ELZAKI TRANSFORM APPLICATIONS FOR SOLUTION OF PROBLEMS ARISING IN PHYSICS AND FINANCE

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***ABSTRACT**. In this research Elzaki transform has been applied to various mathematical problems arising in Physics and Finance. Equivalent mathematical forms of Maxwell model have been derived. Moreover, Linear Volterra type Viscoelastic Hereditary model, load distribution over string model and Steady state heat transfer model solutions have been found using Elzaki transform. In addition, solution of one finance model pertaining collective risk has also been discussed. All results are in agreement with known results. It has been established that Elazki transform is reliable and accurate method which can be effectively employed for various practical problems.*

Key words: Elzaki transform, Maxwell model, Collective risk model, Steady state heat transfer, Volterra integral equation, Viscoelastic model

1 INTRODUCTION

Elzaki transform has been introduced by [7] and its properties as well as applications have been discussed by various researchers [1, 2, 6, 9, 18, 19] . However, there is lot of potential available with respect to finding new applications of Elzaki transform. The emphasis of this research is in finding solution of problems originating in Physics and Finance.

More specifically, linear Volterra type model describing load distribution over string, viscoelastic hereditary model and steady state heat transfer model solutions have been found using Elzaki transform. Moreover, equivalent mathematical forms of Maxwell model have been determined which have been earlier derived using famous Laplace transform. Solution of one finance model arising in collective risk theory has also been discussed. It is considered prudent here to define an integral equation and explicitly Volterra type integral equation as same will be frequently discussed in present research. Integral equation contains unknown function say u(x), which is required to be determined appears under the integral sign. K(x, t) is called 'kernel' function whereas g(x) and h(x) are the limits of integration. Wazwaz [21, 22] has provided excellent introduction on integral equations along with solution methods. General form of an integral equation can be given as under:

$$u(x) = f(x) + \lambda \int_{h(x)}^{g(x)} k(x,t)u(t)\,dt \qquad (1.1)$$

An integro-differential equation is bit different from integral equation as it contains unknown function $u(x)$ which appears under integral sign and also has ordinary derivative of unknown function. For the integro-differential equation, general form can be given as:

$$u^n(x) = f(x) + \lambda \int_{h(x)}^{g(x)} k(x,t)u(t)\,dt \qquad (1.2)$$

The system of integral or integro-differential equations has two or more equations with two or more variables which are required to be determined. Mainly integral equations are classified as Fredholm type and Volterra type based on whether limits of integration are fixed or variable. Specifically, Volterra type integral equation contains at least one variable limit of integration. A brief literature review has been presented in section 2. Relevant details concerning Elzaki transform have been covered in section 3 while practical applications of Elzaki transform have been discussed in section 4.

2 LITERATURE REVIEW

Integro-differential equations with bulge function have been discussed by [1]. Adam [2] conducted comparative study between Adomian Decomposition Method (ADM) and Elzaki transform. Both methods have been used to solve linear partial differential equations with constant coefficients. Elzaki transform method has been used by [8] for solution of systems of linear Integro-differential equations with constant coefficients. Fundamental properties of Elzaki transform have been discussed by [6] and Elzaki transform for comprehensive list of functions has been provided. Furthermore, more general shift theorems have been introduced. Laplace-Elzaki Duality (LED) invoked a complex inverse Elzaki transform, as a Bromwich contour integral formula. Jung and Kim [11] inspected practical formulae for differentiation of integral transforms used for differential equations with variable coefficients. The transforms which have been checked are Laplace, Sumudu and Elzaki. Moreover, it has been argued that proposed formulae can be applied to almost every equation. Kim [13] suggested shifting theorems for the Elzaki transform to solve initial value problems arising in control engineering. Odibat [15] solved Volterra integral equations with separable kernels using the Differential Transform Method (DTM). Approximate solution has been evaluated in form of series. Exact solutions of linear as well as nonlinear integral equations have been presented. Song and Kim [20] solved Volterra integral equations of second kind (convolution type) by using the Elzaki transform. Steady state heat transfer problem has been solved by [24] utilizing Elzaki transform. Two tanks mixing problem has been solved by [16] using Elzaki transform.

