COMPUTATION OF LAX SHOCK TUBE PROBLEM THROUGH LARGE TIME STEP SCHEME WITH ARTIFICIAL COMPRESSION METHOD

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ABSTRACT: *Modeling and simulation of complex flow fields over the space vehicle is not only difficult but also time consuming task. Analytical solutions are not possible while numerical solution contains approximation error. Currently methods for precise and efficient prediction of vehicle response due to aerodynamic loading are greatly under investigation. Work is in progress to reduce computational requirements, speed up simulations and minimize numerical errors.*

Explicit schemes require low computational resources but larger computational time by limiting time step to a certain limit dictated by stability criteria. Combination of explicit large time step scheme with artificial compression method has not been vastly explored to obtain precise and efficient numerical results. In present work Qian's modified form of Harten's large time step scheme with artificial compression method is used to solve Lax shock tube problem. Large time stepping is used to minimize simulation time while artificial compression method is applied to optimize accuracy. Results are very promising and depict the importance of large time stepping *and artificial compression method.*

Keywords: CFL restriction, Explicit scheme, Numerical Dissipation, TVD Scheme, 1D Euler equation

NOMENCLATURE

A	inviscid flux jacobion matrix	n	number of time steps taken
$C_{l\,(x)}$	coefficient functions	p	pressure
D	fractional constant in ACM switch	t	time
E	total energy	Δt	time step
F	physical flux	u	velocity in x-direction
K	CFL restriction parameter	v	local CFL number
P	ACM parameter	x	axial distance
R	eigen vector matrix	Δx	grid spacing
R^{-1}	inverse eigen vector matrix	α	characteristic variable
U	conservative variable vector	β	numerical characteristic speed
a	characteristic speed	ε	entropy fix parameter
c	speed of sound	Φ	numerical dissipation term
f	numerical flux	γ	ratio of specific heat
g	flux correction	λ	mesh ratio
$\breve{g}$	limiter function	μ	CFL parameter
i	grid pointer	ρ	density
ӄ	ACM parameter	σ	limiter function parameter
k	characteristic direction	ψ	entropy correction function
m	number of eigen values	θ	ACM switch

INTRODUCTION

Flow field over the space vehicle contain complex flow features. When the space vehicle is launched from earth to space, it takes long time to reach their destination. Particularly missions for outside earth orbit take time in months to achieve final goal. Computation of such complex flow physics is very costly and is a challenging task. In the field of computational fluid dynamics, Navier-Stocks equation is a smart tool to predict aerodynamics and aero-thermodynamics behavior of space vehicle. Navier-Stocks equation is consisting highly coupled, nonlinear, three dimensional, transient equations [1].

Numerically these equations can be solved in two ways, namely, implicit or explicit formulation [2] [3]. Implicit formulations are robust and lack due to unnecessary hardware and memory requirements for simulation of such large system of discrete equations. Explicit formulation overcomes this issue but deficit due to limitation in time step. Stability criterion for explicit formulations limits time step size which ultimately fallout long computer running times and thus increases computational cost [4-6]. Harten in 1986 presented a second order accurate (2K+3) point Total Variation Diminishing (TVD) scheme with explicit formulation for the computation of weak solutions of hyperbolic conservation laws under a CFL restriction of K [7-9]. Computations for nonlinear wave equation through Harten's large time steps scheme are free of oscillation [10] [11]. However, computation of highly coupled nonlinear system of equations through Harten's large time steps scheme exhibit spurious oscillation in the vicinity of discontinuities [12-15].

Later on Zhan Sen Qian [16] [17] pointed out that this spurious oscillation is due to the improper extension of scalar schemes to coupled system of equations. He suggested that inverse characteristic transformations should be carried out through the local right eigen vector matrix at each cell interface location. His proposed modifications ultimately provide oscillation free results.

Combination of explicit large time step scheme with artificial compression method has not been vastly explored up till now. Behavior of LTS scheme with ACM for shock tube problem using LAX boundary condition is yet to be reported in the literature and still is a hidden corner. Lax boundary condition for shock tube problem is relatively more complex as compared to Sod shock tube problem [18] [19]. Combination of Qian's modified form of Harten's large time step scheme with artificial compression method is used to compute Lax shock tube problem. Large time stepping is used to fulfill the requirement of shorter simulation time while artificial compression method is used to reduce numerical errors.

