ON AREA-BIASED WEIGHTED WEIBULL DISTRIBUTION

Zahida Perveen, Mubbashir Munir, Zulfiqar Ahmed and Munir Ahmad
Lahore Garrison University, Lahore Pakistan
University of Management and Technology Lahore Pakistan
Gift University Gujranwala Pakistan
National College of Business Administration and Economics, Lahore
Corresponding author: drzahida95@gmail.com, mubbasher.munir@umt.edu.pk

ABSTRACT:: *This paper introduces a new distribution based on the Weibull distribution, known as Area biased weighted Weibull distribution (AWWD). Some characteristics of the new distribution are obtained. Plots for the cumulative distribution function, pdf and hazard function, tables with values of skewness and kurtosis are provided. We also provide results of entropies and characterization of AWWD. As a motivation, the statistical application of the results to a problem of ball bearing data has been provided. It is found that our recently proposed distribution fits better than size biased Rayleigh and Maxwell distributions. Since many researchers have studied the procedure of the weighted distributions in the estates of forest, biomedicine and biostatistics etc., we hope in numerous fields of theoretical and applied sciences, the findings of this paper will be useful for the practitioners.*

Keywords: Weighted distribution, Weibull distribution, moments, estimation, recurrence relation, entropy, characterization.

1. INTRODUCTION.

Weighted distributions are suitable in the situation of unequal probability sampling, such as actuarial sciences, ecology, biomedicine, biostatistics and survival data analysis. These distributions are applicable when observations are recorded without any experiment, repetition and random process.

For the collection of suitable model for observed data, the weighted distributions has been used as a device during last 25 years. The idea is most applicable when sampling frame is not available and random sampling is not possible. Firstly the idea of weighted distributions was introduced by Fisher [1]. Cox [2] initially provided the idea of length-biased sampling and after that Rao [3] established a unifying method that can be used for several sampling situations and can be displayed by means of the weighted distributions. Cox [4] estimated mean of the original distribution built on length biased data. Zelen [5] presented the concept of weighted distribution in studying cell kinetics and early discovery of disease. Warren [6] applied the distributions in forest product research. Rao and Patil [7] surveyed on the applications of these distributions correlated to the human population and ecology.

Patil and Rao [8] also discussed weighted binomial distribution to model the human families and estimation of the wildlife family size. Gupta and Keating [9] described the relationship between reliability measures of original and size-biased distribution. Arnold and Nagaraja [10] gave the idea of bivariate weighted distribution whereas Jain and Nanda [11] extended this idea and discussed multivariate aspect of weighted distribution.

Let $f(x;\theta)$ be the pdf of the random variable x and θ be the unknown parameter.

The weighted distribution is defined as;

$$g(x;\theta)=\frac{w(x)\,f(x;\theta)}{E\left[w(x)\right]}. \qquad x\in R,\theta>0 \qquad (1.1)$$

Where w(x) is a weight function. When w(x) = x^m , then these distributions are termed as size - biased distribution of order m. When $m=1$ it is called size biased of order 1 or say length biased distribution, whereas for $m=2$ it is called the area - biased distribution. [Ord and Patil [12]; Patil [13]; Mahfoud and Patil [14]].

In forest product research, equilibrium and length biased distributions have been used as moment distributions. Kochar and Gupta [15] discussed the moment distributional properties in assessment with the actual distributions and derived the bound on the moments of moment distributions. In investigating the area of ecology and wildlife, Gupta [15] used inferential tools to improve the estimates for the reliability measures of size biased inverse Gaussian distribution.

Oluyede [16] described inequalities for the reliability measures of size-biased and the original distributions. Navarro et al. [17] discussed characterization of the original and the size-biased distribution using reliability measures. Gove [18] offered the uses of size-biased distributions in forest science and ecology. Sunoj and Maya [19] established relationships among weighted and original distributions in the situation of repairable system and also characterized the sized-biased and the original distribution. Shen et al. [20] used semi-parametric transformations to model the length biased data. Hussain and Ahmad [21] presented misclassification in the size-biased modified power series distributions and its applications.

Mir and Ahmad [22] derived generalized forms of size-biased discrete distributions and discussed the practical applications in the field of Medical, Zoology and Accidental studies. Mir and Ahmad [23] derived size biased Geeta distribution and size-biased consul distribution respectively, different properties are discussed and contrasts with original distributions are also done. Das and Roy [24] established size-biased form of generalized Rayleigh distribution and apply the consequences to the environmental data.

Das and Roy [25] applied the concept of size-biased sampling in the field of environmental studies. Dara [26] derived reliability measures for size-biased forms of several moment distributions as the special cases of moment distributions. Iqbal and Ahmad [27] found compound scale mixtures of limiting distribution of generalized log Pearson type VII distribution with different continuous and moment distributions. Hasnain [28] introduced a new family of distributions named as exponentiated moment exponential (EME) distribution and developed its properties. Iqbal et al. [29] found a more general class for EME distribution and built up different properties including characterization through conditional moments.

Zahida and Munir [30] worked on Weighted Weibull Distributions (WWD), Double Weibull Distributions (DWD), Weighted Double Weibull Distributions (WDWD), Double Weighted Exponential Distributions (DWED) (both in size biased and area biased). Some basic theoretical properties of all these distributions including cumulative density function, central moments, skewness , kurtosis and moments are studied. Shannon entropy, Renyi entropy, moment generating function and information generating function of all these distributions are derived. Reliability measures including survival function, failure rates, reverse hazard rate function and Mills ratios of these distributions are also obtained. Parameters are evaluated by using method of maximum likelihood estimation along with derivation of practical examples.

2. **Weibull Distribution** is an important and well known distribution which attracted statisticians, working in various fields of applied statistics as well as theory and methods in modern statistic due to its number of special features and ability to fit to data related to various fields like as life testing, biology, ecology, economics, hydrology, engineering and business administration. This distribution is one of the members of the family of extreme value distributions. Weibull distribution is considered as the limit distribution of the smallest or the greatest value, respectively, in a sample with sample size ∞. This distribution comprises the exponential and the Rayleigh distributions as superior cases. In 1939 Swedish physicist introduced a distribution named as Weibull distribution [31]. Estimation procedures were carried out by Richard et al. in [32] for the shifted Weibull distribution, when all its parameters are unidentified. Mudholkar et al. [33] defined a constructive simplification of the Weibull distribution and suitable it to survival data. Marshall and Olkin [34] presented a process for addition a new parameter to an existing two parameter Weibull distribution and this distribution is known as the Marshall-Olkin extended distribution. A model named modified Weibull extension with three parameters was developed by Xie et al. [35]. This model is used for growing, bathtub-shaped, or declining failure rate function and the resulting Weibull probability plot is concaved. Further, Tang et al. [36] have permitted out the statistical analysis of this extension. Ghitany et al. [37] displayed that the Marshall-Olkin extended Weibull distribution could be attained as a compound distribution with collaborating exponential distribution. Nadarajah and Kotz [38] provided products and ratios of Weibull random variables. Extensive form of the Weibull distribution was proposed by Al-Saleh and Agarwal [39] which has two shape parameters.

