ON RELATIVE RIGHT-EQUIVALENCE OF HOLOMORPHIC FUNCTION-GERMS

M. S. J. Al-Bahadeli
Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq.
m.s.jbara@gmail.com

ABSTRACT*: In this paper, we find the necessary and sufficient condition of holomorphic map-germs under $\Theta^{\mathcal{R}}$-equivalence (relative right-equivalence) where Θ is the module of holomorphic vector fields on $(\mathbb{C}^n, 0)$. Also, we give some results on finite relative determinacy and relative stability.*

Keywords: relative right-equivalence, tangent vector field, map-germ.

1. INTRODUCTION

One of the central problems in singularity theory is the classification of function-germs up to changes of coordinates in the source preserving a sub-germ. Since 1970's, Arnol'd, Bruce and many others have made significant progress in the study of this type of equivalence relations. (See for examples [1, 2, 3,4,5].

In [6], we introduce a new version of equivalence relation of holomorphic function-germs which called $\Theta^{\mathcal{R}}$-equivalence where Θ is the module of holomorphic vector fields on $(\mathbb{C}^n, 0)$ such that every vector field in Θ can be integrated to give a deffeomorphism. When Θ is the module of all vector fields on $(\mathbb{C}^n, 0)$. Then $\Theta^{\mathcal{R}}$-equivalence is just the standard right-equivalence ($\mathcal{R}$-equivalence). In addition, $\Theta^{\mathcal{R}}$-equivalence is just $\mathrm{v}^{\mathcal{R}}$-equivalence when Θ is the module of vector fields tangent to a variety $\mathrm{V} \subseteq \mathbb{C}^n$.

In the present paper, we give more results of $\Theta^{\mathcal{R}}$-equivalence as criterion for holomorphic function-germs, relative finite determinacy and relative stability.

2. PRELIMINARIES

In this section, we give some basic notation and preliminary results which will be used throughout this paper, for more details see [7], [8] and [9]. Let $\mathcal{O}_n$ be the local ring of all holomorphic function-germs $(\mathbb{C}^n, 0) \to \mathbb{C}$. This ring contains a unique maximal ideal, denoted by $\mathfrak{M}_\mathrm{n} = \{f \in \mathcal{O}_n | f(0) = 0\}$. We denote by $\mathcal{O}_n^p$ the set of all holomorphic map-germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$. We put $\mathcal{O}_n^0 = \mathfrak{M}_\mathrm{n}\mathcal{O}_n^1$. The group of all automorphisms $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is denoted $\mathbf{Aut}(\mathbb{C}^n, 0)$. Any map-germ $\varphi: (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ induces a ring homomorphism $\varphi^*: \mathcal{O}_n^n \to \mathcal{O}_n^n$ by $\varphi^*(h) = h\phi$.

If $f \in \mathcal{O}_n$, then $j^k f$ will denote the Taylor expansion up to degree k of f at the origin. The set of all k- jets forms a vector space $\mathrm{J}^k(n, 1) = \frac{\mathcal{O}_n^0}{\mathfrak{M}_\mathrm{n}{}^{k+1}}$ and $\pi_k: \mathcal{O}_n \to \mathrm{J}^k(n, 1)$ is the canonical mapping which assigns $j^k f$ to each f. Given $k, p \in \mathbb{N}$ with $k \geq p$, we denote by $\pi_{k,p}: \mathrm{J}^k(n, 1) \to \mathrm{J}^p(n, 1)$ the natural linear projection of $\mathrm{J}^k(n, 1)$ to $\mathrm{J}^p(n, 1)$.

Lemma 2.1: [7, Nakayama's lemma]

Let R be a commutative ring, M an ideal such that for $x \in M$, $1 + x$ is a unit. Let C be an R-module, A and B be R-modules of C with A finitely generated. If $A \subset B + M.A$, $A \subset B$.

Lemma 2.2: [7, Mather's lemma]

Let the Lie group G act smoothly on the manifold M, and suppose that the connected submanifold S satisfies:

i. for all $x \in S$, $T_x S \subset T_x G.x$,

ii. the dimension of $G.x$ is independent of the choice of $x \in S$.