NUMERICAL METHOD

1D Euler equation in conservation form is given below:

$$\frac{\partial U}{\partial t}+\frac{\partial F}{\partial x}=0 \tag{1}$$

$$\frac{\partial U}{\partial t}+A\frac{\partial U}{\partial x}=0 \tag{2}$$

where; $U=\begin{bmatrix}\rho\\ \rho u\\ \rho E\end{bmatrix}$; $F=\begin{bmatrix}\rho u\\ \rho u^2+p\\ (\rho E+p)u\end{bmatrix}$

$$U_i^{n+1}=U_i^n-\lambda\left(f^n_{i+\frac{1}{2}}-f^n_{i-\frac{1}{2}}\right)\ ;\qquad \lambda=\frac{\Delta t}{\Delta x} \tag{3}$$

$$f^n_{i+\frac{1}{2}}=\frac{1}{2}\left[F^n_{i+1}+F^n_i+R^n_{i+\frac{1}{2}}\Phi^n_{i+\frac{1}{2}}\right] \tag{4}$$

Numerical dissipation term $R^n_{i+\frac{1}{2}}\Phi^n_{i+\frac{1}{2}}$ for total variation diminishing (TVD) scheme proposed by Harten is:

$$R^n_{i+\frac{1}{2}}\Phi^n_{i+\frac{1}{2}}=\frac{1}{2\lambda}\sum_{k=1}^{m}R^k_{i+\frac{1}{2}}\left[\left(g^k_{i+1}+g^k_i\right)-\psi^k\left(v^k+\beta^k\right)_{i+\frac{1}{2}}\alpha^k_{i+\frac{1}{2}}\right] \tag{5}$$

Numerical dissipation term $R^n_{i+\frac{1}{2}}\ \Phi^n_{i+\frac{1}{2}}$ for large time stepping proposed by Zhan Sen Qian is [16-17];

$$R^n_{i+\frac{1}{2}}\Phi^n_{i+\frac{1}{2}}=\frac{1}{2\lambda}\sum_{k=1}^{m}R^k_{i+\frac{1}{2}}\left(g^k_{i+1}+g^k_i\right)-\frac{1}{\lambda}\sum_{l=-K+1}^{K-1}R^k_{i+\frac{1}{2}}\left[\sum_{k=1}^{m}C_l\left(v^k+\beta^k\right)_{i+l+\frac{1}{2}}\alpha^k_{i+l+\frac{1}{2}}\right] \tag{6}$$

where m = 3, 4, 5 for one-, two- and three-dimensional problem respectively, and

$$g^k_i=\text{minmod}\left(\tilde{g}^k_{i+\frac{1}{2}},\tilde{g}^k_{i-\frac{1}{2}}\right)$$

$$\tilde{g}^k_{i+\frac{1}{2}}=\sigma^k_{i+\frac{1}{2}}\alpha^k_{i+\frac{1}{2}}$$

$$\sigma^k_{i+\frac{1}{2}}=\frac{K}{2}\left[\psi\left(\frac{v^k_{i+\frac{1}{2}}}{K}\right)\left\{1+\frac{K-1}{2}\psi\left(\frac{v^k_{i+\frac{1}{2}}}{K}\right)\right\}-\frac{K+1}{2}\left(\frac{v^k_{i+\frac{1}{2}}}{K}\right)^2\right]$$

$$v^k_{i+\frac{1}{2}}=\lambda\, a^k_{i+\frac{1}{2}}\ ;\qquad a=u,\ u+c,\ u-c$$

$$\alpha^k_{i+\frac{1}{2}}=R^{-1}\Delta_{i+\frac{1}{2}}U$$

$$\beta^k_{i+\frac{1}{2}}=\begin{cases}\dfrac{\left(g^k_{i+1}-g^k_i\right)}{\alpha^k_{i+\frac{1}{2}}}, & \text{for } \alpha^k_{i+\frac{1}{2}}\neq 0\\ 0, & \text{for } \alpha^k_{i+\frac{1}{2}}=0\end{cases}$$

$$C_{\pm k}(v)=\begin{cases}c_k\left(\mu_{\mp}(v)\right), & \text{for } 1\le k\le K-1\\ \frac{K}{2}\psi\left(\frac{v}{K}\right), & \text{for } k=0\end{cases}$$