McKay [40] presented a bivariate gamma distribution, which is one of the initial processes of the bivariate gamma distributions. A symmetrical bivariate gamma distribution with combined characteristic function was discussed by Kibble [41] and Moran [42]. Sarmanov [43,44] offered asymmetrical bivariate gamma distributions, which are additions of Kibble [41] and Moran [42] bivariate gamma distributions. Jensen [45] and Smith et al. 4 [46] studied Kibble bivariate gamma distributions. Jensen [47] also calculated Moran Recently Nadarajaha and Gupta [48] presented two bivariate gamma distributions constructed on a characterizing property relating products of gamma and beta random variables. Saboor and Ahmad [49] defined a bivariate generalized gamma-type function expending another form of confluent hypergeometric function of two variables and discuss some of its statistical functions. The moment generating function of this distribution was stated in terms of H and G –functions.

Provest and Saboor [49] introduced some properties of three parameter weighted Weibull distribution. He defined the probability density function as

$$f(x; k, \xi, \theta) = \frac{k\,\theta^{\frac{\xi}{k}+1}\, x^{\xi+k-1}\, e^{-\theta x^{k}}}{\Gamma\left(1+\frac{\xi}{k}\right)} \qquad x > 0 \qquad (1.2)$$

with $\xi + k > 0$**,** where ξ is the shape parameter.

2.1 Area Biased Weighted Weibull Distribution

Suppose X has a pdf g (x; ξ, k, θ) with unknown parameters k, ξ, θ. Using Eq. (1.1)and (1.2), the corresponding distribution, called area biased weighted Weibull distribution is of the form:

$$g(x; \xi, k, \theta) = \frac{k\,\theta^{1+\frac{2}{k}+\frac{\xi}{k}}\, x^{1+\xi+k}\, e^{-\theta x^{k}}}{\Gamma\left(1+\frac{2+\xi}{k}\right)}, \quad 0 < x < \infty \text{ and } \xi, k, \theta > 0 \qquad (1.3)$$

here ξ and k are shape parameters and θ is the scale parameter.

Figure (1.1) shows the graphs of AWWD density function for various values of parameters:

Case 1

i) $\xi = 1$, k= $2, \theta = 0.500$
ii) $\xi = 2$, k= $2, \theta = 0.250$
iii) $\xi = 3$, k= $2, \theta = 0.125$

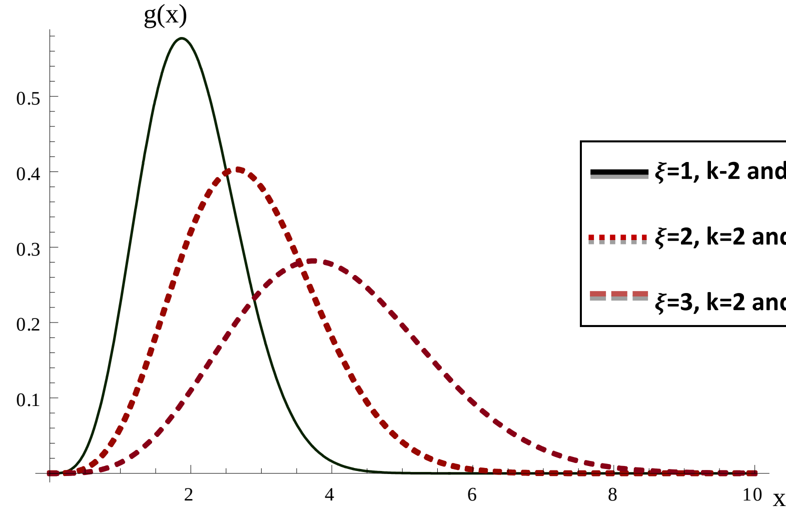

Fig 1.1.1 Probability density function of AWWD for the indicated values of ξ , k and θ

Case 2

i) $\xi = 1$, k= $2, \theta = 0.25$
ii) $\xi = 2$, k= $2, \theta = 0.25$
iii) $\xi = 3$, k= $2, \theta = 0.25$

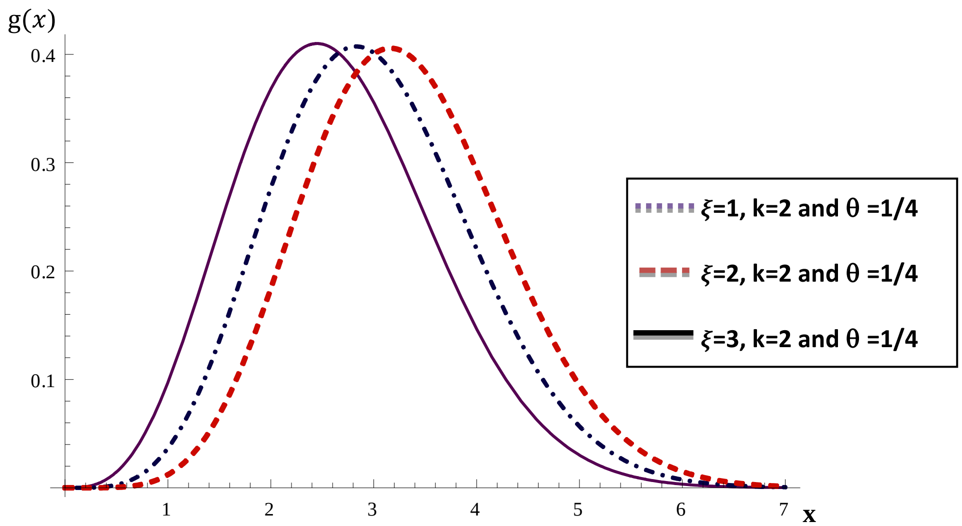

Fig 1.1.2 Probability density function of AWWD for the indicated values of ξ , k and θ

Cumulative Distribution function of AWWD.

Distribution function of a density function is defined as:

$G(x;\xi,k,\theta) = \int_0^x g(t)dt$

$G(x;\xi,k,\theta) = 1 - \frac{1}{\Gamma\left(1+\frac{2+\xi}{k}\right)}\Gamma\left(1+\frac{2+\xi}{k},\theta x^k\right),$

$\theta, k, \xi > 0$ (1.4)

where $\Gamma(a,x) = \int_x^\infty y^{a-1}e^{-y}dy$ denotes an incomplete gamma function.

