USE OF OPTIMAL HOMOTOPY ASYMPTOTIC METHOD FOR FRACTIONAL ORDER NONLINEAR FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

S. Iqbal[1], F. Sarwar[2], Muhammad Rafiq Mufti[3] and I. Siddique[4]

[1] Department of Informatics and System, School of System and Technology, University of Management and Technology, Lahore, Pakistan

[2]Department of Electrical Engineering, University of Management and Technology, C-II Johar Town Lahore, Pakistan.

[3]Department of Computer Sciences, COMSATS Institute of Information Technology, Vehari, Pakistan.

[4] Department of Mathematics, Schol of Sciences, University of Management and Technology, Sialkot Campus, Pakistan

shaukat.iqbal.k@gmail.com, imransmsrazi@gmail.com

ABSTRACT*: In this study, optimal homotopy asymptotic method (OHAM), a semi-numerical technique is formulated for solving nonlinear Fredholm integro-differential equations of fractional order to check the effectiveness and performance of the method. It is observed that the formulation is easy to implement, quite valuable to handle fractional applications and yield tremendous results at minimum computational cost. The computational results of some of the test problems reveal that OHAM is well-organized, very effective, and simple and are in excellent agreement with exact solutions.*

1. INTRODUCTION

The applications of fractional calculus have been revealed by various researchers. Pragmatic modeling of physical phenomenon and the previous history can be successfully achieved by using fractional calculus. Historically, it has achieved enormous attraction among mathematicians and physicists in formulating boundary value problems, integral equations and Fredholm integro-differential equations. Fredholm integro-differential equations (FDIEs) gain an elegant position in numerous fields such as economics, biomechanics, control, elasticity, fluid dynamics, heat and mass transfer, oscillation theory, and airfoil theory; see, for example, [1–2] and the references cited therein. The notion of FIDEs has provoked massive size of research in recent years. During the last decades, several numerical and analytical techniques have been utilized to approximate the solutions of FIDEs such as the neural networks [3], comparison of Adomian decomposition with wavelet Galerkin [4], Differential transform [5], finite differences [6-7], comparison of finite elements and finite difference [8], sinc method [9], Tau method [10-11] and Galerkin method with hybrid functions [12].

In this paper, our study focuses the following class of nonlinear Fredholm fractional integro-differential equations [13]:

$$D^{\alpha}u(x)-\int_0^1 k(x,t)[u(t)]^q dt = g(x),\ q\geq 1, \qquad (1)$$

Subject to the initial conditions

$$u^{(i)}=\delta_i,\quad i=0,1,2,\dots,r-1,$$
$$r-1<\alpha\leq r,\quad r\in N, \qquad (2)$$

where $g(x)$ is the known analytic function, $k(x,t)$ is the linear/nonlinear kernel of the integral, $u(x)$ is the solution of the FIDEs to be determined, D^{α} is the fractional derivative in the Caputo sense, δ is constant and q is the positive integer. There are numerous definitions of fractional derivatives and integrations of order $\alpha>0$. The two most frequently used definitions are the Riemann–Liouville and Caputo [13]. The Riemann–Liouville fractional integration of order α is defined as:

$$I^{\alpha}u(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}u(t)dt,\quad x,\alpha>0 \qquad (3)$$

I^{α} denotes the fractional integral operator of order α in the sense of Riemann-Liouville. The Caputo definition of fractional differential operator of order α is defined as:

$$D^{\alpha}u(x)=\frac{1}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1}u^{(n)}(t)dt,$$
$$n-1<\alpha\leq n, \qquad (4)$$

D^{α} denotes the fractional differential operator of order α in the sense of Caputo. The Caputo fractional derivative first computes an ordinary derivative followed by a fractional integral to achieve the desired order of fractional derivative.

In the present study, OHAM, which is recently introduced by Marinca et al. [16], is formulated for the semi-numerical solutions of fractional order integro-differential equations. It is motivated by the aspiration to obtain exciting solutions of fractional order Fredholm integro-differential equations using newly formulated algorithm of OHAM. The advantage of OHAM is its convergence criteria which is more flexible. In series of papers, authors [17–22] have applied this method effectively to validate the solutions of currently important problems in science and also shown its usefulness, generality and consistency. The solutions of fractional order Fredholm integro-differential equations showed that OHAM is not only useful for differential equations but also for fractional order integro-differential equations, which shows its strength and potential in science and engineering.

The article is organized as follows: we first formulated OHAM for factional order Fredholm Integro-differential equations in section 2. Section 3 exhibits some examples to experiment the proposed formulation. Also a conclusion is presented in last section.

2. OHAM Formulation for Fractional Order Integro-Differential Equations

According to the optimal homotopy asymptotic method [16-22], following is the extended formulation for the solution of fractional order integro differential equations:

a) Write the governing fractional order integro-differential equation as:

$$D^{\alpha}u(x)=f(x)+A(u),0\leq x<1,\ 0<\alpha\leq 1 \qquad (5)$$

with given initial conditions. $D^{\alpha}u(x)=\frac{\partial^{\alpha}u(x)}{\partial x^{\alpha}}$ denotes the Caputo or Riemann-Liouville fraction derivative operator, $A(u)$ is an integral operator, $u(x)$ is unknown function to be determined and $f(x)$ is a known analytic function.

b) Construct an optimal homotopy for fractional order Integro-differential equation, $\phi(x;p):\Omega\times[0,1]\rightarrow\mathbb{R}$, which satisfies:

$$(1-p)\left(\frac{\partial^{\alpha}\phi(x;p)}{\partial x^{\alpha}}-f(x)\right)-$$

$$\mathcal{H}(p)\left\{\frac{\partial^{\alpha}\phi(x;p)}{\partial x^{\alpha}}-A(\phi)-f(x)\right\}=0, \quad (6)$$

where $x\epsilon\Omega$ and $p\epsilon[0,1]$ is an embedding parameter, $\mathcal{H}(p)=\sum_{k\geq 1}p^{k}C_{k}$ is non zero auxiliary function for $p\neq 0$, $C_{k's}$ are auxiliary constants and $\mathcal{H}(0)=0$. The auxiliary function $\mathcal{H}(p)$ provides with a simple way to adjust and control the convergence. It also increases the precision of the results and competence of the method.

