HAMILTONIAN AND EULERIAN CAYLEY GRAPHS OF CERTAIN GROUPS .

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.ABSTRACT *: In this paper we will investigate and study Cayley graphs of certain groups and give some structural properties of Hamiltonian Cayley graphs for symmetric groups Sn.Eulerian for Cayley graphs will be constructed using a certain set of generators.*

KEYWORDS: graph, groups, Hamiltonian, Eulerian graph.

1. INTRODUCTION

The technique of representing groups by graphs invented in nineteenth century by a mathematician named Cayley , where vertices correspond to the elements of a groups and the edges correspond to group generations.

The following definitions which will be used in the squad due to (Harary.F [6]) , (Gallran.J [4]) and (Fraleigh.J [3]) .

Definition 1.1 : A graph Γ = (V ,E) consists of a finite nonempty set V=V (Γ) of n points together with a prescribed set E of q unordered pairs of distinct points of V .

We call V(Γ) the **vertex-set** of ｒ, and E(Γ) the **edge-set** of Γ, often denoted by V and E respectively , the graph Γ will be called an (n,q)-graph where n is the number of vertices and q is the number of edges in Γ .

Each pair x={u,v} of vertices in E (Γ) is an edge of Γ , and x joins u and v. Sometimes, we write x=uv , and say that u and v are adjacent vertices (by u adj .v) , u and x are incident with each other , as are v and x .

Definition 1.2: The order of Γ is the number of vertices of Γ and denoted by | V (Γ) |.

Definition 1.3: A path is an open trail with distinct vertices and edges.

Definition 1.4: A cycle is a closed path.

Definition 1.5: A graph is **K-regular** if every vertex is connected to k other vertices through k-edges

Definition 1.6: Girth of Γ is the length of a shortest cycle (if any) in Γ, denoted by g (Γ).

Definition 1.7: A subgraph of Γ is a graph having all of its vertices and edges in Γ. In other word a graph R is called a subgraph of a graph Γ if V (R) ≤ V (Γ) and E (R) ≤ E (Γ).

Definition 1.8: A simple graph is undirected graph that has no loops and no more than one edge between any two different vertices.

Definition 1.9: two graphs Γ_1 and Γ_2 are said to be **isomorphic** (denoted by $\Gamma_1 \cong \Gamma_2$) if there exists a 1-1 correspondence between their vertex sets , which preserves adjacency .

Definition 1.10: A graph is **connected** if every pair of vertices there is at least the path joining them . A graph is that is not connected is called **disconnected.**

Definition 1.11: A graph is **complete graph K_n** if every pair of its vertices adjacent. Thus K_n is regular of degree n-1.

Definition 1.12 : the **valency** of vertex denoted by val(v_i)= number of edges incident to v_i (sometimes we called it degree of vertex v_i and denoted by d_i or deg v_i) .

Definition 1.13: A connected graph Γ is called **Hamiltonian** if there is a cycle which includes every vertex of Γ , such cycle is called a **Hamiltonian path .**

Definition 1.14: A connected graph Γ is called **Eluerian** if there is a closed trail which includes every edges of Γ , such trial is called an **Eluerian trial.**

Definition 1.15: The distance d_Γ (u,v) between two vertices u and v in Γ is the length of shortest path joining them if any ; otherwise d_Γ (u,v)=∞ , if u=v then d_Γ (u,v) = 0 (in digraph the distance between two vertices u and v is length of any shortest such path)

Definition 1.16: A trail is a walk with distinct edges and distinct vertices.

Definition 1.17: A graph Γ is **n-transitive,** n≥1 if it has an n-rout and if there is always an automorphism of Γ sending each n-rout onto any other n-rout.

Definition 1.18: A group is an order pair (G,*) where G is a non-empty set and * is a binary associative operation on G which contains an identity (the natural element e) and inverse for each element.

Definition 1.19: If a subset H of a group G is itself a group under the operation of G, we say H is **a subgroup** of G.

Definition 1.20: Let G be a group and let g_i ϵ G for i ϵ I if this {g_i : i ϵ I } subgroup is all of G then {g_i : i ϵ I} **generates** G and the g_i are **generates** of G .

Definition 1.21: Let G be a group . A subset **Ω** of G is a **generating set** for group G if every element of G can be expressed as a product of elements of set **Ω.**

Definition 1.22: Let A be the finite set {1,2,…,n}. The group of all permutations of A is the **symmetric group** on n letters and is denoted by S_n.

