GENERALIZED SEVENTH ORDER KORTEWEG-DE VRIES EQUATIONS BY OPTIMAL HOMOTOPY AYSMPTOTIC METHOD

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ABSTRACT:*An advance newly technique known as Optimal Homotopy Asymptotic Method (OHAM) is applied on one of the type of time dependent partial differential equation named as generalize Korteweg- de Vries (gKdV) equation. To observe the standardization of this algorithm, seven ordergKdV is considered with different coefficients to form Lax's 7th order KdV (LsKdV) and 7th order Sawada Kotera (sSK) equations. It is observed that OHAM is effective and reliable during the process; approximate solution obtained by OHAM is compared with exact solution in order to check accuracy.*

Key Word: Optimal Homotopy Asymptotic Method, Korteweg- de Vries equation, Sawada Kotera equation,

1. INTRODUCTION:

Complex and higher order nonlinear differential equations and its analytical solutions and modeling are given priority on the basis of its importance in the field of physical science and engineering. In this category, considerable attention has been given toward the analysis of wave and soliton equations, which are essential feather of fluid mechanic and engineering etc. Several perturbation techniques have been introduced and developed to solve such strongly nonlinear equations although perturbation techniques produced great interest in researchers, so fast improvement was brought by these researchers though perturbation techniques have some limitations, the great drawback of these techniques is the presence of suppose parameter which caused uncertainty during the process of computation and often result tend toward divergence region specially in the case of strongly nonlinear equations. With the requirements of time, many other effective methods was developed which are unperturbed in natures, the approximate solutions obtained by these methods are reliable, like Homotopy Analysis Method[1], Homotopy Perturbation Method [2-3] etc. Recently a new semi analytical method named as Optimal Homotopy Asymptotic Method (OHAM) [4-8]was introduce by Marincaand Harisanu, the background of OHAM is seem to be impressed by HPM tool, but OHAM is considered modified then other all methods. Advancement occurred in the field of nonlinear differential equations by using this tool as a treatment. The main consideration of this paper is to implement OHAM on the following Korteweg- de Vries equations which are as under

$$w_\tau + 35w_\xi^4 + 70(w_\xi^2 w_{\xi\xi\xi} + w_\xi w_{\xi\xi}^2) + 7(2w_\xi w_{\xi\xi\xi\xi\xi} + 3w_{\xi\xi\xi}^2 + 4w_{\xi\xi} w_{\xi\xi\xi\xi}) + w_{\xi\xi\xi\xi\xi\xi\xi} = 0, \quad \text{(a)}$$

$$w_\tau + 63w_\xi^4 + 63(2w_\xi^2 w_{\xi\xi\xi} + w_\xi w_{\xi\xi}^2) + 21(w_\xi w_{\xi\xi\xi\xi\xi} + w_{\xi\xi\xi}^2 + w_{\xi\xi} w_{\xi\xi\xi\xi}) + w_{\xi\xi\xi\xi\xi\xi\xi} = 0, \quad \text{(b)}$$

Equation (a) and (b) are known as Lax's 7th order KdV [15] and 7th order Sk [16] equations respectively. Names of both equations are changed due to the difference in coefficients of (a) and (b). The OHAM extension and modification along with applications can be seen in [10-25]

2. Basic mathematical theory of Optimal Homotopy Asymptotic Method

OHAM is used to the following equation:

$$\mathcal{L}\left(w(\xi,\tau)\right) + \mathcal{N}\left(w(\xi,\tau)\right) + g(\xi,\tau) = 0, \quad \mathcal{B}\left(w, w_\tau\right) = 0 \quad (2.1)$$

Where ξ and τ are independent variables, $\mathcal{L}$ is a linear operator, $w(\xi,\tau)$ is consider as an unknown function, $g(x)$ is consider as known function, the operator $\mathcal{N}(w(\xi,\tau))$ is nonlinear and $\mathcal{B}(w, w_\tau)$ is a taken as boundary operator. Now OHAM can constructs an optimal homotopy, $\varphi(\xi,q)$: $\mathcal{B}$ ×[0,1]→ R that satisfies the following homotopy equation

$$(1-q)\left[\mathcal{L}\left(\varphi(\xi,\tau;q)\right)+g(\xi,\tau)\right]=\mathcal{H}(q)\left[\mathcal{L}\left(\varphi(\xi,\tau;q)\right)+g(\xi,\tau)+\mathcal{N}\left(\varphi(\xi,\tau;q)\right)\right], \quad \mathcal{B}\left(\varphi(\xi,\tau;q)\right) = 0 \quad (2.2)$$