G(x)

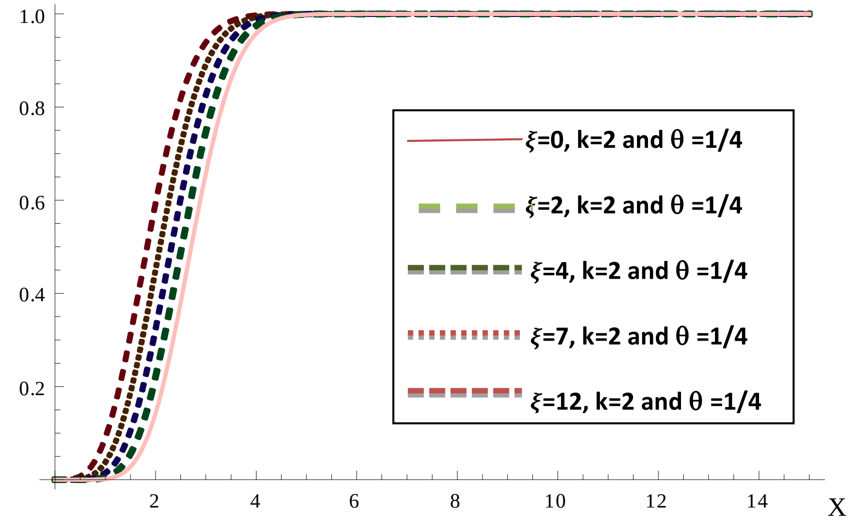

Fig 1.2 Distribution function of AWWD for the indicated values of , k and θ

1.3 Survival Function

The survival function is an important measure in a reliability studies, therefore by definition, the survival function for AWWD is:

$\mathrm{S}(x) = 1 - G(x)$

$$S(x) = \frac{\Gamma\left(1+\frac{2+\xi}{k},\ \theta\, x^{k}\right)}{\Gamma\left(1+\frac{2+\xi}{k}\right)}\ ,\theta, \mathrm{k}, \xi > 0 \qquad (1.5)$$

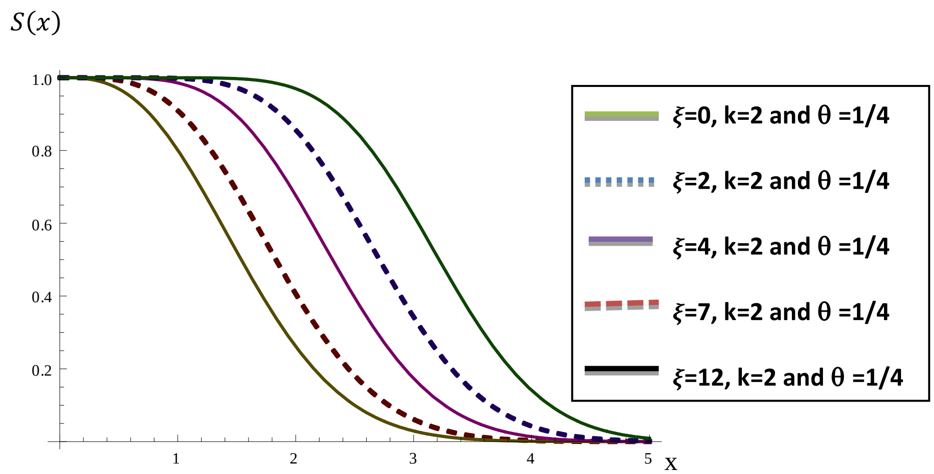

Fig. 1.3.1 Distribution function of AWWD for the indicated values of ξ, k and θ

1.4 Hazard Rate Function of AWWD

The hazard function is the instant level of failure at a certain time. Characteristics of a hazard function are normally related with definite products and applications. Different hazard functions are displayed with different distribution models. The concept of this function was firstly used by Barlow [50] and its properties were firstly investigated by Watson and Leadbetter [51]. Dhillon [52] was another prominent name in providing consciousness about the hazard rate function. Some properties of Hazard rate were pointed out by Nadarajah and Kotz [53]. The reliability measures of weighted distributions were evaluated by Dara [26]. Hazard rate of AWWD is defined as:

$\mathrm{h}\,(x) = \frac{g(x)}{S(x)}$

$$h(x) = \frac{k\ \theta^{1+\frac{2}{k}+\frac{\xi}{k}}\ x^{1+\xi+k}\ e^{-\theta x^{k}}}{\Gamma\left(1+\frac{2+\xi}{k},\ \theta x^{k}\right)}, \qquad \xi, k, \theta > 0 \qquad (1.6)$$

$\mathrm{h}(x)$

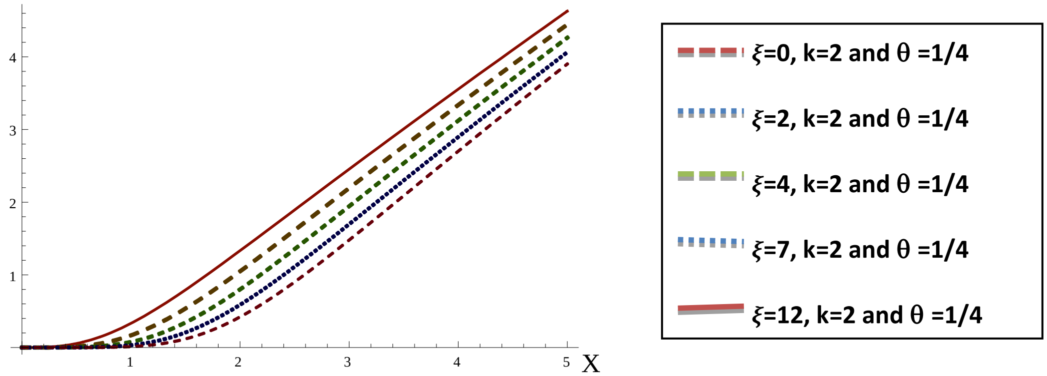

Fig. 1.4.1 Hazard Rate Function of AWWD for the indicated values of ξ, k and θ

1.5 Reverse Hazard Rate Function

This function of AWWD is defined as

$$r(x) = \frac{g(x)}{G(x)}$$

$$r(x) = \frac{\mathrm{k}\,\theta^{1+\frac{2}{\mathrm{k}}+\frac{\xi}{\mathrm{k}}}\, \mathrm{x}^{1+\xi+\mathrm{k}}\, \mathrm{e}^{-\theta \mathrm{x}^{\mathrm{k}}}}{\Gamma\left(1+\frac{2+\xi}{\mathrm{k}}\right) - \Gamma\left(1+\frac{2+\xi}{\mathrm{k}},\ \theta \mathrm{x}^{\mathrm{k}}\right)},\ \xi, k, \theta > 0 \quad (1.7)$$

r(x)

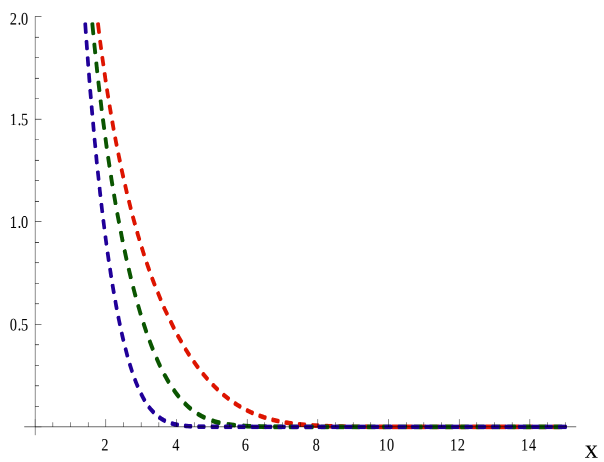

Fig. 1.5.1 Reverse Hazard Rate function of AWWD for the indicated values of ξ, k and θ