Then S is contained in a single G orbit.

Theorem 2.3: [8]

Let $\Phi: G \times M \longrightarrow M$ be a smooth action of a Lie group G on a smooth manifold M. It is assumed that all the orbits are smooth submanifolds of M. For any point $x \in M$ the natural mapping $\Phi_x: G \longrightarrow G.x$ of the group onto the orbit given by $\mathrm{g} \longrightarrow g.x$ is a submersion and the tangent space $T_x G.x$ is the image under the differential $\mathrm{d}\Phi_x: TG_{Id_M} \longrightarrow T_x M$, i.e., $T_x G.x = \mathrm{d}\Phi_x(TG_{Id_M})$.

Theorem 2.4: [9, Artin Approximation Theorem]

Let $f(x, y) = \left(f_1(x, y), \ldots, f_N(x, y)\right) \in \mathbb{C}\{x, y\}^N$, where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_N)$. Suppose that for each $k \in \mathbb{N}$ there exist $y_{k,1}, \ldots, y_{k,N} \in \mathfrak{M}_\mathrm{n}$ such that $f(x, y_k(x)) \in \mathfrak{M}_\mathrm{n}^{\mathrm{k}+1}$, for each i. Then for any $c \in \mathbb{N}$ there exist $y_1, \ldots, y_N \in \mathfrak{M}_\mathrm{n}$ such that $f(x, y_i(x)) = 0$ and for all λ, we have

$y_{k,v}(x) - y_v(x) \in \mathfrak{M}_\mathrm{n}^c$.

3. $\Theta^{\mathcal{R}}$-EQUIVALENCE OF FUNCTION-GERMS

In this section, we give the definition of $\Theta^{\mathcal{R}}$-equivalence on $\mathcal{O}_n^0$ and $\mathrm{J}^k(n, 1)$.

Definition 3.1:

Let Θ be a module of vector fields on $(\mathbb{C}^n, 0)$. Then

i. We define

ii. $\Theta^{\mathcal{R}} = \{\varphi \in \mathbf{Aut}(\mathbb{C}^n, 0) | \exists\ \xi \in \Theta$ that can be integrated to give $\varphi\}$.

iii. For each non-negative integer, we define $\Theta^{\mathcal{R}^k} = \{\, j^k\varphi \mid \varphi \in \Theta^{\mathcal{R}}\,\}$

Definition 3.2:[6]

Suppose that $f, \tilde{f}: (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ are holomorphic function-germs. Let Θ be a module of vector fields on $(\mathbb{C}^n, 0)$.

i. We say that f and $\tilde{f}$ are $\Theta^{\mathcal{R}}$-equivalent (or relative right-equivalent), in short $f \sim_{\Theta^{\mathcal{R}}} \tilde{f}$, if there exists $\in \Theta^{\mathcal{R}}$ such that $\tilde{f} = f \circ \varphi$.

ii. We say that $j^k f$ and $j^k\tilde{f}$ are $\Theta^{\mathcal{R}^k}$-equivalent, in short $f \sim^k_{\Theta^{\mathcal{R}}} \tilde{f}$, if there exists $\varphi^k \in \Theta^{\mathcal{R}^k}$ such that $j^k\tilde{f} = j^k f \circ \varphi^k$.

Theorem 3.3:

Suppose that $f, \tilde{f}: (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ are holomorphic function-germs. Let Θ be a module of vector fields on $(\mathbb{C}^n, 0)$. Then $f \sim^k_{\Theta^{\mathcal{R}}} \tilde{f}$ if and only if there is some $g \in \mathfrak{M}_n{}^{k+1}$ such that $f \sim_{\Theta^{\mathcal{R}}} (\tilde{f} - g)$.