Where $\varphi(\xi,q)$ is an unknown function. $q \in [0, 1]$ is an embedding parameter, while $\mathcal{H}(q)$ is a nonzero auxiliary function for $q \neq 0$ and $\mathcal{H}(0) = 0$, when $q=0$ and $q=1$, it holds that

$$\varphi(\xi,0) = w_0(\xi), \quad \varphi(\xi,1) = w_1(\xi). \quad (2.3$$

By increasing q from 0 to 1, the solution $\varphi(\xi,q)$ varies from $w_0(\xi,\tau)$ to the final solution $w(\xi,\tau)$, where

$w_0(\xi,\tau)$ is evaluated from eqn (2.2) for $q=0$:

$$\mathcal{L}\left(w_0(\xi,\tau)\right) + g(\xi,\tau) = 0, \quad \mathcal{B}\left(w_0, w_\xi\right) = 0. \tag{2.4}$$

The auxiliary function $\mathcal{H}(q)$ is taken in the form of

$$\mathcal{H}(q) = qK_1 + q^2K_2 + ...$$

Where auxiliary constants K_1 and K_2 can be determine in this manner. Next the method expands $\varphi(\xi,\tau;q:K_i)$ in a Taylors series about the parameter q as [2]

$$\varphi(\xi,\tau;\, q:K_i) = w_o(\xi,\tau) + \sum_{k\ge 1} w_k(\xi,\tau;K_i)q^k,\, i = 1,2,... \tag{2.5}$$

auxiliary constants $K_1, K_2, ...$ causes convergence of the series (2.5) and if it converges at $q=1$ then the following form, we have, see [7]

$$\varphi(\xi,\tau;K_i) = w_0(\xi,\tau) + \sum_{k=1}^{\mathcal{M}} w_k(\xi,\tau;K_i)q^k,\, i = 1,2,...,m \tag{2.6}$$

By substituting Eqn (2.5) into Homotopy formula (2.2) and equating like powers of q the original nonlinear problem is converted into a sequence of linear problems, we obtained zeroth order (2.4), first order, secondorder as under.

$$\mathcal{L}\left(w_1(\xi,\tau)\right) = K_1\mathcal{N}_0\left(w_0(\xi,\tau)\right), \quad \mathcal{B}\left(w_0, w_\xi\right) = 0 \tag{2.7}$$

$$\left\{\begin{array}{l} \mathcal{L}\left(w_2(\xi,\tau)\right) - \mathcal{L}\left(w_1(\xi,\tau)\right) = K_2\mathcal{N}_0\left(w_0(\xi,\tau)\right) + K_1\left(\mathcal{L}\left(w_1(\xi,\tau)\right) + \mathcal{N}_1\left(w_0(\xi,\tau)\right), w_1(\xi,\tau)\right) \\ \mathcal{B}\left(w_0, w_\xi\right) = 0 \end{array}\right\} \tag{2.8}$$

$$\left\{\begin{array}{l} \mathcal{L}\left(w_k(\xi,\tau)\right) - \mathcal{L}\left(w_{k-1}(\xi,\tau)\right) = K_k\mathcal{N}_0\left(w_0(\xi,\tau)\right) + \\ \sum_{i=1}^{k-1} K_i\left(\mathcal{L}\left(w_{k-1}(\xi,\tau)\right) + \mathcal{N}_{k-1}\left(w_0(\xi,\tau), w_1(\xi,\tau), ..., w_{k-i}(\xi,\tau)\right)\right) \\ \mathcal{B}\left(w_0, w_\xi\right) = 0 \end{array}\right\} k = 2,3,4,... \tag{2.9}$$

The resulting linear problems can now be solved and their solutions are used to construct k^{th} order solution that involves K_i of the original problem through equation (2.6). Then by inserting equation (2.6) into equation (2.1), they results the following residual:

$$R(\xi,\tau;K_i) = \mathcal{L}(\tilde{w}^{(m)}(\xi,\tau;K_i)) + g(\xi,\tau) + \mathcal{N}(\tilde{w}^{(m)}(\xi,\tau;K_i)) \tag{2.10}$$

When $R(\xi,K_i)=0$, for some values of K_i then $\mathcal{L}(\tilde{w}^{(m)}(\xi,\tau;K_i))$ will coincide with the exact solution. However, this does not happen in general especially in nonlinear problems. Therefore, optimal values of the auxiliary constants $K_1, K_2, ..., K_n$ are calculated aimed at minimizing the following functional J , see [2]

$$J(K_1,K_2,...,K_n) = \int_0^1 R^2(\xi,K_1,K_2,...,K_m)d\xi \quad . \tag{2.11}$$

Therefore, the unknown constants $K_i(i=1,2,...m)$ can be optimally identified from the following conditions, see [2]

$$\frac{\partial J}{\partial K_1} = \frac{\partial J}{\partial K_2} = ... = \frac{\partial J}{\partial K_m} = 0. \tag{2.12}$$

With these known values of the auxiliary constants the approximate solution (2.6) is now well determined.

