE. From the simulation study, when small sample size n = 5 and $\alpha = 4,5,6,7,8$ with the small true parameter $\lambda = 2$, the MAP estimator is the best estimator based on MSE of the estimator, and in other cases for $\lambda = 2$, both BAY estimator and MAP estimator were good estimators based on MSE. For $\lambda = 4$ and n = 5, MSE of BAY estimator and MAP estimator was better than ML estimator, and the MAP estimator is a reasonable working approach as well as a BAY estimator based on MSE with cases n = 10, 15, 20 and $\alpha = 4,5,6,7,8$.

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