SPHERICAL GROWTH SERIES OF THE FREE PRODUCT $\mathbb{Z}_m * \mathbb{Z}_n$

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ABSTRACTS: *The computations of A. Machi for the spherical growth series of* $\mathbb{Z}_2 * \mathbb{Z}_3$ *are given in [1]. We generalize this idea and compute the spherical growth series of the modular group* $\mathbb{Z}_m * \mathbb{Z}_n$.

Keywords: word length, free product, spherical growth series.

1. INTRODUCTION

In geometric group theory the spherical growth function is the study of combinatorial aspect of the elements in a finitely generated group G. Spherical growth series provides an invariant of a group. A. Machi [1] compute the spherical growth series of $\mathbb{Z}_2 * \mathbb{Z}_3$ are given. We generalize this idea and compute the spherical growth series of the modular group $\mathbb{Z}_m * \mathbb{Z}_n$. We start with few basic definitions.

Definition 1. [1] Le G be a finitely generated group and let S and S^{-1} be a finite set of generators of G and their inverses, respectively. The word length of $g \in G$ is denoted by $l_S(g)$ and is defined as the least non-negative integer n for which there exist $s_1, s_2, \dots, s_n \in S \cup S^{-1}$ such that $g = s_1 \cdots s_n$.

Definition 2. [1] The *spherical growth function* of the group G with generating set S associates to a non-negative integer k the number $\sigma(k)$ of the element $g \in G$such that $l_S(g) = k$, the number of elements of G of word length k. The *spherical growth series* (SGS) of G is the formal power series $\mathbf{S}_G(t) = \sum_{k=0}^{\infty} \sigma(k) t^k$, where $\sigma(0) = 1$.

In this paper, for different groups with the given generating set S, we consider the words of length exactly k and find their corresponding spherical growth series.

Definition 3. [1] Let G and H be two groups. The free product $G * H$ of G and H is a group whose elements are the words of the form $g_1 h_1 g_2 h_2 \cdots g_n h_n$, where $g_i \in G$ and $h_i \in H$ with the condition that g_1 and h_n are possibly identities of G and H, respectively.

The free product $G * H$ of two groups is a group that contains the elements of both G and H. In this product G and H are subgroups and the elements of these subgroups are generators of $G * H$. If G and H are non-trivial then $G * H$ is always infinite.

2. Spherical Growth Series of Free Groups and of Free Abelian Groups

There corresponds a growth function to each given group G with a finite generating set S. The growth function, an invariant of the group, generates a series of the group G called the growth series of the group G.

In this section we present SGS of the free groups and the free abelian groups. We start with a famous result regarding the SGS as a proposition.

Proposition 1 [3]. Let G_1 and G_2be two groups generated by finite sets S_1 and S_2, respectively. Then the SGS of the direct product $G_1 \times G_2$, finitely generated by $S = (S_1 \times \{1\}) \cup (\{1\} \times S_2)$ is given by $\mathbf{S}_{G_1 \times G_2}(t) = \mathbf{S}_{G_1} \mathbf{S}_{G_2}$.

The SGS of the free group with one generator is given in the following example.

Example 1. The SGS of $F_1[X]$, the free group generated by $X = \{x_1\}$ is $\mathbf{S}_{F_1[X]}(t) = \frac{1}{1-t}$, with one element of each length.

In the next example the SGS of the free group with n generators is computed by easy computations.

Example 2. The SGS of $F_n[X]$, the free group generated by $X = \{x_i; i = 1, 2, \dots, n\}$ is $\mathbf{S}_{F_n[X]}(t) = \frac{1}{1-nt}$ such that $\sigma(k) = n^k \ \forall\, k \geq 1$.