3. Application of OHAM by Numerical Examples

To analysis the reliability of OHAM, Two models of Lax's7thorder KdV and 7thSK equations are presented by OHAM.

Model 1: ConsideredEqn (a) with the initial condition [9]:

$$w_0(\xi,0) = 2k^2\left(\sec h^2(k\xi)\right), \tag{3.1}$$

Corresponding exact solution is

$$w(\xi,\tau) = 2k^2\left(\sec h^2\left(k\left(\xi - 64k^6\tau\right)\right)\right). \tag{3.2}$$

Where k isarbitrary constant.

Zeroth Order Problem

$$\frac{\partial w_0(\xi,\tau)}{\partial \tau} = 0, \quad (3.3)$$

$$w_0(\xi,0) = 2k^2\left(\sec h^2(k\xi)\right), \quad (3.4)$$

Its solution is

$$w_0(\xi,\tau) = 2k^2\left(\sec h^2(k\tau)\right). \quad (3.5)$$

First Order Problem

$$\frac{\partial w_1(\xi,\tau)}{\partial \tau} = (1+K_1)\frac{\partial w_0(\xi,\tau)}{\partial \tau} + 35K_1\left(\frac{\partial w_0(\xi,\tau)}{\partial \xi}\right)^4 + 70K_1\left(\frac{\partial w_0(\xi,\tau)}{\partial \xi}\right) \left(\frac{\partial^2 w_0(\xi,\tau)}{\partial \xi^2}\right)^2 + 70K_1\left(\frac{\partial w_0(\xi,\tau)}{\partial \xi}\right)^2\left(\frac{\partial^3 w_0(\xi,\tau)}{\partial \xi^3}\right) + 21K_1\left(\frac{\partial^3 w_0(\xi,\tau)}{\partial \xi^3}\right)^2 \quad (3.6)$$
$$+28K_1\left(\frac{\partial^2 w_0(\xi,\tau)}{\partial \xi^2}\right)\left(\frac{\partial^4 w_0(\xi,\tau)}{\partial \xi^4}\right) + 14K_1\left(\frac{\partial w_0(\xi,\tau)}{\partial \xi}\right)\left(\frac{\partial^5 w_0(\xi,\tau)}{\partial \xi^5}\right) + K_1\left(\frac{\partial^7 w_0(\xi,\tau)}{\partial \xi^7}\right),$$

$$w_1(\xi,0) = 0. \quad (3.7)$$

Its solution is

$$w_1(\xi,\tau,K_1) = 128\tau K_1\begin{pmatrix} -28k^{10}\sec h^{10}(k\xi) + 124k^9\sec h^8(k\xi)\tanh(k\xi) - \\ 35k^{11}\sec h^{10}(k\xi)\tanh(k\xi) + 616k^{10}\sec h^8(k\xi)\tan \\ \text{h}^2(k\xi) - 384k^9\sec h^6(k\xi)\tanh^3(k\xi) + 420k^{11}\sec \\ h^8(k\xi)\tanh^3(k\xi) - 868k^{10}\sec h^6(k\xi)\tanh^4(k\xi) + \\ 70k^{12}\sec h^8(k\xi)\tanh^4(k\xi) + 120k^9\sec h^4(k\xi)\tan \\ \text{h}^5(k\xi) - 280k^{11}\sec h^6(k\xi)\tanh^5(k\xi) + 126k^{10}\sec \\ h^4(k\xi)\tanh^6(k\xi) - 2k^9\sec h^2(k\xi)\tanh^7(k\xi) \end{pmatrix} \quad (3.8)$$