c) Expand $\phi(x;p)$ in Taylor's series about p, one can get an approximate solution as:

$$\phi(x;p)=u_0(x)+\sum_{j\geq}u_j(x)p^{j} \quad 7)$$

The series is observed to be convergent at $p=1$, therefore, approximate solution having auxiliary constants is:

$$\tilde{u}(x)=u_0(x)+\sum_{j\geq}u_j(x) \quad (8)$$

d) Compare the coefficients of like powers of embedding parameter p, after substituting $\phi(x;p)$ into optimal homotopy equation to get zeroth-, first-, second- and n^{th}-order (if needed) deformed problems as under:

$$p^0:\ D^{\alpha}u_0(x)-f=0$$
$$p^1:\ D^{\alpha}u_1(x)+C_1A(u_0)+(1+C_1)f$$
$$-(1+C_1)D^{\alpha}u_0(x)=0$$
$$p^2:\ D^{\alpha}u_2(x)+C_1A(u_1)+C_2\big(f+A(u_0)\big)$$
$$-C_2D^{\alpha}u_0(x)-(1+C_1)D^{\alpha}u_1(x)=0 \quad (9)$$

Type equation here.

$$p^n:\ D^{\alpha}u_n(x)+C_1A(u_n)+C_2A(u_{n-1})$$
$$+C_3\big(f+A(u_0)\big)\ldots-C_2D^{\alpha}u_{n-1}(x)$$
$$-(1+C_1)D^{\alpha}u_n(x)=0.$$

e) Execute the fractional operator J^{α} to the series of deformed problems of different orders in step d) to obtain series of solutions. Using these solutions in Eq. (8) will produce the approximate solution $\tilde{u}(x;\ C_i)$.

f) Determine optimal values of auxiliary constants by using methods mentioned in different references therein [16-22].

g) Put optimal values of auxiliary constants evaluated in previous step, in Eq. (8), one can get the approximate solution.

3. ILLUSTRATIVE EXAMPLES

To show the effectiveness and validity of the proposed extended OHAM algorithm for fractional order integro-differential equations, we consider the following four examples chosen from the reference [13-15] and compare the solutions of this proposed algorithm with the exact solutions. We presented the absolute errors in different tables to reveal the accuracy of the extended method.

Example 1

Consider the following linear fractional order Fredholm integro-differential [14] equation with $q=1$:

$$D^{\alpha}u(x)=1-\frac{1}{3}x+\int_0^1 xtu(t)dt,\ \ 0\leq x<1,\ \ 0<\alpha\leq 1 \quad (10)$$

subject to the initial conditions $u(0)=0$ and the exact solutions for $\alpha=1,\frac{3}{4},\frac{1}{2}$ are respectively as: $u(x)=x$, $\frac{4\,x^{3/4}}{3\,\Gamma\left(\frac{3}{4}\right)}-\frac{16\,x^{7/4}}{63\,\Gamma\left(\frac{3}{4}\right)}+\frac{69376\,x^{7/4}}{693\,\Gamma\left(\frac{3}{4}\right)\left(-64+315\,\Gamma\left(\frac{3}{4}\right)\right)}$, $\frac{2\sqrt{x}}{\sqrt{\pi}}-\frac{4\,x^{3/2}}{9\sqrt{\pi}}+\frac{848\,x^{3/2}}{45(-8+21\sqrt{\pi})\sqrt{\pi}}$.

While executing OHAM formulation for the solution of fractional order FIDE, it generates a series of problems: the expressions for zeroth order, first order, second order and third order problems and their solutions are given below as:

- Zeroth order problem and its solution:

$$D^{\alpha}u_0(x)-1+\frac{x}{3}=0 \quad (11)$$

$$u_0(x)=-\frac{x^{\alpha}(x-3(1+\alpha))}{3\,\Gamma(2+\alpha)} \quad (12)$$

- First order problem and its solution:

$$D^{\alpha}u_1(x)+1-\frac{x}{3}+C_1-\frac{xC_1}{3}$$
$$+C_1\int_0^1 txu_0(t)\,dt-(1+C_1)D^{\alpha}u_0(x)=0 \quad (13)$$

$$D^{\alpha}u_1(x)=-\frac{x(7+11\alpha+3\alpha^2)C_1}{3\,\Gamma(4+\alpha)} \quad (14)$$

$$u_1(x)=-\frac{x^{1+\alpha}(7+\alpha(11+3\alpha))C_1}{3\,\Gamma(2+\alpha)\,\Gamma(4+\alpha)} \quad (15)$$

- Second order problem and its solution:

$$D^{\alpha}u_2(x)+C_1\int_0^1 txu_1(t)\,dt+C_2-\frac{xC_2}{3}$$
$$+C_2\int_0^1 txu_0(t)\,dt-C_2D^{\alpha}u_0(x)-(1+C_1)D^{\alpha}u_1(x)=0 \quad (16)$$

$$D^{\alpha}u_2(x)=\frac{x(7+\alpha(11+3\alpha))\left(C_1^2-(3+\alpha)\,\Gamma(2+\alpha)\left(C_1+C_1^2+C_2\right)\right)}{3(3+\alpha)\,\Gamma(2+\alpha)\,\Gamma(4+\alpha)} \quad (17)$$

$$u_2(x)=\frac{x^{1+\alpha}(7+\alpha(11+3\alpha))\left(C_1^2-(3+\alpha)\,\Gamma(2+\alpha)\left(C_1+C_1^2+C_2\right)\right)}{3(3+\alpha)\,\Gamma(2+\alpha)^2\,\Gamma(4+\alpha)} \quad (18)$$