Proof.

$$\begin{aligned} f \sim^k_{\Theta^{\mathcal{R}}} \tilde{f} &\Leftrightarrow j^k\tilde{f} = j^k f \circ \varphi^k \\ &\Leftrightarrow j^k\tilde{f} = j^k f \circ j^k\varphi \\ &\Leftrightarrow j^k\tilde{f} - j^k f \circ j^k\varphi = 0 \\ &\Leftrightarrow j^k(\tilde{f} - f \circ \varphi) = 0 \\ &\Leftrightarrow \tilde{f} - f \circ \varphi \in \mathfrak{M}_n{}^{k+1} \\ &\Leftrightarrow \tilde{f} - f \circ \varphi = g, g \in \mathfrak{M}_n{}^{k+1} \\ &\Leftrightarrow \tilde{f} - g = f \circ \varphi, g \in \mathfrak{M}_n{}^{k+1} \\ &\Leftrightarrow f \sim_{\Theta^{\mathcal{R}}} (\tilde{f} - g). \end{aligned}$$ □

Definition 3.4:[6]

Let $f: (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function-germ and let Θ be a module of vector fields on $(\mathbb{C}^n, 0)$.

(1) The extended $\Theta^{\mathcal{R}}$-tangent space, denoted by $T_{\Theta^{\mathcal{R}_e}}(h)$, is the submodule of $\mathcal{O}_n$ given by

$$T_{\Theta^{\mathcal{R}_e}}(f) = < \xi(f) | \xi \in \Theta >.$$

We also call the Jacobian of f with respect to Θ, denoted by $\mathbf{J}_\Theta(f)$.

(2) The $\Theta^{\mathcal{R}}$-tangent space, denoted by $T_{\Theta^{\mathcal{R}}}(f)$, is the submodule of $\mathcal{O}_n$ given by

$$T_{\Theta^{\mathcal{R}}}(f) = < \xi(f) | \xi \in \Theta \cap \mathfrak{M}_n\mathcal{O}_n^n >.$$

(3) The $\Theta^{\mathcal{R}_e}$-codimension of f, is defined by

$$\Theta^{\mathcal{R}_e} - cod(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{T_{\Theta^{\mathcal{R}_e}}(f)}.$$

Remark 3.5:

(1) If all elements of Θ vanish at the origin, then $T_{\Theta^{\mathcal{R}}}(f) = T_{\Theta^{\mathcal{R}_e}}(f)$.

(2) Suppose that Θ is the set of all vector fields on $(\mathbb{C}^n, 0)$. Then $\Theta^{\mathcal{R}}$-equivalence is just the standard right-equivalence ($\mathcal{R}$-equivalence). For more details see [8].

(3) If Θ is the module of tangent vector fields on a sub-germ $(\mathrm{V}, 0) \subset (\mathbb{C}^n, 0)$, then $\Theta^{\mathcal{R}}$-equivalence is $\mathrm{v}^{\mathcal{R}}$-equivalence. For more details see [2], [3] and [4].

4. AN ALGEBRAIC CRITERION OF $\Theta^{\mathcal{R}}$-EQUIVALENCE.

Definition 4.1:

Suppose that $f, \tilde{f}: (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ are holomorphic function-germs. Let Θ be a module of vector fields on $(\mathbb{C}^n, 0)$. We say that the tangent spaces $T_{\Theta^{\mathcal{R}}}(f)$ and $T_{\Theta^{\mathcal{R}}}(\tilde{f})$ are $\Theta^{\mathcal{R}}$-equivalent, denoted by $T_{\Theta^{\mathcal{R}}}(f) \cong T_{\Theta^{\mathcal{R}}}(\tilde{f})$ if there exists a map-germ $\Phi \in \mathbf{Aut}(\mathbb{C}^n, 0)$ such that $T_{\Theta^{\mathcal{R}}}(f)) = \Phi^*(T_{\Theta^{\mathcal{R}}}(\tilde{f}))$.

Lemma 4.2:

Let Θ be a module of vector fields on $(\mathbb{C}^n, 0)$. Then $T_{Id}\Theta^{\mathcal{R}^k} = \pi_k(\Theta \cap \mathfrak{M}_n\mathcal{O}_n^n)$.

Proof.