$$\mu_{\pm}(v)=\frac{1}{2}\left[\psi\left(\frac{v}{K}\right)\pm\frac{v}{K}\right]$$

$$\psi(v)=\begin{cases}\frac{1}{2}\left(\frac{v^2}{\varepsilon}+\varepsilon\right), & \text{for } |v|<\varepsilon\\ |v|, & \text{for } |v|\ge\varepsilon\end{cases}$$

$$R=\begin{bmatrix}1 & 1 & 1\\ u & u+c & u-c\\ \frac{u^2}{2} & \frac{u^2}{2}+uc+\frac{c^2}{(\gamma-1)} & \frac{u^2}{2}-uc+\frac{c^2}{(\gamma-1)}\end{bmatrix}$$

$$R^{-1}=\begin{bmatrix}1-\frac{(\gamma-1)u^2}{2c^2} & (\gamma-1)\frac{u}{c^2} & -\frac{(\gamma-1)}{c^2}\\ -\frac{u}{2c}+\frac{(\gamma-1)u^2}{4c^2} & \frac{1}{2c}-\frac{(\gamma-1)u}{2c^2} & \frac{(\gamma-1)}{2c^2}\\ \frac{u}{2c}+\frac{(\gamma-1)u^2}{4c^2} & -\frac{1}{2c}-\frac{(\gamma-1)u}{2c^2} & \frac{(\gamma-1)}{2c^2}\end{bmatrix}$$

Table 1 shows the formulas of coefficient functions C_l (x) for different values of K.

Table-01: C_l (x) at different K.

K	C_1	C_2	C_3
2	x^2		
3	$x^2(3-x)$	x^3	
4	$x^2(6-4x+x^2)$	$2x^3(2-x)$	x^4

The artificial compression method (ACM) proposed by Harten [11] is used to control the amount of nonlinear dissipation introduced during discretization process. This method efficiently reduces false diffusion and produce high resolution results. The modification is done in equation (4) by replacing the term $\Phi^n_{i+\frac{1}{2}}$ with $ʓ\theta^n_{i+\frac{1}{2}}\emptyset^n_{i+\frac{1}{2}}$ thus we get;

$$f^n_{i+\frac{1}{2}}=\frac{1}{2}\left[F^n_{i+1}+F^n_i+R^n_{i+\frac{1}{2}}\left(ʓ\theta^n_{i+\frac{1}{2}}\emptyset^n_{i+\frac{1}{2}}\right)\right] \tag{7}$$

$$\theta^n_{i+\frac{1}{2}}=\max\left(\theta^n_{i-K+1},\ldots\ldots,\theta^n_{i+K}\right) \tag{8}$$

$$\theta^n_i=\left|\frac{\left|\alpha^k_{i+\frac{1}{2}}\right|-\left|\alpha^k_{i-\frac{1}{2}}\right|}{\left|\alpha^k_{i+\frac{1}{2}}\right|+\left|\alpha^k_{i-\frac{1}{2}}\right|+D}\right|^P \tag{9}$$

The term $ʓ\,\theta^n_{i+\frac{1}{2}}$ in equation (7) is the key mechanism of

ACM which reduces dissipation. The parameter ɕ is problem dependent having range $0 < ɕ \leq 1$. The values of ɕ used in present studies are 0.9, 0.92, 0.93 & 0.95 for K=1, 2, 3 & 4 respectively. Value of ‘P’ is taken as 0.1

The numerical flux formula for modified large time step total variation diminishing (MLTS TVD) scheme with artificial compression method (ACM) is given by:

$$f_{i+\frac{1}{2}} = \frac{1}{2}\left[F_{i+1} + F_i\right] + \left(ɕ\,\theta^n_{i+\frac{1}{2}}\right)\left[\frac{1}{2\lambda}\sum_{k=1}^{m} R^k_{i+\frac{1}{2}}\left(g^k_{i+1} + g^k_i\right) - \frac{1}{\lambda}\sum_{l=-K+1}^{K-1}\left\{\sum_{k=1}^{m} R^k_{i+l+\frac{1}{2}} C_l\left(v^k + \beta^k\right)_{i+l+\frac{1}{2}} \alpha^k_{i+l+\frac{1}{2}}\right\}\right] \quad (10)$$

Detailed description of TVD scheme, large time step scheme, and artificial compression method can be found in [11-17].