Definition 1.23: A permutation of a set A is one to one function from A onto A.

Definition 1.24: An element of a group **an involution** if it has order 2 (i.e an involution is an element a such that a ≠ e and a^2 = e where e is the identity element).

2. Cayley Graphs of Groups

Cayley graph are of general interest in the field of Algebraic Graph theory and also have certain properties desirable in practical applications.

We present here a brief survey of some of the broader results and conjectures surrounding Cayley graphs.

Theorem 2.1 (kutnar.K. [7]) :

Let G be a finite group. Every connected Cayley graph on G has a Hamiltonian cycle if |G| has any of the following forms (where p,q, and r are distinct primes):

1. kp, where 1 ≤ k < 32 , with k ≠ 24,
2. kpq , where 1 ≤ k ≤ 5,
3. pqr,
4. kp2, where 1 ≤ k ≤ 4,
5. kp^3 , where 1 ≤ k ≤ 2 .

It is clear that every Cayley graph over a group of prime order is Hamiltonian, since all such groups are cyclic.

Theorem 2.2 (Puskey.F. [9]) :

The Cayley graph Cay ($\mathbf{\Omega}:\mathbf{S_n}$) ,where $\mathbf{\Omega}$ is the three element generating set given below , is Hamiltonian.

$\mathbf{\Omega}$ = {(1,2) , (1,2) (3,4) (5,6)…,(2,3)(4,5)(6,7) … }

Theorem 2.3 (pak.I.[8]):

Every finite group G of size $|G| \geq 3$ has a generating set $\mathbf{\Omega}$ of size

$|\mathbf{\Omega}| \leq \log_2 |G|$, such that the corresponding Cayley graph Cay ($\mathbf{\Omega}$: G) , contains a Hamiltonian cycle .

Theorem 2.4 (Alspach.B.[1]) :

Every connected cubic Cayley graph on a dihedral group has a Hamilton cycle.

Theorem 2.5 (Tanakaa.Y.[12]) :

Let S and Ś be two sets of transpositions on {1, 2,…, n} . The Cayley graphs Cay (S: Sn) and Cay (Ś : Sn) are isomorphic if and only if T(S) and T(Ś) are isomorphic .

Theorem 2.6: (Gallian.J.[4]) :

Let G be a group. If N_1 and N_2 are normal subgroups of a finite index in Γ , then $N_1 \cap N_2$ is also a normal subgroup of finite index of G .

Theorem 2.7 (Curran.S.[2]) :

For any minimal generating set $\mathbf{\Omega}$ of transpositions in S_n (n > 4) and for any fixed x ϵ $\mathbf{\Omega}$ there is Hamiltonian cycle in the graph Cay ($\mathbf{\Omega}$:S_n) in which every other edge is x .

3. Group Representation by Cayley Digraph.

We will introduce the Cayley dighrap of group. Provides a method of visualizing the group and its properties. Properties such as commutativity and the multiplication table of group can be recovered from Cayley digraph. Also we introduce some examples for Cayley digraphs of groups, and we will introduce some important theorems related to Cayley graph of groups.

3.1 The Cayley Digraph of a Group:

Definition 4.1.1: A **directed graph** (or **digraph**) is a finite set of points called **vertices,** and a set of arrows called **arcs (edges)** , connecting some of the vertices .

Definition 4.1.2 Cayley digraph of a group:

Let G be finite group G and Ω a set of generators for G .

We define a digraph Cay (Ω: G), called the Cayley digraph of G with generating set as follows.

1- Each element of G is a vertex of Cay (Ω:G)
2- For v and u in G , there is an arc(edge) from u to v if and only if
   u = vx , for some x ϵ Ω

In Cayley digraph method we proposed that each generator by assigned a color , to know which particular generator connects two vertices , and that the arrow joining v to vx be colored with color assigned to x , we called the resulting figure the **color graph of the group** . Rather than use colors to distinguish the different generators, we will use solid arrows, dashed arrows, and dotted arrows.

In general, if there is an arc from v to u , there need not be an arc from u to v , note that there are several ways to draw the digraph of a group given by a particular generating set . However, it is not the appearance of the graph that is relevant but the manner in which the vertices are connected.

These connections are uniquely determined by the generating set . Thus,

Distances between vertices and angles by the arcs have no significance.

It is important to note that Cayley graph of the same group can vary depending on which set generates the group.

The following examples illustrate the representation of certain groups by Cayley digraphs.