- Third order problem and its solution:

$$D^{\alpha}u_3(x)+C_1\int_0^1 txu_2(t)\,dt+C_2\int_0^1 txu_1(t)\,dt$$
$$+C_3-\frac{xC_3}{3}+C_3\int_0^1 txu_0(t)\,dt$$
$$-C_3D^{\alpha}u_0(x)-C_2D^{\alpha}u_1(x)$$
$$-(1+C_1)D^{\alpha}u_2(x)=0 \quad (19)$$

$$D^{\alpha}u_3(x)=\frac{1}{3}x\big(7+\alpha(11+3\alpha)\big)\left(-\frac{C_1^3}{(3+\alpha)^3\,\Gamma(2+\alpha)^2\,\Gamma(3+\alpha)}+\frac{2(2+\alpha)C_1\left(C_1+C_1^2+C_2\right)}{\Gamma(4+\alpha)^2}-\frac{C_2+C_1\left((1+C_1)^2+2C_2\right)+C_3}{\Gamma(4+\alpha)}\right) \quad (20)$$

$$u_3(x)=\frac{x^{1+\alpha}\big(7+\alpha(11+3\alpha)\big)}{3\,\Gamma(2+\alpha)}\left(-\frac{C_1^3}{(3+\alpha)^3\,\Gamma(2+\alpha)^2\,\Gamma(3+\alpha)}+\frac{2(2+\alpha)C_1\left(C_1+C_1^2+C_2\right)}{\Gamma(4+\alpha)^2}-\frac{C_2+C_1\left((1+C_1)^2+2C_2\right)+C_3}{\Gamma(4+\alpha)}\right) \quad (21)$$

Now, one can create $u(x)$ by adding zeroth-order, first-order, second-order and third-order solutions:

$$u(x)=\frac{1}{3}x^{\alpha}\left(\begin{array}{c}\frac{3-x+3\alpha}{\Gamma(2+\alpha)}+\frac{1}{\Gamma(4+\alpha)^4}x(2+\alpha)(3+\alpha)\big(7+\alpha(11+3\alpha)\big)\\ \left(\begin{array}{c}3\big(2+\alpha-\Gamma(4+\alpha)\big)\Gamma(4+\alpha)C_1^2-\\ \big(2+\alpha-\Gamma(4+\alpha)\big)^2C_1^3+\Gamma(4+\alpha)C_1\left(\begin{array}{c}-3\,\Gamma(4+\alpha)\\ +2\big(2+\alpha-\Gamma(4+\alpha)\big)C_2\end{array}\right)\\ -\Gamma(4+\alpha)^2(2C_2+C_3)\end{array}\right)\end{array}\right) \quad (22)$$

By using the procedure mentioned in [16-22], one can calculate the auxiliary constants C_1, C_2 and C_3 presented in Table 1 below:

Table 1: Auxiliary constants for Example 1.

α	C_1	C_2	C_3
1.0	0.3040931004186823	−1.119837333044531	0.5162039301601453
0.75	0.2864458958473718	−1.1225972934673054	0.5019020801907468
0.5	0.26242517703528756	−1.1263540489908248	0.48243493369627943

Solutions for $\alpha=1,\frac{3}{4}$ and $\frac{1}{2}$ can be calculated by using auxiliary constants given in Table 1 respectively:

$$u(x)=\frac{1}{3}(0.5(6.0-x)+0.5\,x)x^1,\ \alpha=1 \quad (23)$$

$$u(x)=\frac{1}{3}\left(\begin{array}{c}0.6217515726462955(5.25-x)\\ +0.7611130429313304x\end{array}\right)x^{\frac{3}{4}},\ \alpha=\frac{3}{4} \quad (24)$$

$$u(x)=\frac{1}{3}x^{\frac{1}{2}}\left(\begin{array}{c}0.7522527780636751(4.5-x)\\ +1.0915074207737452\,x\end{array}\right),\ \alpha=\frac{1}{2} \quad (25)$$

Table 2: Absolute errors of Example 1 for different fractional orders.

x	$\lvert u_{exact}-u_{OHAM}\rvert$		
	$\alpha=1$	$\alpha=\frac{3}{4}$	$\alpha=\frac{1}{2}$
0.0	0.0	0.0	0.0
0.1	0.0	5.551115×10^{-17}	1.110223×10^{-16}
0.2	0.0	5.551115×10^{-17}	0.0
0.3	0.0	5.551115×10^{-17}	1.110223×10^{-16}
0.4	0.0	1.110223×10^{-16}	1.110223×10^{-16}
0.5	0.0	1.110223×10^{-16}	1.110223×10^{-16}
0.6	0.0	3.330669×10^{-16}	0.0
0.7	0.0	3.330669×10^{-16}	2.220446×10^{-16}
0.8	0.0	2.220446×10^{-16}	1.110223×10^{-16}
0.9	0.0	0.0	0.0
1.0	0.0	2.220446×10^{-16}	3.330669×10^{-16}

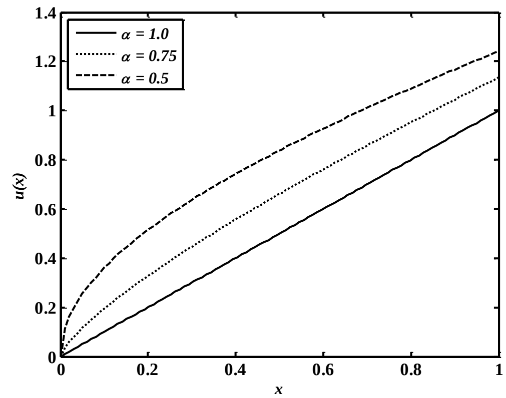

Fig 1. Approximate solutions of Example 1 for different values of α.