Let $\eta \in \pi_k(\Theta \cap \mathfrak{M}_n\mathcal{O}_n^n)$. Then $\eta = j^k\xi$ with $\xi = \sum_{i=1}^n \xi_i \frac{\partial}{\partial x_i} \in \Theta \cap \mathfrak{M}_n\mathcal{O}_n^n$. Consider the one-parameter family φ_t of ξ, it is clear that $\varphi_0 = Id$, $\varphi_t \in \Theta^{\mathcal{R}}$ and $\frac{\partial \varphi_t}{\partial t}(x) = \xi(\varphi_t(x))$. We define

$$\alpha(t) = j^k\varphi_t(x): (-\varepsilon, \varepsilon) \longrightarrow \Theta^{\mathcal{R}^k}.$$

Then we can see that $\alpha(0) = Id$ and

$$\begin{aligned} \dot{\alpha}(0) &= \frac{d}{dt}[j^k\varphi_t(x)]_{|t=0} \\ &= j^k\left[\frac{\partial \varphi_t}{\partial t}(x)\right]_{|t=0} \\ &= j^k\xi \\ &= \eta \in T_{Id}\Theta^{\mathcal{R}^k} \end{aligned}$$

Conversely, given $\xi = \sum_{i=1}^n \xi_i \frac{\partial}{\partial x_i} \in T_{Id}\Theta^{\mathcal{R}^k}$. Then there exists $\alpha(t): (-\varepsilon, \varepsilon) \longrightarrow \Theta^{\mathcal{R}^k}$ with $\alpha(0) = Id$ and $\dot{\alpha}(0) = \xi$. Consider $\varphi_t(x) = \alpha(t)(x) = x + t\xi(x)$. Then $\varphi_t(x) \in \Theta^{\mathcal{R}^k}$and it follows

$\xi \in \pi_k(\Theta \cap \mathfrak{M}_n\mathcal{O}_n^n)$. □

Lemma 4.3:

Let $f: (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function-germ and let Θ be a module of vector fields on $(\mathbb{C}^n, 0)$. Then the tangent space $T_{\Theta^{\mathcal{R}^k}}(f)$ to the $\Theta^{\mathcal{R}^k}$-orbit of j^kf at the point $j^kf \in \mathrm{J}^k(n, 1)$ is given by

$$\pi_k{}^{-1}(T_{\Theta^{\mathcal{R}^k}}(f)) = T_{\Theta^{\mathcal{R}}}(f) + \mathfrak{M}_n^{k+1}.$$

Proof.

From Lemma we have $T_{Id}\Theta^{\mathcal{R}^k} = \pi_k(\Theta \cap \mathfrak{M}_n\mathcal{O}_n^n)$. Let $\eta \in T_{Id}\Theta^{\mathcal{R}^k}$ be a tangent vector, $\eta = j^k\xi$ with $\xi = \sum_{i=1}^n \xi_i \frac{\partial}{\partial x_i} \in \Theta \cap \mathfrak{M}_n\mathcal{O}_n^n$. For $\in \mathbb{R}$, we define $\delta_t = Id + t\xi$. If we consider $\pi_k \circ \delta_t: (-\varepsilon, \varepsilon) \longrightarrow \Theta^{\mathcal{R}^k}$. Then

$$\begin{aligned} T_{\Theta^{\mathcal{R}^k}}(f) &= d\Phi_{j^kf}\left(T_{Id}\Theta^{\mathcal{R}^k}\right) \\ &= \frac{d}{dt}[\pi_k(f \circ \delta_t)]_{|t=0} \\ &= \pi_k\left[\frac{d}{dt}(f \circ \delta_t)\right]_{|t=0} \\ &= \pi_k\left[\sum_{i=1}^n \frac{\partial f}{\partial x_i}(\delta_t)\frac{\partial(\delta_t)_i}{\partial t}(x)\right]_{|t=0} \end{aligned}$$

$$= \pi_k\left[\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(x)\xi_i(x)\right]$$

$$= \pi_k\left[T_{\Theta^{\mathcal{R}}}(f)\right].$$

Hence,

$\pi_k{}^{-1}(T_{\Theta^{\mathcal{R}^k}}(f)) = T_{\Theta^{\mathcal{R}}}(f) + \mathfrak{M}_n^{k+1}$. □

Theorem 4.4:

Suppose that $f,\tilde{f}: (\mathbb{C}^n,0) \to (\mathbb{C},0)$ are holomorphic function-germs. Let Θ be a finitely generated $\mathcal{O}_n$-module of vector fields on $(\mathbb{C}^n,0)$. If $f \sim^k_{\Theta^{\mathcal{R}}} \tilde{f}$ for each non-negative integer k, then $f \sim_{\Theta^{\mathcal{R}}} \tilde{f}$.