3 METHODOLOGY

Definition and properties relating Elzaki transform presented in this section have been discussed by [6, 7, 13]. By definition, Elzaki transform is for given set A as:

$$A = \{f(t) \mid \exists M, k_1, k_2 > 0 \mid f(t) < Me^{\frac{-t}{k_j}}, \text{if } t \in (-1)^j \times [0,\infty)\} \qquad (3.1)$$

For the given function, M should be finite, however, k_1 and

k_2 may be finite or infinite.

$$E[f(t)] = T(p) = p\int_0^\infty f(t)e^{-t/p}\,dt\,, t \geq 0 \quad (3.2)$$

Here $T(p)$ is Elzaki transform of integral function $f(t)$

Let $f(t) = 1$

$$E[1] = T(p) = p\int_0^\infty 1.e^{-t/p}\,dt = p^2 \quad (3.3)$$

Let $f(t) = t$

$$E[t] = T(p) = p\int_0^\infty t.e^{-\frac{t}{p}}\,dt = p^3 \quad (3.4)$$

Generalizing

$$E[t^n] = n!\,p^{n+2} \quad (3.5)$$

3.1 Elzaki Transform of common functions

Elzaki transform of some common functions is given as under

$$E(e^{\mathrm{at}}) = T(p) = p\int_0^\infty e^{\mathrm{at}}.e^{-\frac{t}{p}}\,dt = \frac{p^2}{1-\mathrm{ap}} \quad (3.6)$$

$$E(\sin(\mathrm{at})) = \frac{\mathrm{ap}^3}{a^2p^2+1} \quad (3.7)$$

$$E(\cos(\mathrm{at})) = \frac{p^2}{a^2p^2+1} \quad (3.8)$$

$$E(\sinh(\mathrm{at})) = \frac{\mathrm{ap}^3}{1-a^2p^2} \quad (3.9)$$

Elzaki transform of derivative of function can be given as

$$E(f'(t)) = \frac{T(p)}{p} - \mathrm{v}f(0) \quad (3.10)$$

$$E(f''(t)) = \frac{T(p)}{p^2} - f(0) - \mathrm{v}f'(0) \quad (3.11)$$

Generalizing

$$E(f^n(t)) = \frac{T(p)}{p^n} - \sum_{k=0}^{n-1} p^{2-n+k} f^k(0) \quad (3.12)$$

3.2 Laplace Elzaki Duality (LED)

Inter-conversion between Laplace transform and Elzaki transform is given by LED. Let

$$f(t) \in A = \left\{ \begin{array}{l} f(t) \mid \exists M, k_1, k_2 > 0, \text{such that} \quad \mid \\ f(t) < Me^{\frac{\mid t \mid}{k_j}}, \text{if } t \in (-1)^j \times [0,\infty) \end{array} \right. \quad (3.13)$$

Since Laplace transform can be written as

$$F(s) = L(f(t)) = \int_0^\infty e^{-\mathrm{st}} f(t)\,\mathrm{d}t \quad (3.14)$$

Therefore we obtain two important relations. Writing directly

$$T(v) = \mathrm{vF}\left(\frac{1}{v}\right) \quad (3.15)$$

Where $T(v)$ denotes Elzaki transform. And

$$F(s) = \mathrm{sT}\left(\frac{1}{s}\right) \quad (3.16)$$

4 APPLICATIONS

4.1 Maxwell model under normal loading condition

The Maxwell model can be represented by a spring and a dashpot in a series. Details regarding viscoelastic models have been discussed by [3, 10, 12, 25]. For this model the stresses applied at its opposite ends are equal. i.e.

$$\sigma_1 = \sigma_2 = \sigma \quad (4.1)$$

However, the stresses produced in the spring and the dashpot is not equal.