Example 3. [1] Considering the infinite cyclic group $\mathbb{Z}$ with generating set $\{2, 3\}$, we have $\sigma(1) = 4$, $\sigma(2) = 8$, and $\sigma(k) = 6 \ \forall\, k \geq 3$. Therefore the spherical growth series is

$$\mathbf{S}_Z(t) = \frac{1 + 3t + 4t^2 - 2t^3}{1 - t}.$$

Example 4 . [1] For the infinite cyclic group $\mathbb{Z}$ with natural generator 1, the corresponding SGS is given by

$$\mathbf{S}_Z(t) = \frac{1+t}{1-t}.$$

Example 3 and Example 4 show that the spherical growth series of any particular group G is not unique, it depends greatly on the choice of generating set S.

Example 5 [2]. Consider the free abelian group $\mathbb{Z}^n$ with natural set of generators, if we let $\mathbb{Z}^n = \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ (n times) then by using the Proposition 1 and Example 4 the SGS of $\mathbb{Z}^n$ is given by $\mathbf{S}_{Z^n}(t) = \left(\frac{1+t}{1-t}\right)^n$.

3. Spherical Growth Series of $\mathbb{Z}_2 * \mathbb{Z}_n$

In this section we find the SGS of the free product $\mathbb{Z}_2 * \mathbb{Z}_n$. The computations of A. Machi for the SGS of $\mathbb{Z}_2 * \mathbb{Z}_3$ are given in [1].

Example 6. [1] The SGS of the free product $\mathbb{Z}_2 * \mathbb{Z}_3$ is given by $\mathbf{S}_{Z_2 * Z_3}(t) = \frac{1 + 3t + 2t^2}{1 - 2t^2}$.

In order to compute the SGS of the free product $\mathbb{Z}_m * \mathbb{Z}_n$, we are using the following notations: let $\mathbb{Z}_m = \{1, a, a^2, \dots, a^{m-1}\}$ and $\mathbb{Z}_n = \{1, b, b^2, \dots, b^{n-1}\}$ be two finitely generated abelian groups. To avoid the ambiguity in the products, we set $a^i = x_i$; $i = 1, 2, \dots, m-1$ and $b^j = y_j$; $j = 1, 2, \dots, n-1$. So we have $\mathbb{Z}_m = \{1, x_1, x_2, \dots, x_{m-1}\}$ and $\mathbb{Z}_n = \{1, y_1, y_2, \dots, y_{n-1}\}$. In $\mathbb{Z}_m$, we have $x_i = x_{m-i}^{-1}$; $i = 1, 2, \dots, m-1$ and in $\mathbb{Z}_n$,we have $y_j = y_{n-j}^{-1}$; $j = 1, 2, \dots, n-1$. Let $m' = [\frac{m}{2}]$, $m'' = [\frac{m-1}{2}]$, $n' = [\frac{n}{2}]$ and $n'' = [\frac{n-1}{2}]$. Also let $S = \{x_1, \dots, x_{m'}, y_1, \dots, y_{n'}\}$ be the set of generators for the free product $\mathbb{Z}_m * \mathbb{Z}_n$. Let $X_1 = \{x_1, \dots, x_{m'}, x_1^{-1}, \dots, x_{m''}^{-1}\}$ and $Y_1 = \{y_1, \dots, y_{n'}, y_1^{-1}, \dots, y_{n''}^{-1}\}$. Let us denote by $\sigma_a(k)$ and by $\sigma_b(k)$ the number of words in the free product $\mathbb{Z}_m * \mathbb{Z}_n$ of length k in $X_1 \cup Y_1$ ending at any word of X_1 (respectively at any word of Y_1). Let $l_a(w)$ and $l_b(w)$ denote the word length of $w \in \mathbb{Z}_m * \mathbb{Z}_n$, where w ends at any word of X_1 (respectively at any word of Y_1). Also for $k \geq 1$ we define the following sets: $P = \{w;\ l_a(w) = k+1\}$,

$Q = \{w;\ l_b(w) = k\}$, $U = \{w;\ l_a(w) = k\}$

and $V = \{w;\ l_b(w) = k+1\}$. Suppose that $\mathbf{S}_d(t)$ represents the SGS of a group whose words ends at d and is defined as $\mathbf{S}_d(t) = \sum_{k=1}^{\infty} \sigma_d(k) t^k$.