Second Order Problem

$$\frac{\partial w_2(\xi,\tau)}{\partial \tau} = \frac{\partial w_0(\xi,\tau)}{\partial \tau} + (1+K_1)\frac{\partial w_1(\xi,\tau)}{\partial \tau} + 35K_2\left(\frac{\partial w_0(\xi,\tau)}{\partial \xi}\right)^4 + 140K_1\left(\frac{\partial w_1(\xi,\tau)}{\partial \xi}\right)$$
$$\left(\frac{\partial w_0(\xi,\tau)}{\partial \xi}\right)^3 + 70K_2\left(\frac{\partial w_0(\xi,\tau)}{\partial \xi}\right)\left(\frac{\partial^2 w_0(\xi,\tau)}{\partial \xi^2}\right)^2 + 70K_1\left(\frac{\partial w_1(\xi,\tau)}{\partial \xi}\right)\left(\frac{\partial^2 w_0(\xi,\tau)}{\partial \xi^2}\right)^2 +$$
$$140K_1\left(\frac{\partial w_0(\xi,\tau)}{\partial \xi}\right)\left(\frac{\partial^2 w_0(\xi,\tau)}{\partial \xi^2}\right)\left(\frac{\partial^2 w_1(\xi,\tau)}{\partial \xi^2}\right) + 70K_2\left(\frac{\partial w_0(\xi,\tau)}{\partial \xi}\right)^2\left(\frac{\partial^3 w_0(\xi,\tau)}{\partial \xi^3}\right) +$$
$$140K_1\left(\frac{\partial w_0(\xi,\tau)}{\partial \xi}\right)\left(\frac{\partial w_1(\xi,\tau)}{\partial \xi}\right)\left(\frac{\partial^3 w_0(\xi,\tau)}{\partial \xi^3}\right) + 21K_2\left(\frac{\partial^3 w_0(\xi,\tau)}{\partial \xi^3}\right)^2 + 70K_1\left(\frac{\partial w_0(x,t)}{\partial \xi}\right)^2$$
$$\left(\frac{\partial^3 w_1(\xi,\tau)}{\partial \xi^3}\right) + 42K_1\left(\frac{\partial^3 w_0(\xi,\tau)}{\partial \xi^3}\right)\left(\frac{\partial^3 w_1(\xi,\tau)}{\partial \xi^3}\right) + 28K_2\left(\frac{\partial^2 w_0(\xi,\tau)}{\partial \xi^2}\right)\left(\frac{\partial^4 w_0(\xi,\tau)}{\partial \xi^4}\right)$$
$$+28K_1\left(\frac{\partial^2 w_1(\xi,\tau)}{\partial \xi^2}\right)\left(\frac{\partial^4 w_0(\xi,\tau)}{\partial \xi^4}\right) + 28K_1\left(\frac{\partial^2 w_0(\xi,\tau)}{\partial \xi^2}\right)\left(\frac{\partial^4 w_1(\xi,\tau)}{\partial \xi^4}\right) + 14K_2\left(\frac{\partial w_0(\xi,\tau)}{\partial \xi}\right)$$
$$\left(\frac{\partial^5 w_0(\xi,\tau)}{\partial \xi^5}\right) + 14K_1\left(\frac{\partial w_1(\xi,\tau)}{\partial \xi}\right)\left(\frac{\partial^5 w_0(\xi,\tau)}{\partial \xi^5}\right) + 14K_1\left(\frac{\partial w_0(\xi,\tau)}{\partial \xi}\right)\left(\frac{\partial^5 w_1(\xi,\tau)}{\partial \xi^5}\right)$$
$$K_2\left(\frac{\partial^7 w_0(\xi,\tau)}{\partial \xi^7}\right) + K_1\left(\frac{\partial^7 w_1(\xi,\tau)}{\partial \xi^7}\right), \quad (3.9)$$