1.6 Moment Generating Function of AWWD

The moment generating function of AWWD is defined as:

$$M_X(t) = \int_0^\infty e^{tx} g(x)dx$$

Using Eq. (1.3)

$$M_x(t) = \frac{\mathrm{k}\,\theta^{1+\frac{2}{\mathrm{k}}+\frac{\xi}{\mathrm{k}}}}{\Gamma\left(1+\frac{2+\xi}{\mathrm{k}}\right)} \int_0^\infty e^{tx} x^{1++\xi+\mathrm{k}}\ \mathrm{e}^{-\theta \mathrm{x}^{\mathrm{k}}} \mathrm{d\,x}$$

Using $\boldsymbol{e^{tx}} = \sum_{\boldsymbol{i=0}}^{\infty} \frac{\boldsymbol{(tx)^i}}{\boldsymbol{i!}}$

$$M_x(t) = \frac{\mathrm{k}\,\theta^{1+\frac{2}{\mathrm{k}}+\frac{\xi}{\mathrm{k}}} \sum_{i=0}^{\infty} t^i}{\Gamma\left(1+\frac{2+\xi}{\mathrm{k}}\right) i!} \int_0^\infty x^{i+1++\xi+\mathrm{k}}\ \mathrm{e}^{-\theta \mathrm{x}^{\mathrm{k}}} \mathrm{dx}$$

Using the transformation and then simplification, it becomes:

$$M_x(t) = \frac{\sum_{i=0}^{\infty} t^i}{i!} \frac{\theta^{\frac{-i}{k}}}{\Gamma\left(1+\frac{2+\xi}{\mathrm{k}}\right)} \Gamma\left(1+\frac{2+\mathrm{i}+\xi}{\mathrm{k}}\right) \quad (1.8)$$

1.7 Information Generating Function

The Information Generating Function is defined as:

$$\mathrm{T(s)} = \mathrm{E}\,[g(x)]^{s-1} = \int_0^\infty \left(g(x)\right)^s dx$$

Using Eq. (1.3)

$$= \frac{k^s\, \theta^{s+\frac{2s}{\mathrm{k}}+\frac{\xi s}{\mathrm{k}}}}{\left[\Gamma\left(1+\frac{2+\xi}{\mathrm{k}}\right)\right]^s} \int_0^\infty x^{s(1++\xi+\mathrm{k})}\ \mathrm{e}^{-\mathrm{s}\theta \mathrm{x}^{\mathrm{k}}} \mathrm{d\,x}$$

Using the transformation $\mathrm{s}\,\theta\,\mathrm{x}^{\mathrm{k}} = u$ and applying incomplete gamma function:

$$T(s) = \frac{k^{s-1}\, \theta^{\frac{s-1}{k}}\, \Gamma\left(\mathrm{s}+\frac{\mathrm{s}+\mathrm{s}\xi+1}{\mathrm{k}}\right)}{s^{s+\frac{s}{\mathrm{k}}+\frac{\xi s}{\mathrm{k}}+\frac{1}{\mathrm{k}}}\left[\Gamma\left(1+\frac{2+\xi}{\mathrm{k}}\right)\right]^s} \quad (1.9)$$

Remark 1.7.1 For Shannon entropy $\frac{d}{ds}T(s)|_{s=1}$

3. Limit and Mode of the Function

Note that the limit of the density function given in (1.3) is as follows:

$$\mathrm{x} \xrightarrow{Lim} 0\ \ \mathrm{g}\,(x; \xi, k, \theta) = 0 \quad (2.1)$$

$$\mathrm{x} \xrightarrow{Lim} \infty\ g(x;\ \xi, k, \theta) = 0 \quad (2.2)$$

2.1. Mode of AWWD

Taking log of Eq. (1.3) on both sides:

$$\log \mathrm{g}\,(\mathrm{x};\xi, k, \theta) = \log\left(\frac{\mathrm{k}\,\theta^{1+\frac{2}{\mathrm{k}}+\frac{\xi}{\mathrm{k}}}}{\Gamma\left(1+\frac{2+\xi}{\mathrm{k}}\right)}\right) + (1+\xi+\mathrm{k}) \log$$

$$(x) - \theta\,\mathrm{x}^{\mathrm{k}} \quad (2.1.1)$$

Differentiating Eq. (2.1.1) with respect to x, we obtain:

$$\frac{\partial}{\partial \mathrm{x}}\left(\log \mathrm{g}\,(\mathrm{x};\xi, k, \theta)\right) = \frac{(1+\xi+\mathrm{k})}{x} - \theta\,\mathrm{k}\,x^{k-1} \quad (2.1.2)$$

By putting $\frac{(1+\xi+\mathrm{k})}{x} - \theta\,\mathrm{k}\,x^{k-1} = 0$, we have $\mathrm{x} = \left(\frac{1+\xi+\mathrm{k}}{\theta\,\mathrm{k}}\right)^{\frac{1}{k}}$

and at every value of x, $g''(x) < 0$.

Table 2.1

Mode of AWWD for different values of the parameters

k	ξ	θ	Mode
1	2	0.500	2.000
1	2	0.250	2.828
1	2	0.125	4.000
2	2	0.500	2.236
2	2	0.250	3.162
2	2	0.125	4.472

4. Moments

The moments are used to find the moment ratios.

The moments about origin are as:

$$\mu_r^{/} = \mathrm{E}(\mathrm{x}^{\mathrm{r}}) = \frac{\theta^{\frac{-\mathrm{r}}{\mathrm{k}}}}{\Gamma\left(1+\frac{2+\xi}{\mathrm{k}}\right)} \Gamma\left(1+\frac{2+\mathrm{r}+\xi}{\mathrm{k}}\right) \quad (3.1)$$

$$\mu_r^{/} = a_r \theta^{\frac{-\mathrm{r}}{\mathrm{k}}},$$

$$where \quad a_r = \frac{\Gamma\left(1+\frac{2+r+\xi}{k}\right)}{\Gamma\left(1+\frac{2+\xi}{k}\right)}$$

For $r = 1, 2, 3, 4$, the first four moments about the mean are:

$\mu_1 = 0$

$\mu_2 = a_2\theta^{\frac{-2}{k}} - a_1{}^2\theta^{\frac{-2}{k}}$

$\mu_2 = (a_2 - a_1{}^2)\theta^{\frac{-2}{k}}$

$\mu_3 = a_3\theta^{\frac{-3}{k}} - 3a_1a_2\theta^{\frac{-3}{k}} + 2a_1^3\theta^{\frac{-3}{k}}$

$= \theta^{\frac{-3}{k}}(a_3 - 3a_1a_2 + 2a_1^3)$

$\mu_4 = a_4\theta^{\frac{-4}{k}} - 4a_1a_3\theta^{\frac{-4}{k}} + 6a_1^2a_2\theta^{\frac{-4}{k}} - 3a_1^4\theta^{\frac{-4}{k}}$

$= \theta^{\frac{-4}{k}}(a_4 - 4a_1a_3 + 6a_1^2a_2 - 3a_1^4)$

4.1 Moment Ratios

Table 3.1

Measure of Coefficient of Skewness and Kurtosis for AWWD

θ	k	ξ	$\sqrt{\beta_1}$	β_2
0.5	3.200	1.0	0.0168	2.8561
0.5	3.250	1.0	0.0067	2.8567
0.5	3.280	1.0	0.00073	2.8573
0.5	3.282	1.0	0.00034	2.8573

From Table, it is clear that AWWD is almost symmetrical and platykurtic for $3.200 \leq k \leq 3.282$.