Example 3.1.1

The Cayley digraph for the symmetric group S_3 with the generating set

Ω = {(1,2),(1,2,3)}

The incident function is constructed as follows

(e)(1,2,3) = (1,2,3)	(1,2,3) (1,2) = (1,3)
(1,3,2)(1,2,3) = (e)	(1,3,2) (1,2) = (2,3)
(1,2,3)(1,2,3) = (1,3,2)	(e) (1,2) = (1,2)
(1,3) (1,2,3) = (1,2)	(1,2) (1,2) = (e)
(1,2) (1,2,3) = (2,3)	(1,3) (1,2) = (1,2,3)
(2,3) (1,2,3) = (1,3)	(2,3) (1,2) = (1,3,2)

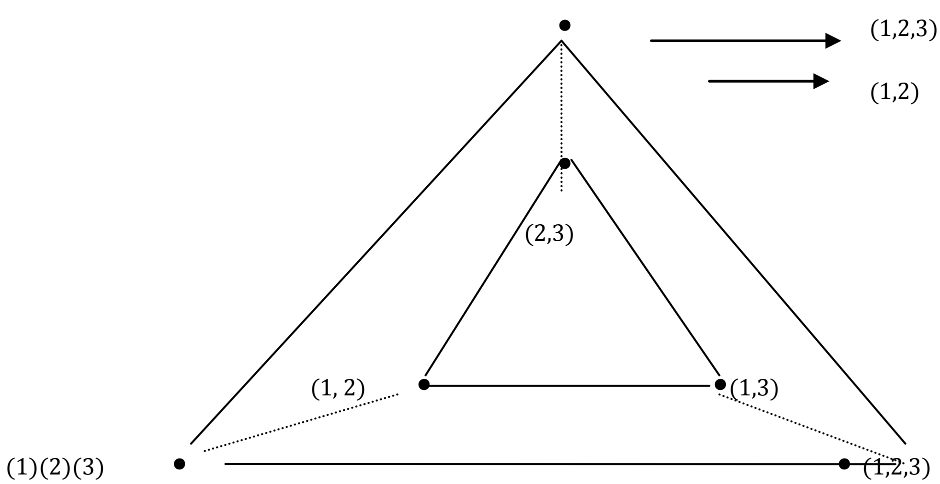


Figure 3.1.1

Cay ({(1,2),(1,2,3)} : S_3)

from this a Cayley digraph for the symmetric group S_3 with the generating set $\Omega = \{ (1,2) , (1,2,3) \}$. by looking at the identity element e , we can deduce that the solid arrow represents multiplication by (1,2,3) , because starting at the identity e , and following the solid arrow yields (1,2,3) . By the same logic, the dashed arrow represent (1,2) . The element (1,2,3) is of order 3 because starting at the identity and following the solid arrow once yields (1,2,3) , follow it again and you get to (1,3,2) , follow it once more and you get back to e, therefore applying the solid arrow three times is equiregular to the identity . By the principal, (1, 2) had order 2 . a quick way to see that an element in the generating set order 2 is to look and see if it has a double-headed arrow, i.e. an arrow on both sides of the arc .

The Cayley digraph illustrates several interesting facts about S_3.

The Cayley digraph shows us that S_3 is non-commutative group . this can be seen by starting at any element , say (1,3) and following the solid arrow and then the dashed arrow , which yields e , then start at (1,3) again and follow the dashed arrow and then the solid arrow , this results in (1,3,2). Since e is different that (1,3,2) then S_3 is non-commutative group .

The Multiplication table of the group can be recovered from the Cayley digraph. As previously stated , (1,2,3) corresponds to traveling the solid arrow , therefore let (1,2,3) = S , and by the same logic , let (1,2) = D .

Then using this SD notation the rest of the elements can be represented in the same manner. (1,3) = SD , (1,3,2) = SS , and (2,3) =DS . Using this notation the multiplication table can be recovered by starting at the identity and traveling the corresponding arrows.

For example (1,3,2)(1,3) = SSSD = (1,2) and (2,3)(1,3) = DSSD = (1,2,3).