TEST CASE DESCRIPTION

Shock tube is one of the few 1D problem for which analytical solution is possible and hence it is often used as a test case for verification and validation purpose. LAX boundary conditions (Table 2) for 1D shock tube problem are used in present computation. Time integration is carried out until physical time 0.15 sec is reached. Here minmod limiter is used and entropy fix parameter value is taken as 0.1. The size of computational domain is $0 \leq x \leq 1$ and number of grids are 1000. Initial discontinuity is centered at $x = x_o$ and $t = 0$ subject to the conditions:

$$U(x,t) = \begin{cases} U_L, & x < x_o \\ U_R, & x \geq x_o \end{cases} \qquad \text{where;} \qquad x_o = 0.5$$

Table-02: LAX Boundary condition

p_R	ρ_R	u_R	p_L	ρ_L	u_L
0.571	0.5	0.0	3.528	0.445	0.698

RESULT AND DISCUSSION

1D shock tube problem for LAX boundary conditions are solved to investigate the behavior of large time step TVD scheme in combination with artificial compression method (ACM). Expansion, contact and shock regions are mainly focused to examine the effectiveness of modified formulation. CFL values are taken as 0.9, 1.8, 2.8, and 3.8 for $K = 1, 2, 3$ and 4, respectively. Higher value of CFL means larger time step size and hence minimum number of time steps are required to perform desired simulation. Computed results are not only compared with analytical results but also with numerical results obtained from large time step scheme without artificial compression method (ACM). Simulations are carried out on Intel(R) Core(TM) 2 CPU @ 2.13 GHz, 2 GB RAM. In all figures, SW=0 means that ACM switch is not applied while SW=1 is vice versa of it.

The 1D shock tube problem with the LAX boundary condition is a relatively complex problem to compute because its Left end face has non zero initial velocity. Computed results of large time step scheme with and without artificial compression method are compared with analytical results from Figure 1 to Figure 4.

Figure 1 shows comparison of results near start of expansion region. Results depict that Artificial Compression Method (ACM) successfully minimize false diffusion in this region without dispersion and stability issues. Results at the end of expansion region are shown in Figure 2. Results show same behavior as at the start of expansion region. Large time step scheme combined with artificial compression method produce less diffusive results with shorter simulation time around expansion region for this case. Complete expansion process is better captured using proposed methodology.

Results near contact discontinuity and shock region are shown in Figure 3 and Figure 4. Results at both regions have been improved by using artificial compression method. 1D shock tube problem with LAX boundary condition is predicted well using combination of large time step scheme and artificial compression method. Slight oscillation at the end of expansion and near shock is overwhelmed by the significant reduction in numerical dissipation.

Overall results for 1D shock tube problem with LAX boundary condition are satisfactory and show the benefit of proposed methodology.

CONCLUSION

Robust and precise computation of hyperbolic conservation laws is a challenging task. Present work demonstrates a methodology based upon the combination of large time step scheme and artificial compression method to compute hyperbolic conservation laws with improved accuracy and shorter simulation time. This methodology is not only helpful for researchers to get precise results within short period of time but also provide researcher a better opportunity to try more possibilities during the prescribed time frame.

Although present studies are focused only on 1D shock tube problem with LAX boundary condition however in future this work should be extended to multi-dimensional flows. Further studies are also required to find optimal values of artificial compression method parameters.