3.2 Mixed Random Variables of AWWD

In the next theorem we show some averages of the mixture random variables comprising algebraic and logarithmic functions with respect to AWWD are found. Some identities from Gradshteyn and Ryzhik [2007], Table of integral, series and products to find these results are used.

Theorem 3.2.1

Let X is the random variable on its support $(0, \infty)$, then $\mu_\beta^{/} = \int_0^\infty \theta(x)g_2(x)dx$ where $\theta(x)$ is a weight function:

(i) $\theta(x) = logx$

(ii) $\theta(x) = \frac{1}{x^2}$

(iii) $\theta(x) = x^k e^{-x^k}$

(iv) $\theta(x) = x^{-2}e^{-x^k}$

Proof: (i) $\mu_\beta^{/} = \int_0^\infty \theta(x)g(x)dx$

From Eq. (1.3)

$$= \frac{k\,\theta^{1+\frac{2}{k}+\frac{\xi}{k}}}{\Gamma\left(1+\frac{2+\xi}{k}\right)}\int_0^\infty logx.\,x^{\xi+k+1}\,e^{-\theta x^k}dx$$

Using the transformation, $\theta\,x^k = t$, $k\,\theta\,x^{k-1}dx = dt$, $0 < t < \infty$

$$= \frac{k\,\theta^{1+\frac{2}{k}+\frac{\xi}{k}}}{\Gamma\left(1+\frac{2+\xi}{k}\right)}\int_0^\infty \log\left(\frac{t}{\theta}\right)^{\frac{1}{k}}\left(\frac{t}{\theta}\right)^{\frac{\xi+k}{k}} e^{-t}\frac{dt}{k\theta\left(\frac{t}{\theta}\right)^{1-\frac{1}{k}}}$$

$$= \frac{1}{k\,\Gamma\left(1+\frac{2+\xi}{k}\right)}\left[\int_0^\infty log\ t\,t^{\frac{2+\xi}{k}}e^{-t}dt - \int_0^\infty log\theta\, t^{\frac{2+\xi}{k}}e^{-t}dt\right]$$

$\int_0^\infty \log x\ x^{\gamma-1}e^{-x}dx = \Gamma^{'}(\gamma)$ (Gradshteyn and Ryzhik [54], 4.352(4))

$$\mu_\beta^{/} = \frac{1}{k\,\Gamma\left(1+\frac{2+\xi}{k}\right)}\left[\Gamma'\left(\frac{2+\xi}{k}+1\right) - log\theta\Gamma\left(\frac{2+\xi}{k}+1\right)\right]$$

ii) $\mu_\beta^{/} = \int_0^\infty \frac{1}{x^2}g(x)dx$

From Eq. (1.3)

$$= \frac{k\,\theta^{1+\frac{2}{k}+\frac{\xi}{k}}}{\Gamma\left(1+\frac{2+\xi}{k}\right)}\int_0^\infty x^{-1+\xi+k}.\ e^{-\theta x^k}dx$$

Using the transformation, $\theta\,x^k = t$, $k\,\theta\,x^{k-1}dx = dt$, $0 < t < \infty$

after simplification, we have:

$$\mu_\beta^{/} = \frac{\theta^{\frac{2}{k}}}{\Gamma\left(1+\frac{2+\xi}{k}\right)}\Gamma\left(1+\frac{\xi}{k}\right)$$

iii) $\theta(x) = x^k e^{-x^k}$

$\mu_\beta^{/} = \int_0^\infty x^k e^{-x^k}g(x)dx$

after simplification, we have:

$$\mu_\beta^{/} = \frac{k\,\theta^{1+\frac{2}{k}+\frac{\xi}{k}}}{(1+\theta)^{2+\frac{2+\xi}{k}}\Gamma\left(1+\frac{2+\xi}{k}\right)}\Gamma\left(2+\frac{2+\xi}{k}\right)$$

iv) $\theta(x) = x^{-2}e^{-x^k}$

$\mu_\beta^{/} = \int_0^\infty x^{-2}e^{-x^k}g(x)dx$

after simplification, we have:

$$\mu_\beta^{/} = \frac{\theta^{1+\frac{2}{k}+\frac{\xi}{k}}}{(1+\theta)^{1+\frac{\xi}{k}}\Gamma\left(1+\frac{2+\xi}{k}\right)}\Gamma\left(1+\frac{\xi}{k}\right)$$

5. Estimation of parameters

Maximum likelihood (ML) Estimation is used to estimate the parameters of AWWD. If $\mathbf{X_1}, \mathbf{X_2} \ldots\ldots \mathbf{X_n}$ be a random sample from a population having pdf $g(x|\boldsymbol{k}, \boldsymbol{\theta}, \boldsymbol{\xi})$, the likelihood function of AWWD distribution may be defined as:

$$L(\theta,\xi,k; x_1, x_2, \ldots, x_n) = \prod_{i=1}^n g(x_i).$$

Here the independent observations are $\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_n}$, then the log likelihood function of the distribution is:

$$\mathbf{L}(\boldsymbol{\theta}, \boldsymbol{\xi}, \mathbf{k}; \boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_n}) = \sum_{i=1}^n \mathbf{log\ g}(\mathbf{x_i}; \boldsymbol{\xi}, \mathbf{k}, \boldsymbol{\theta}) = \mathbf{n\ log(k)} + \mathbf{n}\left(\mathbf{1} + \frac{\mathbf{2}}{\mathbf{k}} + \frac{\boldsymbol{\xi}}{\mathbf{k}}\right)\mathbf{log\ \theta} - \mathbf{nlog}\left(\boldsymbol{\Gamma}\left(\mathbf{1} + \frac{\mathbf{2+\xi}}{\mathbf{k}}\right)\right) + (\mathbf{1} + \mathbf{k} + \boldsymbol{\xi})\sum_{i=1}^n \mathbf{log}(\mathbf{x_i}) - \boldsymbol{\theta}\sum_{i=1}^n \mathbf{x_i^k}; \qquad (5.1)$$