Example 2

Consider the following nonlinear fractional order Fredholm integro-differential equation [13] with $q=2$:

$$D^{\alpha}u(x)=1-\frac{1}{4}x+\int_0^1 xt\big(u(t)\big)^2dt,$$

$$0\le x<1,\ 0<\alpha\le 1 \quad (26)$$

subject to the initial conditions $u(0)=0$ and the exact solutions for $\alpha=1,\frac{3}{4},\frac{1}{2}$ are respectively as: $u(x)=x$,

$$\frac{4\,x^{3/4}}{3\,\Gamma\left(\frac{3}{4}\right)}-\frac{44\,x^{7/4}}{27\,\Gamma\left(\frac{3}{4}\right)}+\frac{231}{64}x^{7/4}\Gamma\left(\frac{3}{4}\right)-\frac{x^{7/4}\sqrt{-323223552+\left(17408-43659\,\Gamma\left(\frac{3}{4}\right)^2\right)^2}}{12096\,\Gamma\left(\frac{3}{4}\right)},$$

$$\frac{2\sqrt{x}}{\sqrt{\pi}}-\frac{5\,x^{3/2}}{2\sqrt{\pi}}+\frac{15}{8}\sqrt{\pi}\,x^{3/2}-\frac{1}{8}\sqrt{\frac{5(-16-312\pi+135\pi^2)}{3\pi}}\,x^{3/2}.$$

OHAM formulation for fractional order FIDE is executed. A series of problems are generated. The expressions for zeroth order, first order, second order and third order problems and their solutions are given below:

- Zeroth order problem and its solution:

$$D^{\alpha}u_0(x)=\frac{4-x}{4} \quad (27)$$

$$u_0(x)=-\frac{x^{\alpha}(x-4(1+\alpha))}{4\,\Gamma(2+\alpha)} \quad (28)$$

- First order problem and its solution:

$$D^{\alpha}u_1(x)+1-\frac{x}{4}+C_1-\frac{xC_1}{4}$$
$$+C_1\int_0^1 txu_0(t)^2\,dt-(1+C_1)D^{\alpha}u_0(x)=0 \quad (29)$$

$$D^{\alpha}u_1(x)=-\frac{x(67+162\alpha+128\alpha^2+32\alpha^3)C_1}{32(2+\alpha)(3+2\alpha)\,\Gamma(2+\alpha)^2} \quad (30)$$

$$u_1(x)=-\frac{x^{1+\alpha}\left(67+2\alpha\left(81+16\alpha(4+\alpha)\right)\right)C_1}{32(3+2\alpha)\,\Gamma(2+\alpha)^2\,\Gamma(3+\alpha)} \quad (31)$$

- Second order problem and its solution:

$$D^{\alpha}u_2(x)+C_1\int_0^1 2txu_0(t)u_1(t)\,dt+C_2$$
$$-\frac{xC_2}{4}+C_2\int_0^1 txu_0(t)^2\,dt-C_2D^{\alpha}u_0(x)$$
$$-(1+C_1)D^{\alpha}u_1(x)=0 \quad (32)$$

$$D^{\alpha}u_2(x)=\frac{\left(\begin{array}{c}x\left(67+2\alpha\left(81+16\alpha(4+\alpha)\right)\right)\\ \left(\frac{(13+22\alpha+8\alpha^2)C_1^2}{\Gamma(2+\alpha)^4}-\frac{4(2+\alpha)(3+2\alpha)\left(C_1+C_1^2+C_2\right)}{\Gamma(2+\alpha)^2}\right)\end{array}\right)}{128(2+\alpha)^2(3+2\alpha)^2} \quad (33)$$

$$u_2(x)=\frac{x^{1+\alpha}\left(67+2\alpha\left(81+16\alpha(4+\alpha)\right)\right)\left(\begin{array}{c}(13+22\alpha+8\alpha^2)C_1^2\\ -4(2+\alpha)(3+2\alpha)\,\Gamma(2+\alpha)^2\left(C_1+C_1^2+C_2\right)\end{array}\right)}{128(2+\alpha)^2(3+2\alpha)^2\,\Gamma(2+\alpha)^5} \quad (34)$$

- Third order problem and its solution:

$$D^{\alpha}u_3(x)+C_1\int_0^1 tx\left(u_1(t)^2+2u_0(t)u_2(t)\right)dt$$
$$+C_2\int_0^1 2txu_0(t)u_1(t)\,dt+C_3-\frac{xC_3}{3}$$
$$+C_3\int_0^1 txu_0(t)^2\,dt-C_3D^{\alpha}u_0(x)-C_2D^{\alpha}u_1(x)$$
$$-(1+C_1)D^{\alpha}u_2(x)=0 \quad (35)$$

$$D^{\alpha}u_3(x)=\frac{\left(\begin{array}{c}x\left(67+2\alpha\left(81+16\alpha(4+\alpha)\right)\right)\\ \left(\begin{array}{c}-\left(877+4\alpha\left(727+\alpha\left(869+40\alpha(11+2\alpha)\right)\right)\right)C_1^3\\ +32(2+\alpha)(3+2\alpha)(13+22\alpha+8\alpha^2)\,\Gamma(2+\alpha)^2C_1\left(C_1+C_1^2+C_2\right)\\ -64(2+\alpha)^2(3+2\alpha)^2\,\Gamma(2+\alpha)^4\left(C_2+C_1\left((1+C_1)^2+2C_2\right)+C_3\right)\end{array}\right)\end{array}\right)}{\left(2048(2+\alpha)^3(3+2\alpha)^3\,\Gamma(2+\alpha)^6\right)} \quad (36)$$

$$u_3(x)=\frac{\left(\begin{array}{c}x^{1+\alpha}\left(67+2\alpha\left(81+16\alpha(4+\alpha)\right)\right)\\ \left(\begin{array}{c}\left(-877-4\alpha\left(727+\alpha\left(869+40\alpha(11+2\alpha)\right)\right)\right)C_1^3\\ +32(2+\alpha)(3+2\alpha)(13+22\alpha+8\alpha^2)\text{Gamma}[2+\alpha]^2C_1(C_1+C_1^2+C_2)\\ -64(2+\alpha)^2(3+2\alpha)^2\text{Gamma}[2+\alpha]^4(C_2+C_1((1+C_1)^2+2C_2)+C_3)\end{array}\right)\end{array}\right)}{(2048(2+\alpha)^3(3+2\alpha)^3\text{Gamma}[2+\alpha]^7)} \quad (37)$$