Proof.

For each non-negative integer k, we have $f \sim^k_{\Theta^{\mathcal{R}}} \tilde{f}$. Thais means, there exist $(\xi^k,\ \varphi^k)$ such that

$$\tilde{f}(\mathrm{x}) - f\left(\varphi^k(x)\right) \in \mathfrak{M}_n{}^{k+1},$$

$$\frac{\partial \varphi^k}{\partial t}(x) - \xi^k\left(\varphi^k(x)\right) \in \mathfrak{M}_n{}^{k+1}.$$

Then by using Artin approximation theorem, the above system has a convergent solution $(\xi,\ \varphi)$ such that

$$\tilde{f}(\mathrm{x}) - f\left(\varphi(x)\right) = 0,$$

$$\frac{\partial \varphi}{\partial t}(x) - \xi\left(\varphi(x)\right) = 0.$$

In addition, we have $\varphi(x) - \varphi^k(x) \in \mathfrak{M}_n{}^2$. It follows $\varphi \in \mathbf{Aut}(\mathbb{C}^n,0)$. □

Theorem 4.5:

Suppose that $f,\tilde{f}: (\mathbb{C}^n,0) \to (\mathbb{C},0)$ are holomorphic function-germs. Let Θ be a finitely generated $\mathcal{O}_n$-module of vector fields on $(\mathbb{C}^n,0)$. If $j^k f - j^k\tilde{f} \in T_{\Theta^{\mathcal{R}^k}}(f) \cong T_{\Theta^{\mathcal{R}^k}}(\tilde{f})$, then $f \sim^k_{\Theta^{\mathcal{R}}} \tilde{f}$.

Proof.

Suppose that $F_t(x) = f(x) - t(f(x) - \tilde{f}(\mathrm{x}))$. Then we can see that $F_0 = f$ and $F_1 = \tilde{f}$.

For all $\mathrm{t} \in \mathbb{C}$, $T_{\Theta^{\mathcal{R}^k}}(F_t)$ is a finitely generated submodule of $\mathcal{O}_n$ with a system of generators $\boldsymbol{m_1}(t),\dots,\boldsymbol{m_r}(t)$ and $\boldsymbol{m_i}(t) = \sum_{j=1}^{r} a_{ij}(t)\,\boldsymbol{m_j}(0)$. Let $\boldsymbol{A}(t) = \left[a_{ij}(t)\right]$, then up to a finite number of values that are the zeros of $\mathbf{det}(\boldsymbol{A}(t))$, $\boldsymbol{A}(t)$ is an invertible matrix and for every point $\mathrm{t} \in \mathrm{U} = \mathbb{C} - \{t_1,\dots,t_s\}$ we have $T_{\Theta^{\mathcal{R}^k}}(F_t) \cong T_{\Theta^{\mathcal{R}^k}}(\tilde{f})$.

Now we need to use Mather's lemma.

(i) we can see that U is open and connected in $\mathbb{C}$. Hence, $\Omega_{\mathrm{U}} = \{j^k F_t \in \mathrm{J}^k(n,1) | \mathrm{t} \in \mathrm{U}\}$ is open and connected submanifold in $\mathrm{J}^k(n,1)$ and then $\mathbf{dim}\, T_{\Theta^{\mathcal{R}^k}}(F_t)$ is independent of the choice of $j^k F_t \in \Omega_{\mathrm{U}}$.

(ii) we can see that $j^k f - j^k\tilde{f} \in T_{\Theta^{\mathcal{R}^k}}(f) \cong T_{\Theta^{\mathcal{R}^k}}(F_t)$ for all $j^k F_t \in \Omega_{\mathrm{U}}$. Therefore, we have that $T_{\mathrm{t}}\Omega_{\mathrm{U}} \subseteq T_{\Theta^{\mathcal{R}^k}}(F_t)$ for all $j^k F_t \in \Omega_{\mathrm{U}}$.