ML estimates can be found by solving Equations

$$\frac{\partial l\,(\theta,\xi,k;\underline{x})}{\partial \xi} = 0, \qquad \frac{\partial l\,(\theta,\xi,k;\underline{x})}{\partial \theta} = 0, \qquad \frac{\partial l\,(\theta,\xi,k;\underline{x})}{\partial k} = 0$$

$$-\frac{n\Psi^{(0)}\left[0,1+\frac{2+\xi}{k}\right]}{k} + \sum_{i=1}^n \log[x_i] + \frac{\log[\theta]n'\left[1+\frac{2}{k}+\frac{\xi}{k}\right]}{k} = 0 \quad (5.2)$$

where $\boldsymbol{\Psi}^{(0)}(\mathbf{z}) = \frac{\boldsymbol{\Gamma}'(\mathbf{z})}{\boldsymbol{\Gamma}(\mathbf{z})}$

$$\frac{\mathbf{n}\left(\hat{\mathbf{k}} - (\mathbf{1}+\hat{\boldsymbol{\xi}})\mathbf{log}(\hat{\boldsymbol{\theta}}) + (\mathbf{1}+\hat{\boldsymbol{\xi}})\boldsymbol{\Psi}^{(0)}\left(\mathbf{1}+\frac{\mathbf{1}+\hat{\boldsymbol{\xi}}}{\hat{\mathbf{k}}}\right)\right)}{\hat{\mathbf{k}}^2} + \sum_{i=1}^n \mathbf{log}(\mathbf{x_i}) - \hat{\boldsymbol{\theta}}\sum_{i=1}^n \mathbf{x_i^{\hat{k}}}\,\mathbf{log}(\mathbf{x_i}) = 0 \qquad (5.3)$$

$$\mathbf{n}\left(\mathbf{1} + \frac{\mathbf{1}}{\hat{\mathbf{k}}} + \frac{\boldsymbol{\xi}}{\hat{\mathbf{k}}}\right).\frac{\mathbf{1}}{\hat{\boldsymbol{\theta}}} - \sum_{i=1}^n \mathbf{x_i^{\hat{k}}} = \mathbf{0}$$

Equations (5.1), (5.2) and (5.3) are nonlinear equations and can be solved through Mathematica software.

The asymptotic variance-covariance matrix is the inverse of $\mathbf{I}(\boldsymbol{\xi}, \mathbf{k}, \boldsymbol{\theta}) = -\mathbf{E}(\mathbf{H}(\mathbf{X}))$

$$\mathbf{H(x)} = \begin{pmatrix} \frac{\partial^2(\log(g(X;\xi,k,\theta))}{\partial\xi^2} & \frac{\partial^2(\log(g(X;\xi,k,\theta))}{(\partial\xi\partial k)} & \frac{\partial^2(\log(g(X;\xi,k,\theta))}{(\partial\xi\partial\theta)} \\ \frac{\partial^2(\log(g(X;\xi,k,\theta))}{(\partial k\partial\xi)} & \frac{\partial^2(\log(g(X;\xi,k,\theta))}{\partial k^2} & \frac{\partial^2(\log(g(X;\xi,k,\theta))}{(\partial k\partial\theta)} \\ \frac{\partial^2(\log(g(X;\xi,k,\theta))}{(\partial\theta\partial\xi)} & \frac{\partial^2(\log(g(X;\xi,k,\theta))}{(\partial\theta\partial k)} & \frac{\partial^2(\log(g(X;\xi,k,\theta))}{\partial\theta^2} \end{pmatrix}$$

$$\frac{\partial^2(\log((g(X;\xi,k,\theta)))}{(\partial\xi)^2} = -\frac{n\Psi^{(0)}\left[1,1+\frac{2+\xi}{k}\right]}{k^2} + \frac{\mathrm{Log}[\theta]n''\left[1+\frac{2}{k}+\frac{\xi}{k}\right]}{k^2}$$

$$\frac{\partial^2(\log(g(X;\xi,k,\theta))}{(\partial k)^2} = -\frac{n}{k^2} - \frac{2n(2+\xi)\Psi^{(0)}\left[0,1+\frac{2+\xi}{k}\right]}{k^3} - \frac{n(2+\xi)^2\Psi^{(0)}\left[1,1+\frac{2+\xi}{k}\right]}{k^4} - \theta\sum_{i=1}^{n}\log[x_i]^2 x_i^k + \left(\frac{4}{k^3}+\frac{2\xi}{k^3}\right)log[\theta]n'\left[1+\frac{2}{k}+\frac{\xi}{k}\right] + \left(-\frac{2}{k^2}-\frac{\xi}{k^2}\right)^2\log[\theta]n''\left[1+\frac{2}{k}+\frac{\xi}{k}\right]$$

$$\frac{\partial^2(\log(g(X;\xi,k,\theta))}{(\partial\theta)^2} = -\frac{n\left[1+\frac{2}{k}+\frac{\xi}{k}\right]}{\theta^2}$$

$$\frac{\partial^2(\log(g(X;\xi,k,\theta)))}{(\partial\xi\,\partial k)} = \frac{n\Psi^{(0)}\left[0,1+\frac{2+\xi}{k}\right]}{k^2} + \frac{n(2+\xi)\Psi^{(0)}\left[1,1+\frac{2+\xi}{k}\right]}{k^3} - \frac{\log[\theta]n'\left[1+\frac{2}{k}+\frac{\xi}{k}\right]}{k^2} + \frac{\left(-\frac{2}{k^2}-\frac{\xi}{k^2}\right)\log[\theta]n''\left[1+\frac{2}{k}+\frac{\xi}{k}\right]}{k}$$

$$\frac{\partial^2(\log(g(X;\xi,k,\theta))}{(\partial k\,\partial\xi)} = \frac{n\Psi^{(0)}[0,1+\frac{2+\xi}{k}]}{k^2} + \frac{n(2+\xi)\Psi^{(0)}\left[1,1+\frac{2+\xi}{k}\right]}{k^3} - \frac{\log[\theta]n'\left[1+\frac{2}{k}+\frac{\xi}{k}\right]}{k^2} + \frac{\left(-\frac{2}{k^2}-\frac{\xi}{k^2}\right)log[\theta]n''[1+\frac{2}{k}+\frac{\xi}{k}]}{k}$$

$$\frac{\partial^2(\log(g(X;\xi,k,\theta))}{(\partial\xi\,\partial\theta)} = \frac{n'[1+\frac{2}{k}+\frac{\xi}{k}]}{k\theta}$$

$$\frac{\partial^2(\log(g(X;\xi,k,\theta))}{(\partial\theta\,\partial\xi)} = \frac{n'[1+\frac{2}{k}+\frac{\xi}{k}]}{k\theta} \quad \frac{\partial^2(\log(g(X;\xi,k,\theta))}{(\partial k\,\partial\theta)}$$

$$= -\sum_{i=1}^{n}\log[x_i]x_i^k + \frac{\left(-\frac{2}{k^2}-\frac{\xi}{k^2}\right)n'\left[1+\frac{2}{k}+\frac{\xi}{k}\right]}{\theta} \quad \frac{\partial^2(\log(g(X;\xi,k,\theta))}{(\partial\theta\,\partial k)}$$

$$= -\sum_{i=1}^{n}\log[x_i]x_i^k + \frac{\left(-\frac{2}{k^2}-\frac{\xi}{k^2}\right)n'\left[1+\frac{2}{k}+\frac{\xi}{k}\right]}{\theta}$$

6. Recurrence Relation of Area Biased Weighted Weibull Distribution

In mathematics, a **recurrence relation** is an equation that recursively expresses a sequence of values, once one or more primary terms are assumed: each further term of the sequence is defined as a function of the preceding terms. Here we derive the recurrence relation in conditional moments.