Now, one can construct $u(x)$ by adding zeroth-order, first-order, second-order and third-order solutions, and other higher order solutions if needed:

$$u(x)=\frac{1}{2048\,\Gamma(2+\alpha)^7}x^{\alpha}\left(\begin{array}{c}-512\left(x-4(1+\alpha)\right)\Gamma(2+\alpha)^6\\ +1/(6+7\alpha+2\alpha^2)^3\,x\left(67+2\alpha\left(81+16\alpha(4+\alpha)\right)\right)\\ \left(\begin{array}{c}-877C_1^3-2908\alpha C_1^3-3476\alpha^2C_1^3-1760\alpha^3C_1^3\\ -320\alpha^4C_1^3+16(2+\alpha)(3+2\alpha)(13+22\alpha+8\alpha^2)\Gamma(2+\alpha)^2C_1\left(\begin{array}{c}C_1(3+2C_1)\\ +2C_2\end{array}\right)\\ -64(2+\alpha)^2(3+2\alpha)^2\,\Gamma(2+\alpha)^4(2C_2+C_1(3+C_1(3+C_1)+2C_2)+C_3)\end{array}\right)\end{array}\right) \quad (38)$$

By using the procedure mentioned in [16-22], one can calculate the auxiliary constants C_1, C_2 and C_3 presented in Table 3 below:

Table 3: Auxiliary constants for Example 2.

α	C_1	C_2	C_3
1.0	0.27614361096200135	−1.1242085344110408	0.49355278429081106
0.75	0.21968735497068775	−1.133038093309415	0.44779885766979954
0.5	0.10519648396762096	−1.1509440608784172	0.35501185260421386

Solutions for $\alpha=1,\frac{3}{4}$ and $\frac{1}{2}$ can be calculated by using auxiliary constants given in Table 3 respectively:

$$u(x)=0.000003814697265625\,x^1$$
$$(-32768.(-8.0+x)+32768.0\,x),\quad \alpha=1 \quad (39)$$

$$u(x)=0.000017538332313929135\,x^{\frac{3}{4}}$$
$$(-8862.752192129661(-7.0+x)$$
$$+13394.778929451515\,x),$$
$$\alpha=\frac{3}{4} \quad (40)$$

$$u(x)=0.00006656051131465182\,x^{0.5}$$

$(-2825.4469624923213(-6.0 + x) +$
$6482.814012506344\, x)$,
$\frac{1}{2}$ (41)

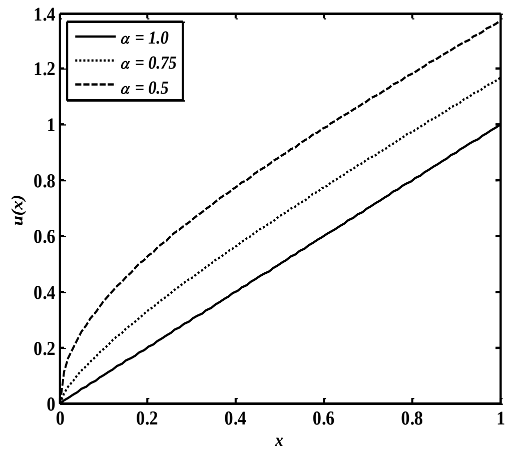

Fig 2. Approximate solutions of Example 2 for different values of α.

Example 3

Consider the following nonlinear fractional order Fredholm integro-differential equation [13] with $q = 4$:

$$D^{\frac{1}{2}}u(x) = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{8}{3}\, x^{\frac{3}{2}} - 2\, x^{\frac{1}{2}}\right) - \frac{1}{1260}x$$
$$+ \int_0^1 xt\big(u(t)\big)^4 dt, \quad 0 \le x < 1, \quad (42)$$

subject to the initial conditions $u(0) = 0$ and the exact solution of the above fractional order Fredholm integro-differential equation is: $u(x) = x^2 - x$.

While executing OHAM formulation for the solution of fractional $\alpha =$ order FIDE, it generates a series of problems: the expressions for zeroth order, first order and second order problems and their solutions are given below as:

- Zeroth order problem and its solution:

$$D^{\alpha}u_0(x) + \frac{x}{1260} = 0 \quad (43)$$

$$u_0(x) = -\frac{x^{1+\alpha}}{1260\, \Gamma(2+\alpha)} \quad (44)$$

- First order problem and its solution:

$$D^{\alpha}u_1(x) - \frac{x}{1260} - \frac{2\sqrt{x}C_1}{\sqrt{\pi}} - \frac{xC_1}{1260} + \frac{8x^{3/2}C_1}{3\sqrt{\pi}}$$
$$+C_1 \int_0^1 txu_0(t)^4\, dt - (1 + C_1)D^{\alpha}u_0(x) = 0 \quad (45)$$

$$D^{\alpha}u_1(x) =$$
$$-\frac{\sqrt{x}\left(\sqrt{\pi}\sqrt{x}+3360631680000(-3+4x)(3+2\alpha)\, \Gamma(2+\alpha)^4\right)C_1}{5040947520000\sqrt{\pi}(3+2\alpha)\, \Gamma(2+\alpha)^4} \quad (46)$$

$$u_1(x) = -\frac{x^{\frac{1}{2}+\alpha}(-3 + 4x - 2\alpha)C_1}{2\, \Gamma\left(\frac{5}{2} + \alpha\right)}$$
$$-\frac{2^{-7+2\alpha}x^{1+\alpha}\, \Gamma\left(\frac{3}{2}+\alpha\right)C_1}{9845600625\sqrt{\pi}\, \Gamma(2+\alpha)^4\, \Gamma(4+2\alpha)} \quad (47)$$

