LOCAL TRUNCATION ERROR AND ASSOCIATED PRINCIPAL ERROR FUNCTION FOR AN ITERATIVE INTEGRATOR TO SOLVE CAUCHY PROBLEMS

***Sania Qureshi, [1]Sajad H. Sandilo, [2]A. H. Sheikh and [3]Asif Ali Shaikh**
*Department of Basic Sciences and Related Studies, Mehran University of Engineering and Technology, Jamshoro.
[1,2]Department of Mathematics and Statistics, Quaid-e-Awam University of Engineering, Science & Technology, Nawabshah.
[3]Department of Basic Sciences and Related Studies, Mehran University of Engineering and Technology, Jamshoro.
*Corresponding Author's Email: sania.qureshi@faculty.muet.edu.pk

ABSTRACT: *The purpose of this paper is to study and derive a generalized formula of local truncation error and its corresponding principal error function for an iterative integrator to solve Cauchy type of problems. As a result, the iterative integrator is found to be second order accurate on the basis of having local truncation error of order* $O(h^3)$ *as confirmed by its comparison with Taylor's series for a function of two variables. The same series has also produced related principal error function for the underlying integrator. The proposed iterative integrator can be used to attain second order accuracy in comparison with single-step Explicit Forward Euler's Integrator to solve initial value problems associated to both scalar and vector-valued ordinary differential equations. Computation of local truncation errors is shown through both linear and nonlinear initial value problems*.

Keywords: Cauchy problems, local truncation error, principal error function.

INTRODUCTION

Ordinary differential equations are being extensively used in various fields of science, engineering, chemistry, business, biomedical and mathematical physics; for number of mathematical models involving such equations form the basis of unlimited application areas in these fields $[1-3]$. Most of these mathematical models specifically containing nonlinearities singularities, oscillations and stiffness; either do not have closed-form solutions or are substantially complicated to obtain in real time domain; for that reason, scholars try to compromise at approximate solutions with related analysis of errors $[4,5]$.

Having great impact on many physical phenomena by such models causes interest among diverse scientific community to explore their solutions and as a result number of algorithms have been developed and the pace of devising better algorithms is on rise as shown in $[6-9]$. With arrival of every new algorithm; analysis of errors can never be neglected which is also a basic purpose of the present paper. Errors called local, global and round-off are needed to be comtemplated for an algorithm to be worthwhile and universally acceptable $[10]$. In addition to this, integrators having larger absolute stability regions have been found to produce better results than those with smaller such regions and accordingly the regions of a few two-step explicit methods are claimed to be expanded as described in $[11]$. Furthermore, extremely important for convergence of general linear methods are the standard order and stability analysis $[12]$.

MATERIALS AND METHODS

First-order ordinary differential equations with an initial condition are of the form:

$$\frac{dy}{dx} = f(x,y);\ y(x_0) = y_0 \quad (1)$$

For the real-valued function *y* of the real variable *x*. In this study, we will mainly be concerned with the error analysis for an explicit linear single-step second order accurate iterative integrator as shown by (2) which was devised and tested upon both linear and nonlinear ordinary differential equations $[6]$.

$$y_n + 1 = y_n + hf\left[\begin{array}{l} x_n + \frac{h}{2}, \\ y_n + \frac{h}{4}\left(\begin{array}{l} f(x_n, y_n) + \\ f\left(\begin{array}{l} x_n + \\ 2h(e^{0.5h}-1), y_n + \\ 2h(e^{0.5h}-1)f_n \end{array}\right) \end{array}\right) \end{array}\right] \quad (2)$$

The Taylor's series expansion for a function of two variables as shown in (3) is employed to analyze local truncation error and associated principal error function for the integrative integrator given by (2).

$$f(x+a, y+b) = f + af_x + bf_y + \frac{1}{2}\left(a^2 f_{xx} + 2abf_{xy} + b^2 f_{yy}\right) + \cdots (3)$$

where $a,\ b \in \Re$ and $f(x,y) \in C^{\infty}$. Furthermore, in order to confirm second order accuracy of the proposed method it is important that Taylor's series expansion of both exact and proposed method must agree with each other up to the term containing h^2. The series for exact result is expressed as:

$$y_{n+1} = y_n + hf(x_n, y_n) + \frac{h^2}{2} f'(x_n, y_n) + \frac{h^3}{6} f''(x_n, y_n) + O(h^3)$$