Theorem 6.1

Let X be the random variable on its support (0,∞). Then recurrence relation through conditional moments for all t >0

$$E\,(X^{nk}|X>t) = \frac{\theta^{n+\frac{2+\xi}{k}}t^{nk+(2+\xi)}e^{-\theta t^k}}{\Gamma\left(1+\frac{2+\xi}{k},\theta t^k\right)\theta^n} + \frac{\left(n+\frac{2+\xi}{k}\right)}{\theta} E(X^{(n-1)k}|X>t) \quad (6.1)$$

where ξ, θ, $k>0$ and n ∈ z^+

Proof:

Let X be the area biased weighted Weibull distribution. Then

Then

$$E\,(X^{nk}|X>t) = \frac{1}{\bar{G}(t)}\int_t^\infty x^{nk}g(x)dx$$

Using $\bar{G}(t) = \frac{\Gamma\left(1+\frac{2+\xi}{k},\ \theta t^k\right)}{\Gamma(1+\frac{2+\xi}{k})}$

$$= \frac{k\theta^{1+\frac{2}{k}+\frac{\xi}{k}}}{\Gamma\left(1+\frac{2+\xi}{k},\theta t^k\right)}\int_t^\infty x^{nk+\xi+k+1}e^{-\theta x^k}dx$$

using the transformation $\theta\, x^k =$ u, $k\,\theta\,x^{k-1}dx = du$ and $\theta\, t^k < u < \infty$

$$= \int_{\theta t^k}^\infty\left(\frac{u}{\theta}\right)^n\left(\frac{u}{\theta}\right)^{\frac{1+\xi+k+nk}{k}} e^{-u}.\frac{du}{\theta\left(\frac{u}{\theta}\right)^{\frac{k-1}{k}}}$$

$$= \frac{\theta^{\frac{2}{k}+\frac{\xi}{k}}}{\Gamma\left(1+\frac{2+\xi}{k},\theta\, t^k\right)\theta^n}\int_{\theta t^k}^\infty (u)^{n+\frac{2}{k}+\frac{\xi}{k}}\,e^{-u}du \quad (6.2)$$

Integration by parts, we get:

$$= \frac{\theta^{n+\frac{2+\xi}{k}}t^{nk+(2+\varepsilon)}e^{-\theta t^k}}{\Gamma\left(1+\frac{2+\xi}{k},\theta\, t^k\right)\theta^n} + \frac{\left(n+\frac{2}{k}+\frac{\xi}{k}\right)}{\theta\Gamma\left(1+\frac{2+\xi}{k},\theta\, t^k\right)\theta^{n-1}}\int_{\theta t^k}^\infty u^{(n-1)+\frac{2+\xi}{k}}\,e^{-u}du \quad (6.3)$$

After some simplification, we will obtain (6.1)

6.1 COROLLARY. IF ξ = 0 then Eq. (6.3) reduces for 2-parameters Weibull distribution.

6.2 Entropy

Entropy is considered as a major tool in every field of science and technology. In Statistics entropy is considered as an amount of incredibility. Different ideas of entropy have been given by Jaynes [55] and the entropies of continuous probability distributions have been approximated by Ma [56]. Shanon entropy is defined as H(X) of a continuous random variable X with a density function f(x)

$$H(X) = E\,[-\log\,(f(X))]$$

$$H\,[g\,(x;\,\xi,k,\theta)] = E[-\log g(x;\,\xi,k,\theta)]$$

$$= E\left[-\log\left\{\frac{k\theta^{1+\frac{2}{k}+\frac{\xi}{k}}x^{1+\xi+k}e^{-\theta^{x^k}}}{\Gamma(1+\frac{2+\xi}{k})}\right\}\right]$$

$$=E\left[\theta x^k - \log(k) - (1+k+\xi)\log(x) - \left(1+\frac{2}{k}+\frac{\xi}{k}\right)\log(\theta) + \log\Gamma\left(1+\frac{2+\xi}{k}\right)\right]$$

$$= E\left(\theta x^k\right) - \log(k) - (1+k+\xi)E\log(x) - \left(1+\frac{2}{k}+\frac{\xi}{k}\right)\log(\theta) + \log\Gamma\left(1+\frac{2+\xi}{k}\right)$$

$$= \log\frac{\Gamma\left(1+\frac{2+\xi}{k}\right)}{k} - \left(1+\frac{2}{k}+\frac{\xi}{k}\right)\log(\theta) - (1+\boldsymbol{\xi}+k)E\log(x) + E\left[\theta x^k\right] \quad (6.2.1)$$

$$E\log(x) = \frac{1}{k}\frac{1}{\Gamma\left(1+\frac{2+\xi}{k}\right)}\left[\int_0^\infty logt.\,t^{\frac{2+\xi}{k}}e^{-t}dt - \int_0^\infty log\theta t^{\frac{2+\xi}{k}}e^{-t}dt\right]$$

$\int_0^\infty \log x.\,x^{\gamma-1}\,e^{-x}dx = \Gamma'(\gamma)$ (Gradshteyn and Ryzhik (2007)) 4.352(4))

$$E\log(x) = \frac{1}{k}\frac{1}{\Gamma\left(1+\frac{2+\xi}{k}\right)}\left[\Gamma'\left(1+\frac{2+\xi}{k}\right) - \log\theta\Gamma\left(1+\frac{2+\xi}{k}\right)\right]$$

(6.2.2)

$$E\left[\theta x^{k}\right]=\frac{k\theta^{2+2+\frac{\xi}{k}}}{\Gamma\left(1+\frac{2+\xi}{k}\right)}\Gamma\left(2+\frac{2+\xi}{k}\right) \qquad (6.2.3)$$

Putting (6.2.2) and (6.2.3) in (6.2.1)

$$H[g(x;\ \xi,k,\theta)]= \log \frac{\Gamma\left(1+\frac{2+\xi}{k}\right)}{k}-\left(1+\frac{2}{k}+\frac{\xi}{k}\right)\log(\theta)-(1+\xi+k)\frac{1}{k}\frac{1}{\Gamma\left(1+\frac{2+\xi}{k}\right)}\left(\Gamma'\left(1+\frac{2+\xi}{k}\right)-\log\theta\Gamma\left(1+\frac{2+\xi}{k}\right)\right)+\frac{k\theta^{2+\frac{2}{k}+\frac{\xi}{k}}}{\Gamma\left(1+\frac{2+\xi}{k}\right)}\Gamma\left(2+\frac{2+\xi}{k}\right)$$

where $\xi, k, \theta > 0$.