- Second order problem and its solution:

$$D^{\alpha}u_2(x) + C_1 \int_0^1 4txu_0(t)^3u_1(t)\, dt - \frac{2\sqrt{x}C_2}{\sqrt{\pi}}$$
$$-\frac{xC_2}{1260} + \frac{8x^{3/2}C_2}{3\sqrt{\pi}} + C_2 \int_0^1 txu_0(t)^4\, dt$$
$$- C_2D^{\alpha}u_0(x)$$
$$-(1 + C_1)D^{\alpha}u_1(x) = 0 \quad (48)$$

$$D^{\alpha}u_2(x) = \frac{2(3-4x)\sqrt{x}C_1}{3\sqrt{\pi}} + \frac{2(3-4x)\sqrt{x}C_1^2}{3\sqrt{\pi}} + \frac{2\sqrt{x}C_2}{\sqrt{\pi}} - \frac{8x^{3/2}C_2}{3\sqrt{\pi}}$$
$$-\frac{xC_1^2}{504189521813376000000(3 + 2\alpha)^2\, \Gamma(2 + \alpha)^8}$$
$$+\frac{2^{-1+2\alpha}x(-5+2\alpha(9+8\alpha))C_1^2}{31255875\sqrt{\pi}\,(11+8\alpha)(13+8\alpha)\, \Gamma(2+\alpha)^2\Gamma(4+2\alpha)} - \frac{x(C_1+C_1^2+C_2)}{5040947520000(3+2\alpha)\, \Gamma(2+\alpha)^4} \quad (49)$$

$$u_2(x) = -\frac{x^{1+\alpha}C_1^2}{504189521813376000000(3 + 2\alpha)^2\, \Gamma(2 + \alpha)^9}$$
$$-\frac{x^{\frac{1}{2}+\alpha}(-3 + 4x - 2\alpha)(C_1 + C_1^2 + C_2)}{2\, \Gamma\left(\frac{5}{2} + \alpha\right)}$$
$$+\frac{2^{-1+2\alpha}x^{1+\alpha}\big(-5 + 2\alpha(9 + 8\alpha)\big)C_1^2}{31255875\sqrt{\pi}\,(11 + 8\alpha)(13 + 8\alpha)\Gamma(2 + \alpha)^3\, \Gamma(4 + 2\alpha)}$$
$$-\frac{x^{1+\alpha}(C_1+C_1^2+C_2)}{5040947520000(3+2\alpha)\, \Gamma(2+\alpha)^5} \quad (50)$$

Now, one can construct $u(x)$ by adding zeroth-order, first-order and second-order solutions, and other higher order solutions if needed:

$$u(x) =$$
$$-\frac{x^{1+\alpha}}{1260\, \Gamma(2+\alpha)} + \frac{1}{504189521813376000000}x^{\alpha}\left(\begin{array}{c} -\frac{312558752^{7+2\alpha}\, x\, \Gamma\left(\frac{3}{2}+\alpha\right)C_1}{\sqrt{\pi}\,\, \Gamma(2+\alpha)^4\, \Gamma(4+2\alpha)]} \\ -\frac{xC_1^2}{(3+2\alpha)^2\, \Gamma(2+\alpha)^9} + \frac{98456006252^{13+2\alpha}x\big(-5+2\alpha(9+8\alpha)\big)C_1^2}{\sqrt{\pi}\,(11+8\alpha)(13+8\alpha)\, \Gamma(2+\alpha)^3\, \Gamma(4+2\alpha)} - \frac{1000188000x\left(C_1+C_1^2+C_2\right)}{(3+2\alpha)\, \Gamma(2+\alpha)^5} \\ -\frac{252094760906688000000\sqrt{x}(-3+4x-2\alpha)(C_1(2+C_1)+C_2)}{\Gamma\left(\frac{5}{2}+\alpha\right)} \end{array}\right)$$

(51)

By using the procedure mentioned in [16-22], one can calculate the auxiliary constants $C_1 = 0.0026392305850217956$ and $C_2 = -1.0050303752467258$.

$$u(x) = -0.00059702601433625x^{\frac{3}{2}}$$

$$+\frac{x^{\frac{1}{2}}\left(\begin{matrix}6.057124487768394\times10^{7}x+\\1.260152318847539\times10^{21}\sqrt{x}(-4.0+4x)\end{matrix}\right)}{5041895218133760000000} \quad (52)$$

Table 4: Absolute errors of Examples 2 & 3.

x	$\lvert u_{exact} - u_{OHAM}\rvert$ (Example 2)			$\lvert u_{exact} - u_{OHAM}\rvert$
	$\alpha = 1$	$\alpha = \frac{3}{4}$	$\alpha = \frac{1}{2}$	$\alpha = \frac{1}{2}$
0.0	0.0	0.0	0.0	0.0
0.1	0.0	3.469447×10^{-17}	1.665335×10^{-16}	4.075011×10^{-6}
0.2	0.0	8.326673×10^{-17}	1.110223×10^{-16}	1.259139×10^{-5}
0.3	0.0	1.387779×10^{-16}	1.110223×10^{-16}	4.454058×10^{-5}
0.4	0.0	5.551115×10^{-17}	2.220446×10^{-16}	8.982461×10^{-5}
0.5	0.0	2.220446×10^{-16}	0.0	1.473177×10^{-4}
0.6	1.110223×10^{-16}	1.110223×10^{-16}	0.0	2.162603×10^{-4}
0.7	0.0	1.110223×10^{-16}	0.0	2.960947×10^{-4}
0.8	2.220446×10^{-16}	5.551115×10^{-17}	4.440892×10^{-16}	3.863888×10^{-4}
0.9	0.0	2.220446×10^{-16}	0.0	4.867951×10^{-4}
1.0	2.220446×10^{-16}	4.440892×10^{-16}	0.0	5.970260×10^{-4}