Below is the multiplication table for S_3 . The multiplication table is dependent on the group , not on the generating set

Table (1)

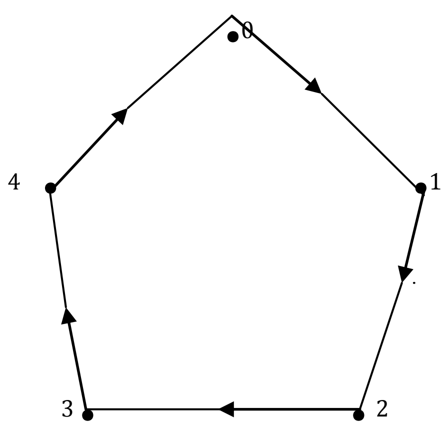
Multiplication table of S_3

	e	(1,2)=D	(1,2,3)=S	(1,3)=SD	(1,3,2)=SS	(2,3)=DS
E	e	(1,2)	(1,2,3)	(1,3)	(1,3,2)	(2,3)
(1,2)=D	(1,2)	e	(2,3)	(1,3,2)	(1,3)	(1,2,3)
(1,2,3)=S	(1,2,3)	(1,3)	(1,3,2)	(2,3)	e	(1,2)
(1,3)=SD	(1,3)	(1,2,3)	(1,2)	e	(2,3)	(1,3,2)
(1,3,2)=SS	(1,3,2)	(2,3)	e	(1,2)	(1,2,3)	(1,3)
(2,3)=DS	(2,3)	(1,3,2)	(1,3)	(1,2,3)	(1,2)	e

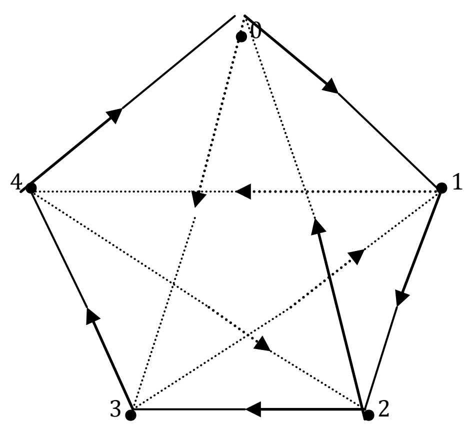
Example 3.1.2

The Cayley digraph for the cyclic group Z_5 with generating set

$\Omega = \{1\}$ ($Z_5=\langle 1\rangle$)



Cay ({1}: Z_5)



Cay({1,3}:Z_5)

Figure 3.1.2

Cay ({1}:Z_5) and Cay ({1,3}:Z_5)

On the left figure the Cayley digraph of Z_5 with generating set {1} .

In the Cayley digraph each of the elements of the group Z_5 are the vertices of the digraph . The solid arrow represents addition by 1 , which is the only element in the generating set .The Cayley digraph illustrates several things about Z_5 with generating set {1}. The first point of interest is that 1 is order 5 , because if you start at the identity 0, and add 1 five times which equivalent to following the solid arrow five times , you get back to the identity . Another property that should be noted is Z_5 is cyclic because there is only one kind of arrows , which implies that there exist a generating set with only one element . (we can see Z_5 in [18])

Note : In general the Cayley digraph for $Z_n = <1>$ is C_n .

3.2 The basic properties of Cayley Digraphs of Groups

In this section we will introduce some important theorems related to Cayley graphs of groups contain the basic constructive properties .

Definition 3.2.1 (Gross.J.[5])

An arbitrary graph Γ is said to be a Cayley graph if there exist a group G and a generating set Ω such that Γ is isomorphic to the Cayley graph for G and Ω.

Theorem 3.2.1

The complete graph K_{2n+1} is a Cayley graph for group $Z_{2n+1\,+}$ with generating set {1, 2,...,n} .

Proof

Let $\Gamma = K_{2n+1}$ is complete graph that's mean every vertex v_i in V (Γ)

Must adjacent with 2n vertices of Γ . Now when draw Cayley graph for group $Z_{2n=1}$ with generating set {1,2,...,n} . WLOS first ny generating 1 we draw the cycle C_{2n+1} from v_0 and traverse all the vertices in closed path to $v_{2n+1} = v_0$ (since in Z_{2n+1} , 0 equal 2n+1)

(i.e v_0 adj. v_1 and v_0 adj. v_{2n})

Now by generating 2 we draw the all edges in the following form

v_0 adj. v_2 v_1 adj. v_3
v_2 adj. v_4 v_3 adj. v_6
. .
. .
. .
v_{2n} adj. v_1 v_{2n-1} adj. v_0

Following the same argument, every vertex of Γ adjacent with all vertices of Γ. Thus, the constructed graph is a complete graph K_{2n+1} .