The hypotheses of Mather's lemma are satisfied and then Ω_{U} is contained in a single $\Theta^{\mathcal{R}^k}$-orbit. Hence $f \sim^k_{\Theta^{\mathcal{R}}} \tilde{f}$. □

Theorem 4.6:

Suppose that $f,\tilde{f}: (\mathbb{C}^n,0) \to (\mathbb{C},0)$ are holomorphic function-germs with $f - \tilde{f} \in T_{\Theta^{\mathcal{R}}}(f)$. Let Θ be a finitely generated $\mathcal{O}_n$-module of vector fields on $(\mathbb{C}^n,0)$. Then f and $\tilde{f}$ are $\Theta^{\mathcal{R}}$-equivalent if and only if $T_{\Theta^{\mathcal{R}}}(f) \cong T_{\Theta^{\mathcal{R}}}(\tilde{f})$.

Proof.

Suppose that f and $\tilde{f}$ are $\Theta^{\mathcal{R}}$-equivalent. Then there exists vector field $\xi \in \Theta$ that can be integrated to give a map-germ $\varphi \in \mathbf{Aut}(\mathbb{C}^n,0)$ such that $\tilde{f} = f \circ \varphi$.

By Chain Rule we have that

$$\frac{\partial(f \circ \varphi)}{\partial x_i} = \sum_{j=1}^{n}\frac{\partial \varphi_i}{\partial x_i}\left(\frac{\partial f}{\partial x_j} \circ \varphi\right)$$

$$= \left(\frac{\partial f}{\partial x_1} \circ \varphi,\dots,\frac{\partial f}{\partial x_n} \circ \varphi\right).\mathrm{D}\varphi,$$

where $\mathrm{D}\varphi$ is the Jacobian matrix of φ, which is invertible since $\varphi \in \mathbf{Aut}(\mathbb{C}^n,0)$. It follows that $T_{\Theta^{\mathcal{R}}}(f \circ \varphi) = \varphi^*\left(T_{\Theta^{\mathcal{R}}}(f)\right)$, i.e., $T_{\Theta^{\mathcal{R}}}(\tilde{f}) = \varphi^*\left(T_{\Theta^{\mathcal{R}}}(f)\right)$

Conversely, suppose that $T_{\Theta^{\mathcal{R}}}(f)$ and $T_{\Theta^{\mathcal{R}}}(\tilde{f})$ are $\Theta^{\mathcal{R}}$-equivalent. Then there exists $\Phi \in \mathbf{Aut}(\mathbb{C}^n,0)$ such that $\Phi^*(T_{\Theta^{\mathcal{R}}}(f)) = T_{\Theta^{\mathcal{R}}}(\tilde{f})$.

By replacing f by $f \circ \Phi$ we may assume that $T_{\Theta^{\mathcal{R}}}(f) = T_{\Theta^{\mathcal{R}}}(\tilde{f})$ holds. For each non-negative integer k, we have $T_{\Theta^{\mathcal{R}^k}}(f) = T_{\Theta^{\mathcal{R}^k}}(\tilde{f})$ and from Theorem4.5 we get $f \sim^k_{\Theta^{\mathcal{R}}} \tilde{f}$. Then from Theorem4.4 we have that $f \sim_{\Theta^{\mathcal{R}}} \tilde{f}$. □

5. RELATIVE FINITE DETERMINACY

Definition 5.1:[6]

Let $f: (\mathbb{C}^n,0) \to (\mathbb{C},0)$ be a holomorphic function-germ and let Θ be a module of vector fields on $(\mathbb{C}^n,0)$. We say that f is k-$\Theta^{\mathcal{R}}$-determined if f is $\Theta^{\mathcal{R}}$-equivalent to any map-germ $g: (\mathbb{C}^n,0) \to (\mathbb{C},0)$ such that $j^k f = j^k g$. If f is k-$\Theta^{\mathcal{R}}$-determined for some k, then f is said to be finitely $\Theta^{\mathcal{R}}$-determined.