Example 4

Consider the following nonlinear fractional order FIDE [15] with $q = 2$:

$$D^{\alpha}u(x) = 1 - \tfrac{1}{3}x + \int_0^1 x\big(u(t)\big)^2 dt, 0 \le x < 1,\ 0 < \alpha \le 1 \quad (53)$$

subject to the initial conditions $u(0) = 0$ and the exact solutions for $\alpha = 1, \frac{3}{4}, \frac{1}{2}$ are respectively as: $u(x) = x$, $\frac{4\,x^{3/4}}{3\,\Gamma\left(\frac{3}{4}\right)} - \frac{12\,x^{7/4}}{7\,\Gamma\left(\frac{3}{4}\right)} + \frac{189}{64}x^{7/4}\Gamma\left(\frac{3}{4}\right)$

$$-\frac{x^{7/4}\sqrt{\frac{1}{5}\left(-262144 - 8655360\left(\Gamma\left(\frac{3}{4}\right)\right)^2 + 8751645\left(\Gamma\left(\frac{3}{4}\right)\right)^4\right)}}{448\,\Gamma\left(\frac{3}{4}\right)},$$

$$\frac{2\sqrt{x}}{\sqrt{\pi}} - \frac{8\,x^{3/2}}{3\sqrt{\pi}} + \frac{3}{2}\sqrt{\pi}\,x^{3/2} - \frac{1}{6}\sqrt{\frac{-32-240\,\pi+81\,\pi^2}{\pi}}\,x^{3/2}.$$

OHAM formulation for fractional order FIDE is executed. A series of problems are generated. The expressions for zeroth order, first order, second order and third order problems and their solutions are given below:

- Zeroth order problem and its solution:

$$D^{\alpha}u_0(x) - 1 + \frac{x}{3} = 0 \quad (54)$$

$$u_0(x) = -\frac{x^{\alpha}(x-3(1+\alpha))}{3\,\Gamma(2+\alpha)} \quad (55)$$

- First order problem and its solution:

$$D^{\alpha}u_1(x) + 1 - \frac{x}{3} + C_1 - \frac{xC_1}{3} + C_1\int_0^1 xu_0(t)^2\,dt - (1 + C_1)D^{\alpha}u_0(x) = 0 \quad (56)$$

$$D^{\alpha}u_1(x) = -\frac{x(19+50\alpha+51\alpha^2+18\alpha^3)C_1}{9(3+8\alpha+4\alpha^2)\,\Gamma(2+\alpha)^2} \quad (57)$$

$$u_1(x) = -\frac{x^{1+\alpha}(19+\alpha(50+3\alpha(17+6\alpha)))C_1}{9(3+4\alpha(2+\alpha))\,\Gamma(2+\alpha)^3} \quad (58)$$

- Second order problem and its solution:

$$D^{\alpha}u_2(x) + C_1\int_0^1 2xu_0(t)u_1(t)\,dt + C_2$$

$$-\frac{xC_2}{3} + C_2\int_0^1 xu_0(t)^2\,dt - C_2D^{\alpha}u_0(x)$$

$$-(1 + C_1)D^{\alpha}u_1(x) = 0 \quad (59)$$

$$D^{\alpha}u_2(x) = \frac{\left(\begin{matrix}x\big(19+\alpha\big(50+3\alpha(17+6\alpha)\big)\big)\\(7C_1^2+6\alpha C_1^2-3(3+2\alpha)\,\Gamma(2+\alpha)^2(C_1+C_1^2+C_2))\end{matrix}\right)}{27(1+2\alpha)(3+2\alpha)^2\,\Gamma(2+\alpha)^4} \quad (60)$$

$$u_2(x) = \frac{\left(\begin{matrix}x^{1+\alpha}\big(19+\alpha\big(50+3\alpha(17+6\alpha)\big)\big)\\\big((7+6\alpha)C_1^2-3(3+2\alpha)\,\Gamma(2+\alpha)^2\big(C_1+C_1^2+C_2\big)\big)\end{matrix}\right)}{27(1+2\alpha)(3+2\alpha)^2\,\Gamma(2+\alpha)^5} \quad (61)$$

- Third order problem and its solution:

$$D^{\alpha}u_3(x) + C_1\int_0^1 x\big(u_1(t)^2 + 2u_0(t)u_2(t)\big)\,dt$$

$$+C_2\int_0^1 2xu_0(t)u_1(t)\,dt + C_3 - \frac{xC_3}{3}$$

$$+ C_3\int_0^1 xu_0(t)^2\,dt$$

$$-C_3D^{\alpha}u_0(x) - C_2D^{\alpha}u_1(x) - (1 + C_1)D^{\alpha}u_2(x) = 0 \quad (62)$$

$$D^{\alpha}u_3(x) = \frac{\left(\begin{matrix}x\big(19+\alpha\big(50+3\alpha(17+6\alpha)\big)\big)\\\left(\begin{matrix}6(1+2\alpha)(3+2\alpha)(7+6\alpha)\,\Gamma(2+\alpha)^2C_1\big(C_1+C_1^2+C_2\big)\\-68C_1^3-232\alpha C_1^3-255\alpha^2C_1^3-90\alpha^3C_1^3\\-9(1+2\alpha)(3+2\alpha)^2\,\Gamma(2+\alpha)^4\big(C_2+C_1\big((1+C_1)^2+2C_2\big)+C_3\big)\end{matrix}\right)\end{matrix}\right)}{(81(1+2\alpha)^2(3+2\alpha)^3\Gamma(2+\alpha)^6)} \quad (63)$$

$$u_3(x) = \frac{\left(\begin{matrix}x^{1+\alpha}\big(19+\alpha\big(50+3\alpha(17+6\alpha)\big)\big)\\\left(\begin{matrix}6(1+2\alpha)(3+2\alpha)(7+6\alpha)\,\Gamma(2+\alpha)^2C_1\big(C_1+C_1^2+C_2\big)\\-68C_1^3-232\alpha C_1^3-255\alpha^2C_1^3-90\alpha^3C_1^3\\-9(1+2\alpha)(3+2\alpha)^2\,\Gamma(2+\alpha)^4\big(C_2+C_1\big((1+C_1)^2+2C_2\big)+C_3\big)\end{matrix}\right)\end{matrix}\right)}{(81(1+2\alpha)^2(3+2\alpha)^3\,\Gamma(2+\alpha)^7)} \quad (64)$$