Alfred Renyi [1921-1970] entropy is usually known as the generalized procedure of Shannon entropy. The Renyi entropy is named after. It is useful in ecology and statistics. It is defined as

$$I_R(\beta)=\frac{1}{1-\beta}\log\left(\int_0^{\infty} g^{\beta}(x)dx\right) \quad \beta>0, \beta\neq 1)$$

Putting value of g(x) from Eq. (1.3) in above equation, we get:

$$=\frac{1}{1-\beta}\log\left[\int_0^{\infty}\left[\frac{k\theta^{1+\frac{2}{k}+\frac{\xi}{k}}x^{1+\xi+k}e^{-\theta^{x^k}}}{\Gamma\left(1+\frac{2+\xi}{k}\right)}\right]^{\beta}\right]$$

Take $\quad g^{\beta}(x;\ \xi,k,\theta)=\left[\frac{k\theta^{1+\frac{2}{k}+\frac{\xi}{k}}x^{1+\xi+k}e^{-\theta^{x^k}}}{\Gamma\left(1+\frac{2+\xi}{k}\right)}\right]^{\beta}$

$$\int_0^{\infty} g^{\beta}(x;\ \xi,k,\theta)dx$$

$$=\frac{k^{\beta}\theta^{\beta+\frac{2\beta}{k}+\frac{\beta\xi}{k}}}{\left[\Gamma\left(1+\frac{2+\xi}{k}\right)\right]^{\beta}}\int_0^{\infty} x^{\beta+\beta\xi+k\beta}\ e^{-\beta\theta^{x^k}}dx$$

$$I_R(\beta)=-\log k+\frac{\beta-k-1}{k}\log\theta+\log\Gamma\left(\beta+\frac{1+\beta+\beta\xi}{k}\right)-\left(\beta+\frac{1}{k}+\frac{\beta\xi}{k}+\frac{\beta}{k}\right)\log\beta-\beta\log\Gamma\left(1+\frac{2+\xi}{k}\right)$$

7. Characterization of Area Biased Weighted Weibull Distribution

A characterization is a definite distributional property of statistics that uniquely defines the related stochastic model. There are some functions related to a probability distribution that uniquely classify it. Such functions are called characterizing functions. Here we are characterizing the AWWD distribution through conditional moments by using the characterizing function $f(x)$.

Theorem 7.1

Let X be the random variable on its support $(0, \infty)$. Then the following condition holds for characterizing the AWWD,

iff $E[(u(x)|X>t)=\frac{pe^{-\theta t^k}}{\theta k\bar{F}(t)}, \qquad \theta,\ k>0$

where $u(x)=x^{-\xi-2}$ and p is constant.

Proof.

$$E[(u(x)|X>t)]=\frac{1}{\bar{F}(t)}\int_t^{\infty}x^{-1-\xi}g(x)dx$$

$$=\frac{k\,\theta^{1+\frac{1}{k}+\frac{\xi}{k}}}{\Gamma\left(1+\frac{1+\xi}{k}\right)\bar{F}(t)}\int_t^{\infty}x^{-1+k}\,e^{-\theta x^k}dx$$

$$=\frac{p}{\theta k\bar{F}(t)}\int_t^{\infty}\theta\,k\,x^{-1+k}\,e^{-\theta x^k}dx$$

Using the transformation, $\theta x^k = s$, we have:

$$=\frac{p}{\theta k\bar{F}(t)}\int_{\theta t^k}^{\infty}e^{-s}ds$$

$$=\frac{p}{\theta k\bar{F}(t)}e^{-\theta t^k}$$

Conversely

$$\frac{1}{\bar{F}(t)}\int_t^{\infty}x^{-2-\xi}\,g_2(x)dx=\frac{P}{\theta k\bar{F}(t)}e^{-\theta t^k}$$

Differentiating both sides w.r.t, 't', we get

$$-t^{-1-\xi}\,g_2(t)=\frac{Pe^{-\theta t^k}}{\theta k}\left(-\theta k t^{k-1}\right)$$

after simplification

$$g_2(t)=pe^{-\theta t^k}t^{\xi+k}$$

where $p=\frac{k\theta^{1+\frac{2}{k}+\frac{\xi}{k}}}{\Gamma\left(1+\frac{2+\xi}{k}\right)}$ is constant.

8. Numerical Example.

8.1. The Ball Bearing Data Set

See for data set published in Lawless [57].

Table 8.1
Ball Bearing Data Set

17.88	28.92	33.0	41.52	42.12
45.6	48.8	51.84	51.96	54.12
55.56	67.8	68.44	68.88	84.12
93.12	98.64	105.12	105.84	105.84
127.92	128.04	173.4		

Table 8.2
Goodness-of-Fit Statistics and Parameters' Estimates

Distributions	$\hat{\theta}$	$\hat{k}$	$\hat{\xi}$	$\hat{\alpha}$	$\hat{\beta}$	A_0^2	W_0^2
Size biased Rayleigh	-	-	-	-	46.764	0.708	0.134
Size biased Maxwell	-	-	-	40.50	-	1.693	0.278
Weighted Weibull(area biased)	0.8151	0.6047	3.7599	-	-	0.060	0.059
Weighted Weibull (size biased)	0.8151	0.6047	4.7599	-	-	0.1909	0.0332

In Table 8.2, the approximations of the parameters are specified. For goodness-of-fit statistics Anderson-Darling and Cramer-von Mises tests have been used, the weighted Weibull model proposals the best fitting:

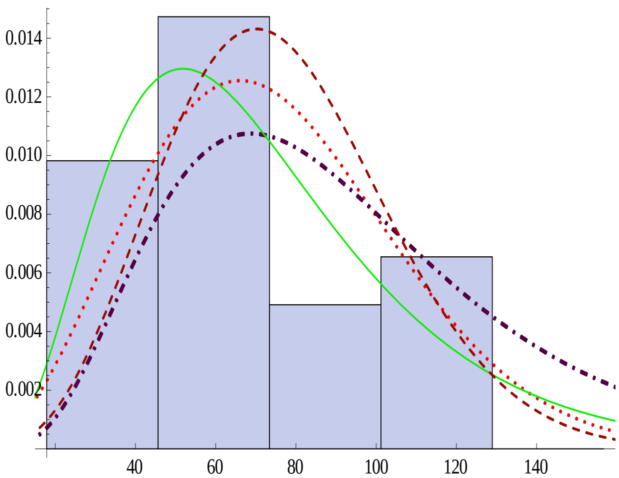

Fig.8.1 Weighted Weibull size biased (dashed line), Weighted Weibull area biased (dotted line), Maxwell (Solid Line) and Rayleigh (dotted dashed Line) on the Histogram for the Ball Bearing Data

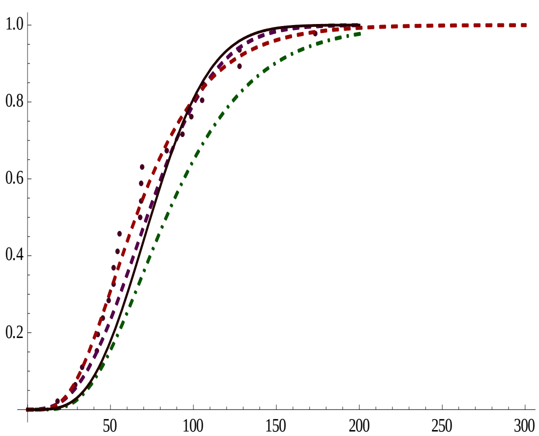

Fig.8.2 Weighted Weibull Density Estimates, cdf Estimates and Empirical cdf

Concluding Remarks

In this paper**,** we discussed the Area Biased Weighted Weibull Distribution (AWWD).Some characteristics of the newly proposed distribution are obtained. The plots for the cdf, pdf and hazard function and tables for skewness and kurtosis for different values of parameters have been derived. Also the results of entropies and characterization have been proved. The estimators of the parameters are discussed through maximum likelihood estimation technique. The statistical application of the results to a problem of ball bearing data has been provided. It is found that our newly proposed distribution fits better than size biased Rayleigh and Maxwell distributions.

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