Theorem 5.2:

Let p and k be non-negative integers with $k \le p$. Let $f: (\mathbb{C}^n,0) \to (\mathbb{C},0)$ be a holomorphic function-germ and let Θ be a module of vector fields on $(\mathbb{C}^n,0)$. If f is k-$\Theta^{\mathcal{R}}$-determined, then

$$\mathfrak{M}_n^{k+1} \subseteq T_{\Theta^{\mathcal{R}}}(f) + \mathfrak{M}_n^{p+1}.$$

Proof.

Let $\Omega = \left\{z \in \mathrm{J}^p(n,1) \middle| \pi_{p,k}(z) = \pi_{p,k}(j^p f)\right\}$ where $\pi_{p,k}: \mathrm{J}^p(n,1) \to \mathrm{J}^k(n,1)$ the natural linear projection. We have that Ω is an affine subspace of $J^p(n,1)$. It follows, $T_{j^p f}\Omega = \pi_p(\mathfrak{M}_n^{k+1})$.

By hypothesis f is k-$\Theta^{\mathcal{R}}$-determined it follows Ω a subset of the $\Theta^{\mathcal{R}^p}$-orbit of $j^p f$. Therefore, $T_{j^p f}\Omega \subset T_{\Theta^{\mathcal{R}^p}}(f)$ and this implies that

$$\mathfrak{M}_n^{k+1} \subset \pi_p{}^{-1}(T_{\Theta^{\mathcal{R}^p}}(f)) = T_{\Theta^{\mathcal{R}}}(f) + \mathfrak{M}_n^{p+1}.$$

□

Corollary 5.3:

Let k be a non-negative integer. Let $f:(\mathbb{C}^n,0) \to (\mathbb{C},0)$ be a holomorphic function-germ and let Θ be a module of vector fields on $(\mathbb{C}^n,0)$. If f is k-$\Theta^{\mathcal{R}}$-determined, then

$$\mathfrak{M}_n^{k+1} \subseteq T_{\Theta^{\mathcal{R}}}(f).$$

Proof.

Since f is k-$\Theta^{\mathcal{R}}$-determined. Then $\pi_{k+1}(f)$ is k-$\Theta^{\mathcal{R}^{k+1}}$-determined. By using Theorem 5.2 with $p = k+1$, we obtain

$$\mathfrak{M}_n^{k+1} \subseteq T_{\Theta^{\mathcal{R}}}(f) + \mathfrak{M}_n^{k+2}.$$

By Nakayama's lemma, it follows that

$$\mathfrak{M}_n^{k+1} \subseteq T_{\Theta^{\mathcal{R}}}(f).$$

□

6. RELATIVE STABILITY

Definition 6.1:

Let $f:(\mathbb{C}^n,0) \to (\mathbb{C},0)$ be a holomorphic function-germ and let Θ be a module of vector fields on $(\mathbb{C}^n,0)$. Let U be a neighbourhood of 0 in $\mathbb{C}^n$ with $\mathcal{O}_U(f) = \{g \in \mathcal{O}_n | f_{|U} = g_{|U}\}$. We say that f is $\Theta^{\mathcal{R}}$-stable if f is $\Theta^{\mathcal{R}}$-equivalent to any function-germ $g \in \mathcal{O}_U(f)$. In other words, if the $\Theta^{\mathcal{R}}$-orbit of f contains $\mathcal{O}_U(f)$.

Theorem 6.2:

Let $f:(\mathbb{C}^n,0) \to (\mathbb{C},0)$ be a holomorphic function-germ and let $\Theta=\{\xi_j\}_{j=1}^{r}$ be a finitely generated $\mathcal{O}_n$-module of vector fields on $(\mathbb{C}^n,0)$. f is $\Theta^{\mathcal{R}}$-stable if and only if $\Theta^{\mathcal{R}_e} - cod(f) = 0$.

Proof.

See the proof of Theorem 1.3 in [2]. It is only necessary to replace the tangent space $E(f)$ by our tangent space $T_{\Theta^{\mathcal{R}_e}}(f)$.
□

Definition 6.3.[6]

Let $\Theta_i = \{\xi_j^i\}_{j=1}^{r_i}$ be a set of vector fields on $(\mathbb{C}^{n_i},0)$, $i = 1,2$. Then the product of $\Theta_1$and Θ_2, denoted$\Theta_1 \times \Theta_2$, is the set of vector fields on $(\mathbb{C}^{n_1} \times \mathbb{C}^{n_2}, 0 \times 0)$ define by
$\Theta_1 \times \Theta_2 = \{\xi_1^1, \dots, \xi_{r_1}^1, \xi_1^2, \dots, \xi_{r_2}^2\}$.