Now, one can build $u(x)$ by adding zeroth-order, first-order, second-order and third-order solutions, and other higher order solutions if needed:

$$u(x)=\frac{1}{81\,\Gamma(2+\alpha)^7}x^{\alpha}\left(\begin{array}{c}-27\big(x-3(1+\alpha)\big)\Gamma(2+\alpha)^6\\ +1/((1+2\alpha)^2(3+2\alpha)^3)\,x\left(19+\alpha\big(50+3\alpha(17+6\alpha)\big)\right)\\ \left(\begin{array}{c}-68C_1^3-232\alpha C_1^3-255\alpha^2C_1^3-90\alpha^3C_1^3\\ +3(1+2\alpha)(3+2\alpha)(7+6\alpha)\,\Gamma(2+\alpha)^2C_1(C_1(3+2C_1)+2C_2)\\ -9(1+2\alpha)(3+2\alpha)^2\,\Gamma(2+\alpha)^4(2C_2+C_1(3+C_1(3+C_1)+2C_2)+C_3)\end{array}\right)\end{array}\right) \quad (65)$$

By using the procedure mentioned in [16-22], one can calculate the auxiliary constants C_1, C_2 and C_3 presented in Table 5 below:

Table 5: Auxiliary constants for Example 2.

α	C_1	C_2	C_3
1.0	0.25406684306091626	−1.1276612628827687	0.4756610767937577
0.75	0.1565604724796715	−1.1429109146157186	0.396638845398501
0.5	−0.4477250663735092	−1.2374190371090146	−0.09309317308818105

Solutions for $\alpha=1,\frac{3}{4}$ and $\frac{1}{2}$ can be calculated by using auxiliary constants given in Table 5 respectively:

$u(x)=0.00009645061728395061x^1$
$(-1728.(-6.+x)+1728.0x),\quad \alpha=1$ (66)
$u(x)=0.0004434383281348996$
$x^{\frac{3}{4}}(-467.3716976318376(-5.25+x)+798.8518432553515\,x),\qquad \alpha=\frac{3}{4}$ (67)
$u(x)=0.0016829126811408262$
$x^{\frac{1}{2}}(-148.998179662681(-4.5+x)+629.8598193676037\,x),\qquad \alpha=\frac{1}{2}$ (68)

Table 6: Absolute errors of Example 4 for different fractional orders.

x	$\lvert u_{exact}-u_{OHAM}\rvert$		
	$\alpha=1$	$\alpha=\frac{3}{4}$	$\alpha=\frac{1}{2}$
0.0	0.0	0.0	0.0
0.1	1.387779×10^{-17}	4.510281×10^{-17}	5.551115×10^{-17}
0.2	2.775558×10^{-17}	9.714451×10^{-17}	1.110223×10^{-16}
0.3	0.0	8.326673×10^{-17}	1.110223×10^{-16}
0.4	5.551115×10^{-17}	1.665335×10^{-16}	2.220446×10^{-16}
0.5	1.110223×10^{-16}	2.775558×10^{-16}	2.220446×10^{-16}
0.6	1.110223×10^{-16}	1.110223×10^{-16}	8.881784×10^{-16}
0.7	1.110223×10^{-16}	4.440892×10^{-16}	6.661338×10^{-16}
0.8	2.220446×10^{-16}	0.0	6.661338×10^{-16}
0.9	2.220446×10^{-16}	4.440892×10^{-16}	8.881784×10^{-16}
1.0	2.220446×10^{-16}	6.661338×10^{-16}	8.881784×10^{-16}

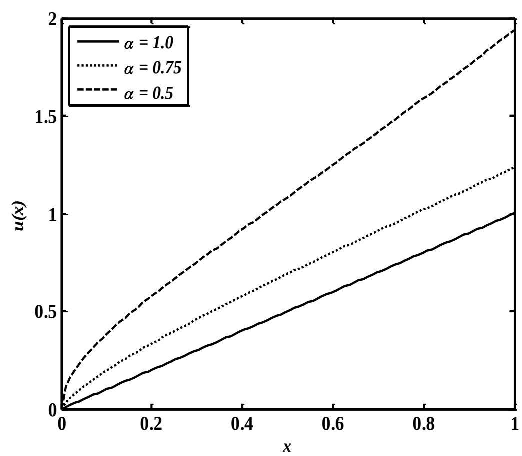

Fig 3. Approximate solutions of Example 4 for different values of α.

Tables 1, 3 and 5 show the values of auxiliary constants of examples 1, 2, & 4 for different values of α. Tables 2, 4 and 6 show the absolute errors of all examples, reflecting that the solutions of all the examples are in excellent agreement with the exact solutions and method is very trustworthy. Numerical solutions of examples 1, 2 and 4 for different values of α are presented in Figures 1, 2 and 3.

CONCLUSION

The objective of this work is to show the utility and usefulness of OHAM, formulated for the solutions of nonlinear Fredholm integro-differential equations of fractional order semi-numerically. It is suitable for solving the linear and nonlinear fractional order problems.

The illustrative examples demonstrate the powerfulness of the presented method. This method is simple in applicability, as it does not require discretization like numerical methods. Furthermore, this method provides a convenient way to control the convergence by optimally determining the auxiliary constants. Moreover, this method converges rapidly at lower order of approximations. Therefore, OHAM for fractional order Fredholm integro-differential equations shows its concealed strength and potential for the solution of nonlinear problems in real life applications. It is worth mentioning that the obtained results are in excellent agreement with exact solutions.

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