Net strain produced will be equal to

$$\epsilon = \epsilon_1 + \epsilon_2 \quad (4.2)$$

After necessary calculations we obtain the Maxwell model as

$$\sigma + \frac{\eta\dot{\sigma}}{E} = \eta\dot{\epsilon} \quad (4.3)$$

We can derive same relation for Maxwell model using Elzaki transform. It is to be noted that "." over the symbol represents the derivative while "-" represents the transformed variable obtained after application of Elzaki transform.

Since strain produced in spring can be written as

$$\epsilon_1 = \frac{1}{E}\sigma \quad (4.4)$$

Taking Elzaki transform on both sides of Eq. (4.4)

$$\bar{\epsilon}_1 = \frac{\bar{\sigma}}{E} \quad (4.5)$$

Similarly, for the dashpot we have

$$\dot{\epsilon}_2 = \frac{1}{\eta}\sigma \quad (4.6)$$

Taking Elzaki transform on both sides of Eq. (4.6)

$$E(\dot{\epsilon}_2) = \frac{\bar{\epsilon}_2[v]}{v} - \epsilon_2(0) \quad (4.7)$$

Here, it is to be noted that

$$\epsilon_2(0) = 0 \quad (4.8)$$

$$\frac{\bar{\epsilon}_2}{v} = \frac{\bar{\sigma}}{\eta} \quad (4.9)$$

Also, if we take Elzaki transform of Eq. (4.2), we get

$$\bar{\epsilon} = \bar{\epsilon}_1 + \bar{\epsilon}_2 \quad (4.10)$$

Substituting the values from Eq. (4.5) and Eq.(4.9) in Eq.(4.10)

$$\bar{\epsilon} = \frac{\bar{\sigma}}{E} + \bar{\sigma}\frac{v}{\eta} \quad (4.11)$$

By taking Inverse Elzaki transform on both sides of Eq. (4.11), we get,

$$\sigma + \frac{\eta\dot{\sigma}}{E} = \eta\dot{\epsilon} \quad (4.12)$$

Hence, form obtained at Eq.(4.12) for Maxwell model by application of Elzaki
transform is compatible with form at Eq. (4.3).

4.2 Maxwell model under sudden loading condition

Now we consider the Maxwell model for sudden load response which is given as

$$\epsilon(t) = \frac{\sigma_0}{E}H(t) + \frac{\sigma_0}{\eta}t \quad (4.13)$$

Let the load be

$$\sigma = \sigma_0 H(t) \quad (4.14)$$

Where

$H(t) =$ Unit Heaviside function

$$\dot{H} = \frac{dH}{dt} = \delta \quad (4.15)$$

Writing the relation

$$\sigma_0 H(t) + \frac{\eta}{E}\sigma_0\delta = \eta\dot{\epsilon} \quad (4.16)$$

Taking Elzaki transform on both sides of Eq. (4.16)

$$E[\sigma_0 H(t)] + E[\frac{\eta}{E}\sigma_0\delta] = E[\eta\dot{\epsilon}] \quad (4.17)$$

$$\bar{\epsilon} = \frac{\sigma_0}{\eta}v^3 + \frac{\sigma_0}{E}v^2 \quad (4.18)$$

Taking inverse Elzaki transform on both sides of Eq. (4.18)

$$E^{-1}[\bar{\epsilon}] = E^{-1}[\frac{\sigma_0}{\eta}v^3] + E^{-1}[\frac{\sigma_0}{E}v^2] \quad (4.19)$$

Hence, we obtain the same desired form for sudden load application as given at Eq. (4.13)

$\epsilon(t) = \frac{\sigma_0}{\eta} t + \frac{\sigma_0}{E} H(t)$ (4.20)

4.3 Load distribution over string model

Load distribution over a string modeled as Volterra integral equation has been discussed by [17]. Problem given below has also been solved by [15] using DTM. Consider the string for length 0 to a point x over the string

$u(x) = x + \int_0^x ((t-x)u(t))\, dt$ (4.21)

Take Elzaki transform on both sides:

$E[u(x)] = E[x] + E[\int_0^x ((t-x)u(t))\, dt]$(4.22)

Simplifying

$T[v] = v^3 + \frac{1}{v}[-v^3 T[v]]$ (4.23)

$T[v] = v^3 - v^2 T[v]$ (4.24)

$T[v] = \frac{v^3}{1+v^2}$ (4.25)

Taking inverse Elzaki transform on both sides of Eq. (4.25), We get

$E^{-1}[T[v]] = E^{-1}[\frac{v^3}{1+v^2}]$ (4.26)

$u(x) = \mathrm{Sin}(t)$ (4.27)

This is the required analytic solution.