Note: the complete K_{2n} is a Cayley graph for group Z_{2n} with generating set {1,2,...,n} such that appear bidirected –arc with generator n .

Definition 3.2.2

A graph Γ is vertex-transitive if for all vertex pairs u,v ϵ V (Γ) , there is an automorphism of Γ that maps u to v .

(i.e. if $\Phi: \Gamma \to \Gamma$; automorphism if u adjacent to v , then Φ (u) adjacent to v)

Lemma 3.2.1

For any group G , the Cayley digraph is vertex-transitive .

(i.e every Cayley digraph is vertex-transitive)

Proof

Let a and b be any two elements from the group G .

We must show there is an adjacency-preserving automorphism Φ of G mapping a to b.Define $\Phi(x) = (ba^{-1})x$, for all x ϵ G , clearly Φ maps a to b since $\Phi(a) = (ba^{-1\cdot})a = b(a^{-1}a) = be = b$ (associativity)

The map Φ is injective since if Φ (x) = Φ (y) then $(ba^{-1})x = (ba^{-1})y$ and so $x = (ba^{-1})^{-1}(ba^{-1})x = (ba^{-1})^{-1}(ba^{-1})y = y$ (inverses)

Similarly Φ is surjective since for any x in G ,

$\Phi(ab^{-1}x) = (ba^{-1})(ab^{-1})x = x$, so Φ is a bijection .

Finally , Φ maps vertices adjacent to a to vertices adjacent to b .

$\Phi(ag_i) = (ba^{-1})(ag_i) = bg_i$ for all $g_i \in \Omega$.

While every Cayley graph is vertex-transitive, but a vertex-transitive graph Γ is Cayley if there exists an automorphism group G of Γ that acts regularly on V (Γ) , i.e for each u , v ϵ V (Γ) , there exists exactly one automorphism

Φ (u) = v .

Example 3.2.2

The Petersen graph is smallest vertex-transitive but is not a Cayley graph, since its automorphism group has no transitive subgroup of order 10

Theorem 3.2.2

Every Cayley digraph is strongly-connected.

Proof

In view of lemma 3.2.1, it is sufficient to show the existence of a path from the identity, e, to any other group element g. This is easy since there is , by definition of a generating set , a sequence of generators $g_1, g_2, \ldots, g_k$ whose product is g .

These lemmas imply that every (undirected) Cayley graph is vertex-transitive and connected.

Notes:

The Cayley graph Cay (Ω: G) is:

1. loopless if and only if $e \notin \Omega$.
2. undirected if and only if $\Omega = \Omega^{-1}$.
3. connected of and only if Ω generates G.
4. simple if and only if undirected.
5. if |Ω| = k , then Cay (Ω:G) is k-regular.

4. Hamiltonian and Eulerian Cayley Graphs

We will introduce Hamiltonian cycles and paths in Cayley digraph and graph of a group and we will introduce if Cayley digraph and graph of a group is Eulerian graph .

4.1 Hamiltonian cycles and paths in Cayley digraph and graph :

Finding Hamiltonian cycles and paths in graph is difficult problem.

The classical question raised by lováse asks whether every Cayley graph is Hamiltonian .

In this section, we introduce some results on Cayley graph of symmetric groups S_n.

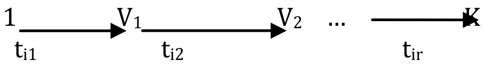
Theorem 4.1.1 (Ruskey.F.[9])

A set $\Omega = \{t_1, t_2, \ldots, t_k\}$ of transpositions generates S_n if and only if undirected graph associated with Ω (denoted by G (Ω) with n vertices where each edge denotes one of transpositions) is connected .

Proof

Consider the elements 1 and k . we can construct a permutation σ such that σ (1) = k . Let $\sigma = t_{ir} \ldots t_{i3}\, t_{i2}\, t_{i1}$. WLOS, assume 1 appears for the first time in t_{i1} and k appears for the first time in t_{ir} .

Then , there is path from 1 to k as follows :



Conversely, suppose that G (Ω) is connected. There is no loss of generality in assuming that G (Ω) is a tree . The proof is by induction on n . The result is true for n =2 .

Inductively , assume the result is true for all m < n . since G (Ω) is a tree , it has a pendant vertex , and there is no loss of generality in labelling this vertex n and the vertex connected to it n -1 .