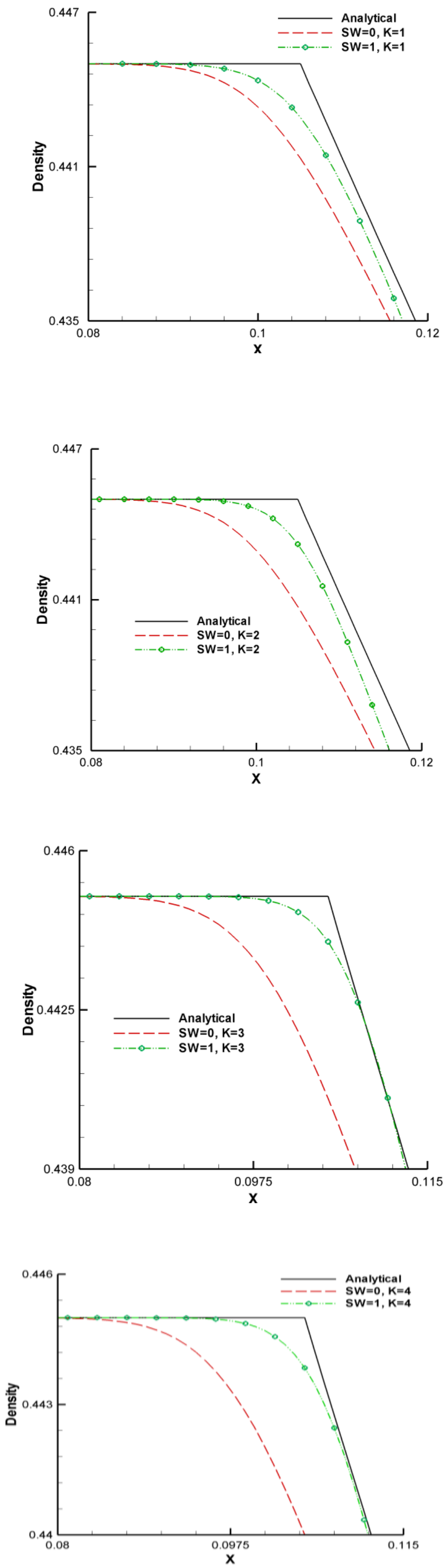

Fig. 1. : Switch vs. without switch, near start of expansion region

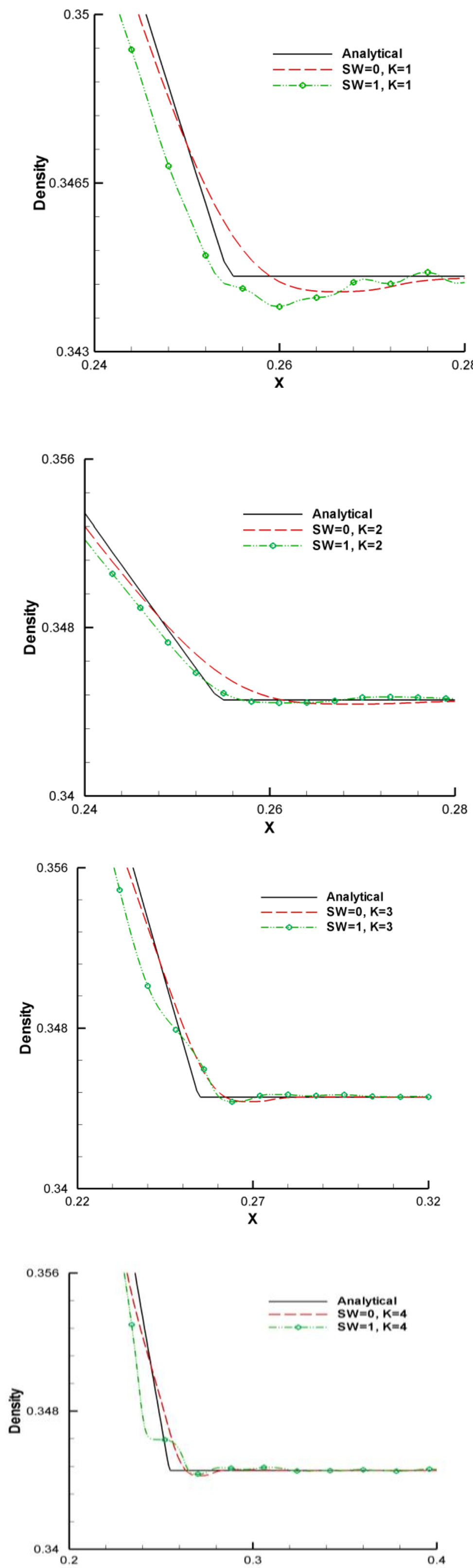

Fig. 2. Switch vs. without switch, near end of expansion region