4.4 Steady state heat transfer model

The problem has been discussed by [24]. We shall solve same problem but written in Volterra type integral equation form. Consider an object placed in oven. We assume case of convective heat transfer where heat of fluid is transferred to the object. Then we have,

$-\mathrm{hM}T(x) = \rho \mathrm{V} \mathrm{C}_p T'(x),\ \ T(0) = \gamma$ (4.28)

$h =$ Convective heat transfer coefficient

$M =$ Surface area of body

$\rho =$ Density of body

$V =$ Volume

$C_p =$ Specificheat

$T(x)$ = Temperature (dependent on independent variable x: position)

Let $-\frac{\mathrm{hM}}{\rho \mathrm{VC}_p} = a$ (4.29)

$\Rightarrow aT(x) = \frac{\mathrm{dT}}{\mathrm{dx}}$ (4.30)

Solving Steady heat transfer problem Volterra integral form Integrating both sides of Eq. (4.30)

$\int_0^x \frac{\mathrm{dT}}{\mathrm{dx}} dx = a \int_0^x T(t)\, dt$ (4.31)

$T(x) - T(0) = a \int_0^x T(t)\, dt$ (4.32)

$T(x) = \gamma + a \int_0^x T(t)\, dt$ (4.33)

Take Elzaki transform on both sides of Eq. (4.33)

$E[T(x)] = E[\gamma] + E[a \int_0^x T(t)\, dt]$ (4.34)

$T[v] = \gamma v^2 + \mathrm{av}\mathrm{T}[v]$ (4.35)

$T[v](1 - \mathrm{av}) = \gamma \mathrm{v}^2$ (4.36)

Take Inverse Elzaki transform on both sides of Eq. (4.36)

$E^{-1}[T[v]] = \gamma E^{-1}[\frac{v^2}{1-\mathrm{av}}]$ (4.37)

$T(x) = \gamma e^{ax}$ (4.38)

$T(x) = \gamma e^{\frac{-hMx}{\rho V C_p}}$ (4.39)

This is the required result.

4.5 Collective risk theory model

Consider risk factor involved in an insurance company. We assume that loaded risk premiums are collected in some fund. We call this risk reserve and this reserve is used to pay risk sums which get due. We define probability of such a claim occurring in some interval defined as (t, t + d t) as p.dt. Probability that claim results in a payment is ≤ z is S(z).

We assume that claims are independent of each other and function S(z) is constant in time. Details can be seen at [5, 14].

Mean value of claim can be given as,

pdt $\int_0^\infty z\, dS(z)$ (4.40)

By taking mean risk sum as unity,

$\int_0^\infty z\, dS(z) = 1$ (4.41)

Net risk premium is given by,

$\mathrm{dP} = \mathrm{p.dt}$ (4.42)

Risk reserve U at point t can be given as,

$U_t = -\sum_{n=1}^{k} z_k + \lambda \mathrm{P} + U_0$ (4.43)

Where λ is security factor and λP is loaded risk premium. z_k is the risk sum which is paid in interval (0,t).

Probability that 'k' claims occur,

$\frac{e^{-p} p^k}{k!}$ (4.44)

Hence, probability that k claims together result in payment which is ≤ z can be given as,

$S_k(z) = \int_0^z S_{k-1}(z-y)\, dS(y)\,,\ S_0(z) = 1$ (4.45)

Above is recurrence formula.