Lemma 1. The SGS of the free product $\mathbb{Z}_2 * \mathbb{Z}_n$ is given by

$$\mathbf{S}_{Z_2 * Z_n}(t) = \frac{1 + nt + (n-1)t^2}{1 - (n-1)t^2}.$$

Proof. Since $\mathbb{Z}_2 = \{1, a\}$ and $\mathbb{Z}_n = \{1, b, b^2, \dots, b^{n-1}\}$ are finitely generated abelian groups generated by a and b, respectively and $S = \{x_1, y_1, y_2, \dots, y_{n'}\}$ be the generating set for the free product $\mathbb{Z}_2 * \mathbb{Z}_n$. For $p \in \mathbb{N}$, let $X_{2p} = \{yw : y \in Y_1,\ w \in X_{2p-1}\}$, $X_{2p+1} = \{x_1 w : w \in X_{2p}\}$, $Y_{2p} = \{x_1 y : y \in Y_{2p-1}\}$ and $Y_{2p+1} = \{yw :\ y \in Y_1,\ w \in Y_{2p}\}$. The sets of words of length k ending at any word of X_1 are given in the table given below.

Table 1.

k	Set of words	$\sigma_a(k) = n(X_i)$
1	X_1	1
2	X_2	$n-1$
3	X_3	$n-1$
4	X_4	$(n-1)^2$
5	X_5	$(n-1)^2$
6	X_6	$(n-1)^3$
⋮	⋮	⋮

Also the sets of words of length k ending at any word of Y_1 are given in the following table:

Table 2.

k	Set of words	$\sigma_b(k) = n(Y_i)$
1	Y_1	$n-1$
2	Y_2	$n-1$
3	Y_3	$(n-1)^2$
4	Y_4	$(n-1)^2$
5	Y_5	$(n-1)^3$
⋮	⋮	⋮

Now to find recurrence relations between the words of Table 1 and Table 2, we define a function $f : Q \to P$ by $f(w) = wx_1$. It is clear that f is bijective. It follows that $\sigma_a(k+1) = \sigma_b(k)$ for all $k \geq 1$. By defining another bijective function, $h : U \to V$ by $h(w) = wy_i;\ \forall\ y_i \in Y_1 = \{y_1, y_2, \dots, y_{n-1}\}$, it is clear that $\sigma_b(k+1) = (n-1)\sigma_a(k)$ for all $k \geq 1$.

In terms of the series $\mathbf{S}_a(t)$ and $\mathbf{S}_b(t)$ the recurrence relations above are used to find the following linear system.

$$\begin{aligned}
\mathbf{S}_a(t) &= \sum_{k=1}^{\infty} \sigma_a(k) t^k \\
&= \sigma_a(1)t + \sigma_a(2)t^2 + \sigma_a(3)t^3 + \cdots \\
&= t + t\left(\sigma_a(2)t + \sigma_a(3)t^2 + \sigma_a(4)t^3 + \cdots\right) \\
&= t\left(1 + \sum_{k=1}^{\infty} \sigma_a(k+1) t^k\right) \\
&= t\left(1 + \sum_{k=1}^{\infty} \sigma_b(k) t^k\right).
\end{aligned}$$

Hence $\mathbf{S}_a(t) = t\left(1 + \mathbf{S}_b(t)\right)$. Also

$$\mathbf{S}_b(t) = \sum_{k=1}^{\infty} \sigma_b(k) t^k$$
$$= \sigma_b(1)t + \sigma_b(2)t^2 + \sigma_b(3)t^3 + \cdots$$
$$= (n-1)t + t\left(\sigma_b(2)t + \sigma_b(3)t^2 + \cdots\right)$$
$$= (n-1)t + t\left(\sum_{k=1}^{\infty} \sigma_b(k+1)t^k\right)$$
$$= (n-1)t + t\sum_{k=1}^{\infty}(n-1)\sigma_a(k)t^k.$$