$$w_2(\xi,0) = 0. \quad (3.10)$$

Its solution is

$$
\begin{aligned}
w_2(\xi,\tau;K_1,K_2) = {} & -128(28k^{10}\tau \operatorname{sech}^{10}(k\xi)(K_1+K_1^2+K_2)+14873104K_1^2 k^{16}\tau^2 \operatorname{sech}^{16}(k\xi)-26434240K_1^2k^{18}\tau^2 \operatorname{sech}^{18}(k\xi)+113680K_1^2k^{20}\tau^2 \operatorname{sech}^{20}(k\xi) \\
& -124k^9\tau \operatorname{sech}^8(k\xi)\tanh(k\xi)(K_1+K_1^2+K_2)+35K_1k^{11}\tau \operatorname{sech}^{10}(k\xi)\tanh(k\xi)(K_1+K_1^2+K_2)+506862720K_1^2k^{17}\tau^2 \operatorname{sech}^{16}(k\xi)\tanh(k\xi) \\
& -57529920K_1^2k^{19}\tau^2 \operatorname{sech}^{18}(k\xi)\tanh(k\xi)-616K_1k^{10}\tau \operatorname{sech}^8(k\xi)\tanh^2(k\xi)(K_1+K_1^2+K_2)-700801664K_1^2k^{16}\tau^2 \operatorname{sech}^{14}(k\xi)\tanh^2(k\xi) \\
& +1800411424K_1^2k^{18}\tau^2 \operatorname{sech}^{16}(k\xi)\tanh^2(k\xi)-23402400K_1^2k^{20}\tau^2 \operatorname{sech}^{18}(k\xi)\tanh^2(k\xi)+384k^9\tau \operatorname{sech}^6(k\xi)\tanh^3(k\xi)(K_1+K_1^2+K_2) \\
& -420k^{11}\tau \operatorname{sech}^8(k\xi)\tanh^3(k\xi)(K_1+K_1^2+K_2)-7652342208K_1^2k^{17}\tau^2 \operatorname{sech}^{14}(k\xi)\tanh^3(k\xi)+1467630080K_1^2k^{19}\tau^2 \operatorname{sech}^{16}(k\xi)\tanh^3(k\xi) \\
& -2195200K_1^2k^{21}\tau^2 \operatorname{sech}^{18}(k\xi)\tanh^3(k\xi)+868k^{10}\tau \operatorname{sech}^6(k\xi)\tanh^4(k\xi)(K_1+K_1^2+K_2)-70k^{12}\tau \operatorname{sech}^8(k\xi)\tanh^4(k\xi)(K_1+K_1^2+K_2) \\
& +3600455232K_1^2k^{16}\tau^2 \operatorname{sech}^{12}(k\xi)\tanh^4(k\xi)-13511852032K_1^2k^{18}\tau^2 \operatorname{sech}^{14}(k\xi)\tanh^4(k\xi)+382811520K_1^2k^{20}\tau^2 \operatorname{sech}^{16}(k\xi)\tanh^4(k\xi) \\
& -120k^9\tau \operatorname{sech}^4(k\xi)\tanh^5(k\xi)(K_1+K_1^2+K_2)+280k^{11}\tau \operatorname{sech}^6(k\xi)\tanh^5(k\xi)(K_1+K_1^2+K_2)+22201243008K_1^2k^{17}\tau^2 \operatorname{sech}^{12}(k\xi)\tanh^5(k\xi) \\
& -6805388800K_1^2k^{19}\tau^2 \operatorname{sech}^{14}(k\xi)\tanh^5(k\xi)+37004800K_1^2k^{21}\tau^2 \operatorname{sech}^{16}(k\xi)\tanh^5(k\xi)-126k^{10}\tau \operatorname{sech}^4(k\xi)\tanh^6(k\xi)(K_1+K_1^2+K_2) \\
& -4368350080K_1^2k^{16}\tau^2 \operatorname{sech}^{10}(k\xi)\tanh^6(k\xi)+24832670912K_1^2k^{18}\tau^2 \operatorname{sech}^{12}(k\xi)\tanh^6(k\xi)-1255654400K_1^2k^{20}\tau^2 \operatorname{sech}^{14}(k\xi)\tanh^6(k\xi) \\
& +1254400K_1^2k^{22}\tau^2 \operatorname{sech}^{16}(k\xi)\tanh^6(k\xi)+2k^9\tau \operatorname{sech}^2(k\xi)\tanh^7(k\xi)(K_1+K_1^2+K_2)-17610018048K_1^2k^{17}\tau^2 \operatorname{sech}^{10}(k\xi)\tanh^7(k\xi) \\
& +8395376640K_1^2k^{19}\tau^2 \operatorname{sech}^{12}(k\xi)\tanh^7(k\xi)-93452800K_1^2k^{21}\tau^2 \operatorname{sech}^{14}(k\xi)\tanh^7(k\xi)+1423223840K_1^2k^{16}\tau^2 \operatorname{sech}^8(k\xi)\tanh^8(k\xi) \\
& -13228890752K_1^2k^{18}\tau^2 \operatorname{sech}^{10}(k\xi)\tanh^8(k\xi)+1024343040K_1^2k^{20}\tau^2 \operatorname{sech}^{12}(k\xi)\tanh^8(k\xi)-2508800K_1^2k^{22}\tau^2 \operatorname{sech}^{14}(k\xi)\tanh^8(k\xi) \\
& +3912615168K_1^2k^{17}\tau^2 \operatorname{sech}^8(k\xi)\tanh^9(kx)-2880640000K_1^2k^{19}\tau^2 \operatorname{sech}^{10}(k\xi)\tanh^9(k\xi)+41395200K_1^2k^{21}\tau^2 \operatorname{sech}^{12}(k\xi)\tanh^9(k\xi) \\
& -107454336K_1^2k^{16}\tau^2 \operatorname{sech}^6(k\xi)\tanh^{10}(k\xi)+1906080512K_1^2k^{18}\tau^2 \operatorname{sech}^8(k\xi)\tanh^{10}(k\xi)-180633600K_1^2k^{20}\tau^2 \operatorname{sech}^{10}(k\xi)\tanh^{10}(k\xi) \\
& -195500929K_1^2k^{17}\tau^2 \operatorname{sech}^6(k\xi)\tanh^{11}(k\xi)+213032960K_1^2k^{19}\tau^2 \operatorname{sech}^8(k\xi)\tanh^{11}(k\xi)+1047616K_1^2k^{16}\tau^2 \operatorname{sech}^4(k\xi)\tanh^{12}(k\xi) \\
& -50308608K_1^2k^{18}\tau^2 \operatorname{sech}^6(k\xi)\tanh^{12}(k\xi)+1048320K_1^2k^{17}\tau^2 \operatorname{sech}^4(k\xi)\tanh^{13}(k\xi)-128K_1^2k^{16}\tau^2 \operatorname{sech}^2(k\xi)\tanh^{14}(k\xi)).
\end{aligned}
\tag{3.12}
$$