Then (n -1 , n) ϵ { t_1, t_2,…,t_k} , and we may denote it as t_k . Remove the vertex n and the edge from n -1 to n ; the resulting graph is still connected, since n was a pendant vertex. By the inductive hypothesis

$<t_1,t_2,\dots,t_{k-1}> = S_{n-1}$. Then $\Gamma = <S_{n-1}, t_k>$. Since (1, 2,… , n-1) ϵ S_{n-1} ,

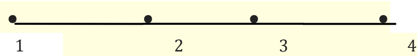
(1,2,…,n-1)(n-1 ,n)=(1,2,…,n-1,n) ϵ Γ.

Also, (1,2) ϵ S_{n-1} . Thus $S_n = <(1, 2),(1,2,\dots,n)> \subseteq \Gamma$, and we have equality . The result follows by induction.

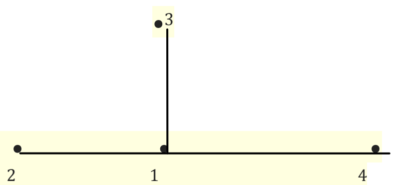
Example 4.1.1

G (Ω) with 4 vertices (associated with S_4) there two forms either

1- Line graph (p4)



2 Star graph ($k_{1,3}$)



That's mean S_4 generated by the set of transpositions

1- Ω = {(1,2),(2,3),(3,4)}
2- Ω ={(1,2)(1,3),(1,4)} (this graph we can see in [9])

Theorem 4.1.2

The graph Cay (Ω:S_n) is Hamiltonian whenever Ω is a generating set for S_n consisting of transpositions , then the resulting Cayley graph contains a Hamiltonian cycle .

Theorem 4.1.3

If n is even (for n ≥ 4), the Cayley graph of S_n generated by {(1,2),(1,2,…,n)} does not contain a Hamiltonian cycle .

Proof

Let a = (1,2) and b = 1,2,…,n)

Then $c=ab^{-1}$=(1,2)(1,n,n-1,n-2,…,3,2)=(1,n,n-1,…,3) is of order

n-1 , which is odd .

But $[G:<a>]=\frac{n!}{2}$, which contradicts Rankin and swans theorem (4.1.3) .

Thus the resulting Cayley graph does not contain a Hamiltonian cycle.

The existence of Hamiltonian cycles in Cayley graph of S_3 and S_5 generated by {(1,2),(1,2,3)} and {(1,2),(1,2,3,4,5)} respectively , suggests the possibility of Hamiltonian cycles existing in S_n Cayley graphs when n is odd.

4.2 Eulerian Graphs and Cayley Graphs

The problem of finding Eulerian trail (closed) is perhaps the oldest problem in graph theory ; it was originated by Euler in the 18th century .

We can imagine that Cayley graph of group is Eulerian graph relying on the basis of the following theory.

Theorem 4.2.1 (Harary.F.[6])

Graph Γ (V.E) is Eulerian if and only if Γ is connected and the degree of every vertex in Γ is even.

Theorem 4.2.2

Let Γ be a Cayley graph of the group G with set generator Ω, and let |Ω| is even. Then Γ is Eulerian graph.

Proof:

Let Ω be a set of generators of a group G with 2n generators, where n is an integer, so the constructed Cayley graph in

2n-regular. Thus valancy (v_i) is even for all v_i in Γ.

By previous theorem,then the graph constructed is Eulerian graph.

We can back to figure 3.2.3 and note Cay ({(1,2),(1,2,3)}:S_s) is Eulerian graph since |Ω|=2 is even and note Eulerian closed trail as the following {(1,3,2),e,(1,2),e,(1,2,3),(1,3)(1,2,3),(1,3,2),(2,3),(1,3),(1,2),(2,3),(1,3,2)}.

Lemma 4.2.1 (Ruskey.f.[9]).

A connected directed multigraph is Eulerian if and only if out-degree of each vertex is the same as its in-degree.

Note : Let Γ be an undirected Cayley graph generated by a transposition generating tree on n vertices . If Π is an even permutation, then the outdegree and indegree of Π are ⌈(n-1)/2⌉ and ⌊(n-1)/2⌋ ,respectively . If is Π an odd permutation, then the indegree and outdegree of Π are ⌈(n-1)/2⌉ and ⌊(n-1)/2⌋ ,respectively .

(i.e the indegree is greater than or equal to the outdegree for an odd vertex).

Moreover, Cayley digraph is regular if n is odd.

We can say, not every Cayley graph is Eulerian since we have example explain Cayley graph is not Eulerian:

Cay ({(1,2),(1,3),(2,3)}:S_3).

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