Therefore $\mathbf{S}_b(t) = (n-1)t\left(1+\mathbf{S}_a(t)\right)$. By using Equation 2 in Equation 1 we get

$$\mathbf{S}_a(t) = t\left(1+(n-1)t\left(1+\mathbf{S}_a(t)\right)\right)$$
$$= t + (n-1)t^2 + (n-1)t^2\mathbf{S}_a(t)$$
$$\Rightarrow \quad \mathbf{S}_a(t) = \frac{t+(n-1)t^2}{1-(n-1)t^2}.$$

Also putting Equation 1 in Equation 2 we have

$$\mathbf{S}_b(t) = (n-1)t\left(1+t\left(1+\mathbf{S}_b(t)\right)\right)$$
$$= (n-1)t + (n-1)t^2 + (n-1)t^2\mathbf{S}_b(t)$$
$$\Rightarrow \mathbf{S}_b(t) = \frac{(n-1)t(1+t)}{1-(n-1)t^2}.$$

Hence the corresponding SGS of the free product $\mathbb{Z}_2 * \mathbb{Z}_n$ is

$$\mathbf{S}_{Z_2*Z_n}(t) = 1 + \mathbf{S}_a(t) + \mathbf{S}_b(t)$$
$$= 1 + \frac{t+(n-1)t^2}{1-(n-1)t^2} + \frac{(n-1)t(1+t)}{1-(n-1)t^2}$$
$$= \frac{1+nt+(n-1)t^2}{1-(n-1)t^2}.$$

4. Spherical Growth Series of $\mathbb{Z}_m * \mathbb{Z}_n$

Theorem. The SGS of the free product $\mathbb{Z}_m * \mathbb{Z}_n$ is given by

$$\mathbf{S}_{Z_m*Z_n}(t) = \frac{1+(m+n-2)t+(m-1)(n-1)t^2}{1-(m-1)(n-1)t^2}.$$ Proof.

Since $\mathbb{Z}_m = \{1, a, a^2, \dots, a^{m-1}\}$ and $\mathbb{Z}_n = \{1, b, b^2, \dots, b^{n-1}\}$ are finitely generated abelian groups generated by a and b, respectively and $S = \{x_1, x_2, \dots, x_{m'}, y_1, y_2, \dots, y_{n'}\}$ be the set of generators for the free product $Z_m * Z_n$. First of all for $p \in \mathbb{N}$, let us define

$$X_{2p} = \{yw : \ y \in Y_1, \ w \in X_{2p-1}\}$$
$$X_{2p+1} = \{xw : \ x \in X_1, \ w \in X_{2p}\}$$
$$Y_{2p} = \{xy : \ x \in X_1, \ y \in Y_{2p-1}\}$$
$$Y_{2p+1} = \{yw : \ y \in Y_1, \ w \in Y_{2p}\}.$$

Now the sets of words of length k ending at any word of X_1 are written in the table given below.

Table 3.

k	Set of words	$\sigma_a(k) = n(X_i)$
1	X_1	$m-1$
2	X_2	$(n-1)(m-1)$
3	X_3	$(n-1)(m-1)^2$
4	X_4	$(n-1)^2(m-1)^2$
5	X_5	$(n-1)^2(m-1)^3$
$\vdots$	$\vdots$	$\vdots$

And the words of length k ending at any word of Y_1 are shown in the following table:

Table 4.

k	Set of words	$\sigma_b(k) = n(Y_i)$
1	Y_1	$n-1$
2	Y_2	$(n-1)(m-1)$
3	Y_3	$(m-1)(n-1)^2$
4	Y_4	$(m-1)^2(n-1)^2$
5	Y_5	$(m-1)^2(n-1)^3$
$\vdots$	$\vdots$	$\vdots$