Or

$$y_{n+1} = y_n + hf + \frac{h^2}{2} S + \frac{h^3}{6}\left(T + Sf_y\right) + O(h^4) \quad (4)$$

where $f'(x_n, y_n) = f_x + ff_y = S$ and $f''(x_n, y_n) = f_{xx} + 2ff_{xy} + f^2 f_{yy} + f_y\left(f_x + ff_y\right) = Sf_y + T$. As stated in $[1]$, if exact result is assumed at previous integration steps then a local truncation error is defined to be the error caused by current single integration step. Mathematically,

$$T_n = \frac{y(x_{n+1}) - y(x_n)}{h} - f\left[\begin{array}{l} x_n + \frac{h}{2}, \\ y_n + \frac{h}{4}\left(\begin{array}{l} f(x_n, y_n) + \\ f\left(\begin{array}{l} x_n + 2h(e^{0.5h}-1), \\ y_n + 2h(e^{0.5h}-1)f_n \end{array}\right)\end{array}\right)\end{array}\right] \quad (5)$$

where $y(x_{n+1})$ and y_{n+1} are the exact and approximate solutions of a differential equation at the current stage respectively and it has been assumed that no error occurs at previous integration step; $x = x_n$.

The derivation of required local truncation error of the proposed method using (3) goes as follow:

$$y_{n+1} = y_n + hf\left[\begin{array}{l} x_n + \frac{h}{2}, y_n + \frac{h}{4} \\ \left(\begin{array}{l} 2f + 2h(e^{0.5h}-1)(f_x + ff_y) + \\ 2h^2(e^{0.5h}-1)^2 \\ (f_{xx} + 2ff_{xy} + f^2 f_{yy}) \end{array}\right)\end{array}\right]$$

Let $S = f_x + ff_y$ and $T = f_{xx} + 2ff_{xy} + f^2 f_{yy}$.

Thus,

$$y_{n+1} = y_n + hf\left[\begin{array}{l} x_n + \frac{h}{2}, y_n + \\ \frac{h}{4}\left(\begin{array}{l} 2f + 2h(e^{0.5}-1)S \\ + 2h^2(e^{0.5}-1)^2 T \end{array}\right)\end{array}\right]$$

$$y_{n+1} = y_n + hf\left[\begin{array}{l} x_n + \frac{h}{2}, y_n + \\ \frac{h}{2} f + \frac{h^2}{2}(e^{0.5h}-1)S + \\ \frac{h^3}{2}(e^{0.5h}-1)^2 T \end{array}\right]$$

$$y_{n+1} = y_n + h\left[\begin{array}{l} f + \frac{h}{2} f_x + \left(\begin{array}{l} \frac{h}{2} f + \frac{h^2}{2}(e^{0.5h}-1)S + \\ \frac{h^3}{2}(e^{0.5h}-1)^2 T \end{array}\right) f_y + \\ \frac{1}{2}\left(\begin{array}{l} \frac{h^2}{4} f_{xx} + h\left(\begin{array}{l} \frac{h}{2} f + \frac{h^2}{2}(e^{0.5h}-1)S + \\ \frac{h^3}{2}(e^{0.5h}-1)^2 T \end{array}\right) f_{xy} \\ + \left(\begin{array}{l} \frac{h}{2} f + \frac{h^2}{2}(e^{0.5h}-1)S + \\ \frac{h^3}{2}(e^{0.5h}-1)^2 T \end{array}\right)^2 Sf_y \end{array}\right)\end{array}\right]$$

$$y_{n+1} = y_n + h\left[\begin{array}{l} f + \frac{h}{2} f_x + \frac{h}{2} ff_y + \\ \frac{h^2}{2}(e^{0.5h}-1)Sf_y + \\ \frac{1}{2}\left(\frac{h^2}{4} f_{xx} + \frac{h^2}{2} ff_{xy} + \frac{h^2}{4} f^2 f_{yy}\right) \end{array}\right] + O(h^4)$$

where all higher powers of *h* have been ignored. Thus,

$$y_{n+1} = y_n + h\left[\begin{array}{l} f + \frac{h}{2}(f_x + ff_y) + \\ \frac{h^2}{2}(e^{0.5h}-1)Sf_y + \\ \frac{h^3}{8}(f_{xx} + 2ffxy + f^2 f_{yy}) \end{array}\right]$$

$$y_{n+1} = y_n + hf + \frac{h^2}{2} S + \frac{h^3}{2}(e^{0.5h}-1)Sf_y + \frac{h^3}{8} T + O(h^4)$$