Definition 6.4.[6]

Let $f:(\mathbb{C}^{n_1},0) \to (\mathbb{C},0)$ and $\tilde{f}:(\mathbb{C}^{n_2},0) \to (\mathbb{C},0)$ be holomorphic function-germs. We define the direct sum $f \oplus \tilde{f}:(\mathbb{C}^{n_1} \times \mathbb{C}^{n_2}, 0 \times 0) \to (\mathbb{C},0)$ by $(f \oplus \tilde{f})(x,y) = f(x) + \tilde{f}(y)$.

Theorem 6.5.

Let $\Theta_i = \{\xi_j^i\}_{j=1}^{r_i}$be a finitely generated $\mathcal{O}_n$-module of vector fields on $(\mathbb{C}^{n_i},0)$, $i = 1,2$. Let $f:(\mathbb{C}^{n_1},0) \to (\mathbb{C},0)$ and $\tilde{f}:(\mathbb{C}^{n_2},0) \to (\mathbb{C},0)$ be holomorphic function-germs. Then $f \oplus \tilde{f}$ is $\Theta_1 \times \Theta_2{}^{\mathcal{R}}$-stable if and only if f is $\Theta_1{}^{\mathcal{R}}$-stable or $\tilde{f}$ is $\Theta_2{}^{\mathcal{R}}$-stable.

Proof.

We have

$$\Theta_1 \times \Theta_2{}^{\mathcal{R}_e} - cod(f \oplus \tilde{f}) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{n_1+n_2}}{T_{\Theta_1 \times \Theta_2{}^{\mathcal{R}_e}}(f \oplus \tilde{f})}$$

$$= \dim_{\mathbb{C}} \frac{\mathcal{O}_{n_1+n_2}}{<\xi(f\oplus\tilde{f})|\xi\in(\Theta_1\times\Theta_2)>}$$

$$= \dim_{\mathbb{C}} \frac{\mathcal{O}_{n_1+n_2}}{<\xi_1^1(f),\dots,\xi_{r_1}^1(f),\xi_1^2(\tilde{f}),\dots,\xi_{r_2}^2(\tilde{f})>}$$

From [10], page 181, we can see that

$$\frac{\mathcal{O}_{n_1+n_2}}{< \xi_1^1(f), \dots, \xi_{r_1}^1(f), \xi_1^2(\tilde{f}), \dots, \xi_{r_2}^2(\tilde{f}) >} \cong \frac{\mathcal{O}_{n_1}}{< \xi_1^1(f), \dots, \xi_{r_1}^1(f) >} \otimes \frac{\mathcal{O}_{n_2}}{\langle \xi_1^2(\tilde{f}), \dots, \xi_{r_2}^2(\tilde{f}) \rangle}$$

It follows, we have that

$$\Theta_1 \times \Theta_2{}^{\mathcal{R}_e} - cod(f \oplus \tilde{f}) = \dim_{\mathbb{C}} \left(\frac{\mathcal{O}_{n_1}}{< \xi_1^1(f), \dots, \xi_{r_1}^1(f) >} \otimes \frac{\mathcal{O}_{n_2}}{\langle \xi_1^2(\tilde{f}), \dots, \xi_{r_2}^2(\tilde{f}) \rangle} \right)$$

$$= \dim_{\mathbb{C}} \left(\frac{\mathcal{O}_{n_1}}{<\xi_1^1(f),\dots,\xi_{r_1}^1(f)>} \right) . \dim_{\mathbb{C}} \left(\frac{\mathcal{O}_{n_2}}{\langle \xi_1^2(\tilde{f}),\dots,\xi_{r_2}^2(\tilde{f}) \rangle} \right)$$

$$= \Theta^{\mathcal{R}_e} - cod(f). \Theta^{\mathcal{R}_e} - cod(\tilde{f}).$$

□

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