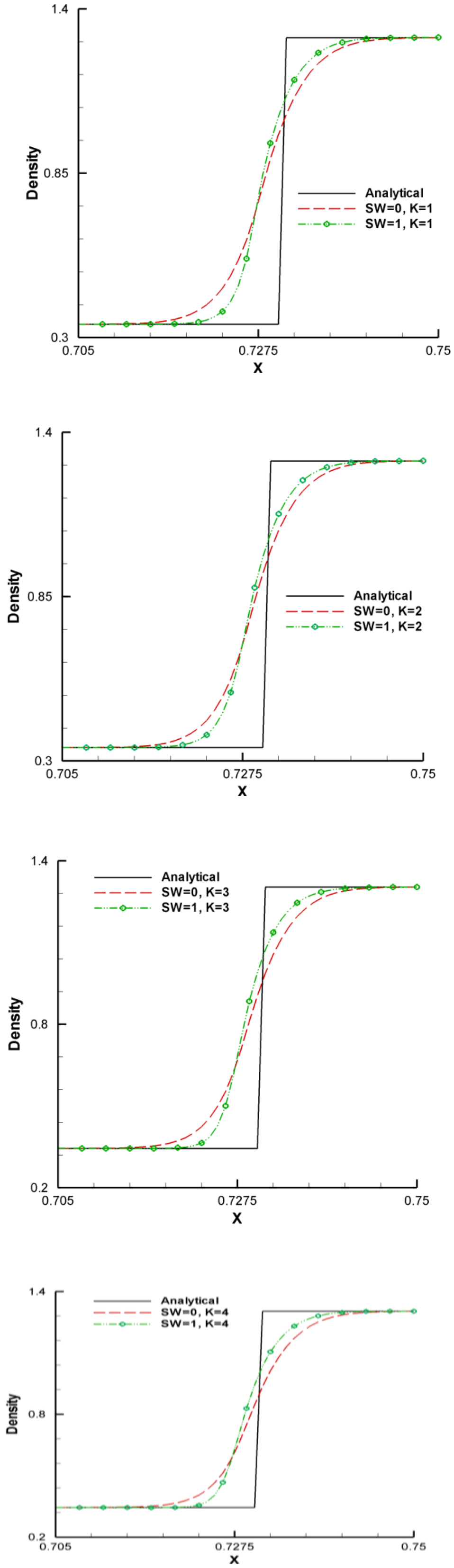

Fig. 3. Switch vs. without switch, near contact region

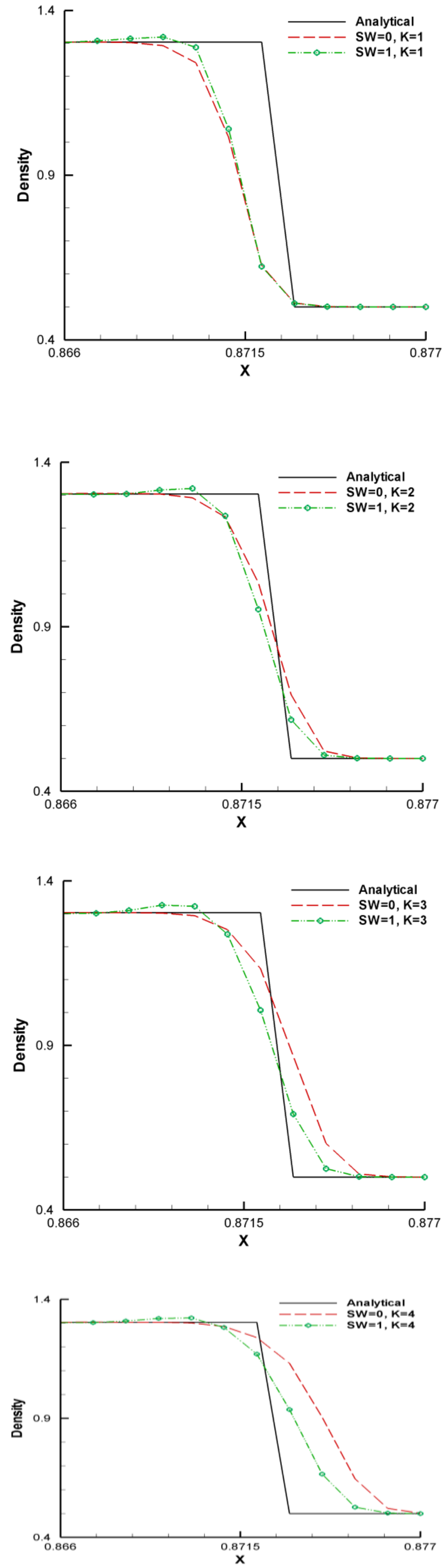

Fig. 4. Switch vs. without switch, near shock region

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