To find recurrence relations between the words of Table 3 and Table 4, we define a bijective function $f : Q \to P$ by $f(w) = wx_i; \ \forall \ x_i \in X = \{x_1, x_2, \dots, x_{m-1}\}$. It follows that $\sigma_a(k+1) = (m-1)\sigma_b(k)$ for all $k \geq 1$. We define another bijective function $h : U \to V$ by $h(w) = wy_i; \ \forall \ y_i \in Y = \{y_1, y_2, \dots, y_{n-1}\}$. Now it is clear that $\sigma_b(k+1) = (n-1)\sigma_a(k)$ for all $k \geq 1$. In terms of the series $\mathbf{S}_a(t)$ and $\mathbf{S}_b(t)$ the recurrence relations above are used to find the following linear system.

$$\mathbf{S}_a(t) = \sum_{k=1}^{\infty} \sigma_a(k)t^k$$
$$= \sigma_a(1)t + \sigma_a(2)t^2 + \sigma_a(3)t^3 + \cdots$$
$$= (m-1)t + t\left(\sigma_a(2)t + \sigma_a(3)t^2 + \cdots\right)$$
$$= t\left((m-1) + \sum_{k=1}^{\infty}\sigma_a(k+1)t^k\right)$$
$$= t\left((m-1) + \sum_{k=1}^{\infty}(m-1)\sigma_b(k)t^k\right).$$

Therefore $\mathbf{S}_a(t) = (m-1)t\left(1+\mathbf{S}_b(t)\right)$. Also we have

$$
\begin{aligned}
S_b(t) &= \sum_{k=1}^{\infty} \sigma_b(k) t^k \\
&= \sigma_b(1)t + \sigma_b(2)t^2 + \sigma_b(3)t^3 + \sigma_b(4)t^4 + \cdots \\
&= (n-1)t + t\left(\sigma_b(2)t + \sigma_b(3)t^2 + \sigma_b(4)t^3 + \cdots\right) \\
&= (n-1)t + t\left(\sum_{k=1}^{\infty} \sigma_b(k+1) t^k\right) \\
&= (n-1)t + t\sum_{k=1}^{\infty} (n-1)\sigma_a(k) t^k .
\end{aligned}
$$

Hence $S_b(t) = (n-1)t\left(1 + S_a(t)\right)$. Putting Equation 3 in Equation 4 we have

$$
\begin{aligned}
S_a(t) &= (m-1)t\left(1 + (n-1)t\left(1 + S_a(t)\right)\right) \\
&= (m-1)t + (m-1)(n-1)t^2 + (m-1)(n-1)t^2 S_a(t)
\end{aligned}
$$

Hence we have $S_a(t) = \dfrac{(m-1)t\left(1 + (n-1)t\right)}{1 - (m-1)(n-1)t^2}$.

Also putting Equation 4 in Equation 3 we get

$$
\begin{aligned}
S_b(t) &= (n-1)t\left(1 + (m-1)t\left(1 + S_b(t)\right)\right) \\
&= (n-1)t + (m-1)(n-1)t^2 + (m-1)(n-1)t^2 S_b(t).
\end{aligned}
$$

Therefore we have

$$
S_b(t) = \frac{(n-1)t(1 + (m-1)t)}{1 - (m-1)(n-1)t^2}.
$$

Finally the SGS of the free product $\mathbb{Z}_m * \mathbb{Z}_n$ is given as

$$
\begin{aligned}
S_{Z_m * Z_n}(t) &= 1 + S_a(t) + S_b(t) \\
&= 1 + \frac{(m-1)t(1 + (n-1)t)}{1 - (m-1)(n-1)t^2} + \frac{(n-1)t(1 + (m-1)t)}{1 - (m-1)(n-1)t^2} \\
&= \frac{1 + (m+n-2)t + (m-1)(n-1)t^2}{1 - (m-1)(n-1)t^2}.
\end{aligned}
$$

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!Unexpected End of Formula

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