By adding (3.3) to (3.12) to obtained approximate solution in the form of

$$\tilde{w}(\xi,\tau;K_1,K_2) = w_0(\xi,\tau) + w_1(\xi,\tau;K_1) + w_2(\xi,\tau;K_1,K_2). \tag{3.13}$$

Using least square method to compute the constant values K_1 and K_2 which are as below

$$K_1 = -0.769634099690644, \quad K_2 = 0.5206897343781481. \tag{3.14}$$

2^{nd}order approximate solution of OHAM is evaluated by putting constants values K_1 and K_2.

Table 1(a): Comparison of Absolute errors of HPM and OHAM with exact solution for $\tau = 0.1$.

ξ	Absolute error HPM[17]	Absolute error OHAM
0.1	1.523567×10^{-4}	7.09824×10^{-9}
0.2	3.046766×10^{-4}	1.19166×10^{-11}
0.3	4.569652×10^{-4}	7.13055×10^{-9}
0.4	6.092284×10^{-4}	1.42267×10^{-8}
0.5	7.614719×10^{-4}	2.12695×10^{-8}

Table 1(b): Comparison of Absolute errors of HPM and OHAM with exact solution for $\tau = 0.3$

ξ	Absolute error HPM [17]	Absolute error OHAM
0.1	1.51770×10^{-4}	1.49674×10^{-8}
0.2	3.03446×10^{-4}	5.96959×10^{-8}
0.3	4.55033×10^{-4}	2.68558×10^{-8}
0.4	6.06538×10^{-4}	4.75981×10^{-8}
0.5	7.57965×10^{-4}	6.81050×10^{-8}

Table 1(c): Comparison of Absolute errors of HPM and OHAM with exact solution for $\tau = 0.5$.

ξ	Absolute error HPM [17]	Absolute error OHAM
0.1	1.50294×10^{-4}	1.37177×10^{-8}
0.2	3.00437×10^{-4}	2.12036×10^{-8}
0.3	4.50436×10^{-4}	5.59122×10^{-8}
0.4	6.00296×10^{-4}	9.02518×10^{-8}
0.5	7.50021×10^{-4}	1.24071×10^{-7}

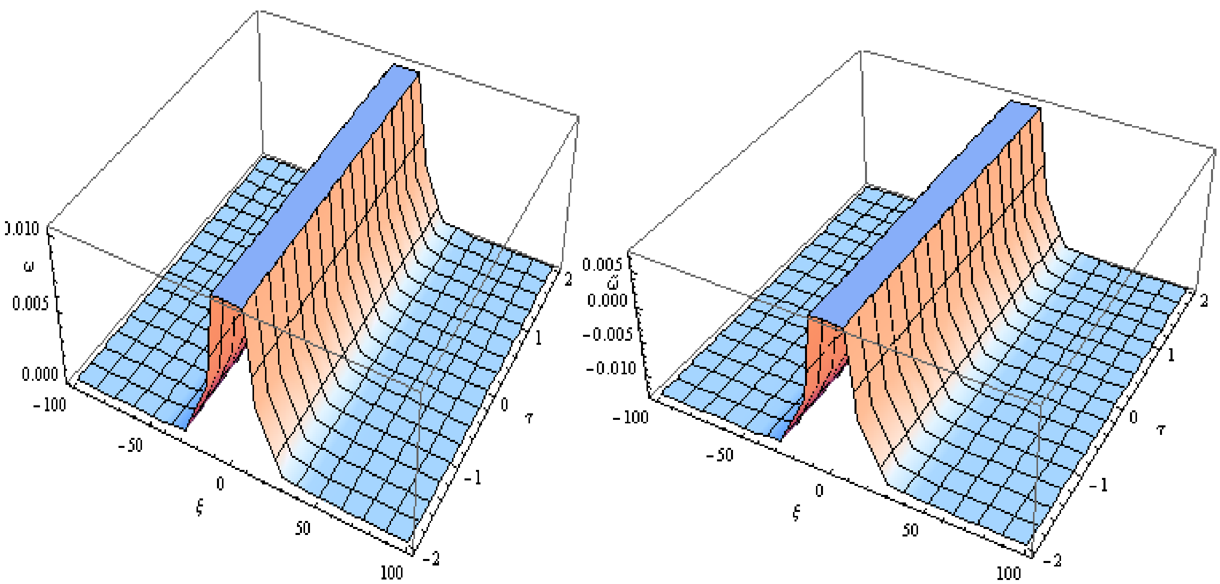

Fig 1(a): The graphs of exact solution in comparison with numerical results for $\tilde{\omega}(\xi,\tau)$ through OHAM of model 1.