Substituting all of this into (5) in which the term $y(x_{n+1})$ constitutes the exact solution obtained by Taylor's series expansion; we have

$$T_n = h^3 Sf_y\left[\frac{1}{6} - \frac{1}{2}(e^{0.5h}-1)\right] + h^3 T\left(\frac{1}{6} - \frac{1}{8}\right)$$

$$T_n = \left[\left(\frac{1}{3} - \frac{1}{2} e^{0.5h}\right)Sf_y + \frac{1}{24} T\right]h^3$$

In this way, the required principal error function for the iterative integrator proposed in [1] is found to be:

$$PEF = \left(\frac{1}{3} - \frac{1}{2} e^{0.5h}\right)Sf_y + \frac{1}{24} T$$

RESULTS AND DISCUSSION

This section carries out computation of local truncation error for all proposed integrator in case of both linear and nonlinear initial value problem. The first initial value problem chosen to be linear, goes as follows:

$$\frac{dy}{dx} = x + y; \quad y(0) = 0$$

On $x \in [0,1]$ with the exact solution given by: $y(x) = e^x - x - 1$.Third column of the table 1 has been computed using proposed method but every next integration step uses the exact value from previous integration step in accordance with definition of local truncation error which assumes no error is incurred at previous step.

As a second example, a nonlinear problem is selected which is given below with its exact solution defied in the closed interval $[0,1]$:

$$\frac{dy}{dx} = y(1-y); \quad y(0) = 0.5$$

where the exact solution is as follows: $y(x) = \frac{1}{1+e^{-x}}$. In this example, it has also been observed from table 3 that whenever step size (*h*) is halved; ratio of absolute relative errors is approaching to a constant (4) which verifies the claim of second order accuracy of the proposed iterative integrator.

Table 1 Percentile Relative Local Truncation Error for the linear case.

x	y_{exact}	$y_{proposed}$	y_{latest}	Local
0	0	0	0	-
0.1	0.0050	0.0052	0.0050	3.3054
0.2	0.0211	0.0214	0.0213	0.4801
0.3	0.0493	0.0499	0.0497	0.3185
0.4	0.0910	0.0918	0.0917	0.1358
0.5	0.1476	0.1487	0.1485	0.1488
0.6	0.2206	0.2221	0.2219	0.0985
0.7	0.3118	0.3138	0.3135	0.0805
0.8	0.4231	0.4255	0.4253	0.0566
0.9	0.5566	0.5596	0.5592	0.0720
1.0	0.7146	0.7183	0.7179	0.0532

Table 2 Percentile Relative Local Truncation Error for the nonlinear case.

x	y_{exact}	$y_{proposed}$	y_{latest}	Local
0	0.5	0.5	0.5	0
0.1	0.5250	0.5250	0.5250	-0.0040
0.2	0.5498	0.5499	0.5499	-0.0120
0.3	0.5744	0.5745	0.5744	-0.0074
0.4	0.5987	0.5987	0.5986	-0.0146
0.5	0.6225	0.6225	0.6225	-0.0065
0.6	0.6457	0.6457	0.6457	-0.0068
0.7	0.6682	0.6682	0.6682	-0.0018
0.8	0.6900	0.6900	0.6900	-0.0037
0.9	0.7109	0.7110	0.7100	-0.1336
1.0	0.7311	0.7311	0.7310	-0.0080

Table 3 Second Order Accuracy .

Step size	Absolute Relative Error	Ratios
0.2	1.302923136046465e-04	---------------
0.1	2.520690258883109e-05	5.1689e+00
0.05	5.381800699231935e-06	4.6837e+00
0.025	1.2299624284058600e-06	4.3756e+00
0.0125	2.930089350772398e-07	4.1977e+00
0.0063	7.143874621179230e-08	4.10154e+00

CONCLUSIONS AND FUTURE WORK

The present paper offers discussion upon derivation of local truncation error and its corresponding principal error function associated with a proposed iterative integrator. This truncation error is claimed to have achieved second order accuracy for the integrator, that is; each time the step size h is halved, the truncation error is reduced by a factor of 4. In order to serve this purpose, Taylor's series expansion for a function of two variables has been employed that agrees with proposed method up to the term involving h^2. Both linear and nonlinear models are solved with the proposed integrator for the computations of local errors as shown by tables thereof approximate result are also in compliance with exact results to certain amount of digits. In the time to come for the proposed iterative integrator, investigation for the decay of global truncation error will be carried out and possible error bounds would be constructed which are considered to be the essence of an algorithm to have been accepted. Apart from this, its consistency and stability analysis will also be major theme of the future research work.

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