[4] discussed following collective risk theory model,

$u(x) = 1 + \int_0^x ((t-x)u(t))\, dt$ (4.46)

This can also be written as

$u(x) = 1 - \int_0^x ((x-t)u(t))\, dt$ (4.47)

Take Elzaki transform on both sides of Eq. (4.47)

$E[u(x)] = E[1] - E[\int_0^x ((x-t)u(t))\, dt]$ (4.48)

$T[v] = v^2 - \frac{1}{v}[v^3 T[v]]$ (4.49)

$T[v] = v^2 - v^2 T[v] = \frac{v^2}{1+v^2}$ (4.50)

Taking Inverse Elzaki transform on both sides of Eq. (4.50), we get

$u(x) = \mathrm{Cos}(x)$ (4.51)

This is the required result.

4.6 Viscoelastic hereditary model

It is a known fact that viscoelastic materials exhibit properties of both viscous and elastic materials when subject to deforming force. From hereditary perspective, the material retains memory or effect of previous deformations. Hence, each time a material undergoes loading and unloading, previous effects add up to newer loading/ unloading. Such a model can be effectively described through hereditary integral and same has also been discussed by [23]. Solution of same has been acquired by using Elzaki transform.

$g(t) = 1. - 0.75 e^{-t/4}$ (4.52)

$R(t) = 0.25 + 0.75e\text{^}(-t/2)$ (4.53)

$f(t) = g(t) - \int_0^t \dot{R}(t-s)f(s)\,ds$ (4.54)

Taking Elzaki transform of Eq. (4.54)

$E(f(t)) = E(1.-0.75e^{-t/4}) - E\left(\int_0^t \dot{R}(t-s)f(s)\,ds\right)$(4.55)

$T(v) = E[1] - 0.75E[e^{-t/4}] - \frac{1}{v}(E(\dot{R}(t))E(f(t)))$ (4.56)

$T(v) = v^2 - 0.75[\frac{v^2}{1+\frac{v}{4}}] - \frac{1}{v}(\frac{R(v)}{v} - vR(0))T(v)$ (4.57)

$$T(v) = \frac{v^4 - \frac{3v^4}{v+4}}{0.25v^2 + \frac{1.5v^2}{v+2}} \quad (4.58)$$

Expanding Eq. (4.58) as Taylor series and taking inverse Elzaki transform, we obtain

$f(t) =$
$0.25 + 0.28125t + 0.0058594t^2 - 0.00219726t^3$
$+0.000190735t^4 - 0.0000108719t^5$ (4.59)

$f(t) = 4.+1.5\left(1 - \frac{t}{4} + \frac{t^2}{32} - \frac{t^3}{384} + \frac{t^4}{6144} - \frac{t^5}{122880}\right)$

$-5.25(1 - \frac{t}{8} + \frac{t^2}{128} - \frac{t^3}{3072} + \frac{t^4}{98304} - \frac{t^5}{3932160})$ (4.60)

Writing Eq. (4.60) in closed form

$f(t) = 4 + 1.5e^{-t/4} - 5.25e^{-t/8}$ (4.61)

This is the required result.

5 CONCLUSION

In this research work, Elzaki transform has been employed to derive equivalent mathematical forms of Maxwell model. Moreover, Linear Volterra type viscoelastic hereditary model, load distribution over string model and Steady state heat transfer model solutions have been found using Elzaki transform. In addition, solution of one finance model pertaining collective risk has also been discussed. It has been confirmed that Elzaki transform is a simple and robust compatible alternative to well-known Laplace transform method which can be effectively applied to tackle problems arising in Physical sciences and Finance as well. Results presented in research substantiate this claim. The scope of research can be further extended to linear models arising in Physical sciences which have not been earlier solved using Elzaki transform. Furthermore, other transformation methods like Aboodh transform, Kamal transform etc. can also be applied to similar problems.

6 DECLARATIONS

6.1 Availability of data and material

Not applicable.

6.2 Competing interests

The authors declare that they have no competing interests.

6.3 Funding

Not applicable.

6.4 Authors' contributions

All authors equally contributed in writing this research paper.

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