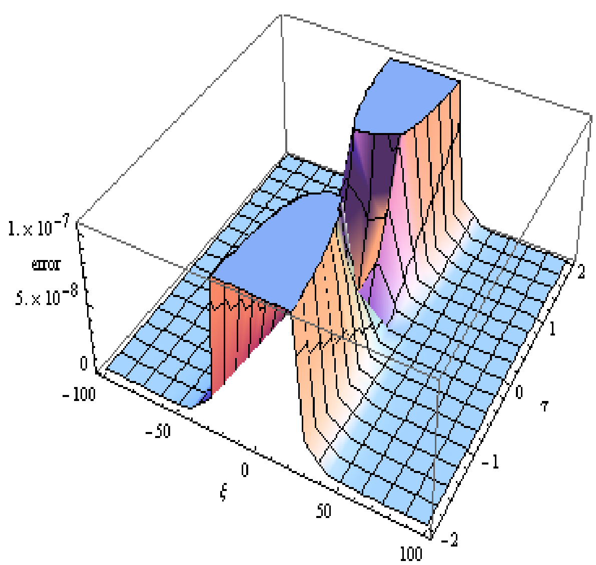

Fig 1(b): Graph of absolute error of OHAM with exact solution of model 1.

Model 2: ConsideredEqn (b)withthe initial condition [9]:

$$w(\xi,0)=\frac{4}{3}k^2\left(2-3\left(\tan^2(k\xi)\right)\right), \quad (3.15)$$

with exact solution is

$$w(\xi,\tau)=.75k^2\left(2-3\tan h^2\left(k\left(\xi-\frac{256k^6\tau}{3}\right)\right)\right).$$

(3.16)

Where k is arbitrary constant.

By substituting Eqn(a) with corresponding initial condition in Eqn (2.2), Zeorth, first, second order can be obtained

(3.19)

Zeroth Order Problem

$$\frac{\partial w_0(\xi,\tau)}{\partial\tau}=0, \quad (3.17)$$

$$w_0(\xi,0)=\frac{4}{3}k^2\left(2-3\left(\tan^2(k\xi)\right)\right), \quad (3.18)$$

Its solution is

$$w_0(\xi,\tau)=\frac{4}{3}k^2\left(2-3\left(\tan^2(k\xi)\right)\right).$$

First Order Problem

$$\frac{\partial w_1(\xi,\tau)}{\partial\tau}=(1+K_1)\frac{\partial w_0(\xi,\tau)}{\partial\tau}+63K_1\left(\frac{\partial w_0(\xi,\tau)}{\partial\xi}\right)^4+63K_1\left(\frac{\partial w_0(\xi,\tau)}{\partial\xi}\right)\left(\frac{\partial^2 w_0(\xi,\tau)}{\partial\xi^2}\right)^2$$
$$+126K_1\left(\frac{\partial w_0(\xi,\tau)}{\partial\xi}\right)^2\left(\frac{\partial^3 w_0(\xi,\tau)}{\partial\xi^3}\right)+21K_1\left(\frac{\partial^3 w_0(\xi,\tau)}{\partial\xi^3}\right)^2+21K_1\left(\frac{\partial^2 w_0(\xi,\tau)}{\partial\xi^2}\right) \quad (3.20)$$
$$\left(\frac{\partial^4 w_0(\xi,\tau)}{\partial\xi^4}\right)+21K_1\left(\frac{\partial w_0(\xi,\tau)}{\partial\xi}\right)\left(\frac{\partial^5 w_0(\xi,\tau)}{\partial\xi^5}\right)+K_1\left(\frac{\partial^7 w_0(\xi,\tau)}{\partial\xi^7}\right),$$

$$w_1(\xi,0)=0. \quad (3.21)$$

Its solution is

$$w_1(\xi,\tau,K_1)=256\tau K_1\begin{pmatrix}-42k^{10}\sec h^{10}(k\xi)+124k^9\sec h^8(k\xi)\tanh(k\xi)-126k^{11}\\ \sec h^{10}(k\xi)\tanh(k\xi)+1365k^{10}\sec h^8(k\xi)\tanh^2(k\xi)-384\\ k^9\sec h^6(k\xi)\tanh^3(k\xi)+2520k^{11}\sec h^8(k\xi)\tanh^3(k\xi)-\\ 1932k^{10}\sec h^6(k\xi)\tanh^4(k\xi)+1008k^{12}\sec h^8(k\xi)\tanh^4(k\\ \xi)+120k^9\sec h^4(k\xi)\tanh^5(k\xi)-1512k^{11}\sec h^6(k\xi)\tanh^5(k\\ \xi)+252k^{10}\sec h^4(k\xi)\tanh^6(k\xi)-2k^9\sec h^2(k\xi)\tanh^7(k\xi)\end{pmatrix} \quad (3.22)$$

Second Order Problem

$$\frac{\partial w_2(\xi,\tau)}{\partial\tau}=\frac{\partial w_0(\xi,\tau)}{\partial\tau}+(1+K_1)\frac{\partial w_1(\xi,\tau)}{\partial\tau}+63K_2\left(\frac{\partial w_0(\xi,\tau)}{\partial\xi}\right)^4+252K_1\left(\frac{\partial w_1(\xi,\tau)}{\partial\xi}\right)$$
$$\left(\frac{\partial w_0(\xi,\tau)}{\partial\xi}\right)^3+63K_2\left(\frac{\partial w_0(\xi,\tau)}{\partial\xi}\right)\left(\frac{\partial {w_0}^2(\xi,\tau)}{\partial\xi^2}\right)^2+63K_1\left(\frac{\partial w_1(\xi,\tau)}{\partial\xi}\right)\left(\frac{\partial {w_0}^2(\xi,\tau)}{\partial\xi^2}\right)^2+$$
$$126K_1\left(\frac{\partial w_0(\xi,\tau)}{\partial\xi}\right)\left(\frac{\partial^2 w_0(\xi,\tau)}{\partial\xi^2}\right)\left(\frac{\partial^2 w_1(\xi,\tau)}{\partial\xi^2}\right)+126K_2\left(\frac{\partial w_0(\xi,\tau)}{\partial\xi}\right)^2\left(\frac{\partial^3 w_0(\xi,\tau)}{\partial\xi^3}\right)+$$
$$252K_1\left(\frac{\partial w_0(\xi,\tau)}{\partial\xi}\right)\left(\frac{\partial w_1(\xi,\tau)}{\partial\xi}\right)\left(\frac{\partial^3 w_0(\xi,\tau)}{\partial\xi^3}\right)+21K_2\left(\frac{\partial^3 w_0(\xi,\tau)}{\partial\xi^3}\right)^2+126K_1\left(\frac{\partial w_0(x,t)}{\partial\xi}\right)^2$$
$$\left(\frac{\partial^3 w_1(\xi,\tau)}{\partial\xi^3}\right)+42K_1\left(\frac{\partial^3 w_0(\xi,\tau)}{\partial\xi^3}\right)\left(\frac{\partial^3 w_1(\xi,\tau)}{\partial\xi^3}\right)+21K_2\left(\frac{\partial^2 w_0(\xi,\tau)}{\partial\xi^2}\right)\left(\frac{\partial^4 w_0(\xi,\tau)}{\partial\xi^4}\right)$$
$$+21K_1\left(\frac{\partial^2 w_1(\xi,\tau)}{\partial\xi^2}\right)\left(\frac{\partial^4 w_0(\xi,\tau)}{\partial\xi^4}\right)+21K_1\left(\frac{\partial^2 w_0(\xi,\tau)}{\partial\xi^2}\right)\left(\frac{\partial^4 w_1(\xi,\tau)}{\partial\xi^4}\right)+21K_2\left(\frac{\partial w_0(\xi,\tau)}{\partial\xi}\right)$$
$$\left(\frac{\partial^5 w_0(\xi,\tau)}{\partial\xi^5}\right)+21K_1\left(\frac{\partial w_1(\xi,\tau)}{\partial\xi}\right)\left(\frac{\partial^5 w_0(\xi,\tau)}{\partial\xi^5}\right)+21K_1\left(\frac{\partial w_0(\xi,\tau)}{\partial\xi}\right)\left(\frac{\partial^5 w_1(\xi,\tau)}{\partial\xi^5}\right) \quad (3.23)$$
$$K_2\left(\frac{\partial^7 w_0(\xi,\tau)}{\partial\xi^7}\right)+K_1\left(\frac{\partial^7 w_1(\xi,\tau)}{\partial\xi^7}\right),$$
$$w_2(\xi,0)=0. \quad (3.24)$$

Its solution is

$$w_2(\xi,\tau;K_1,K_2)=-256(42k^{10}\tau\sec h^{10}(k\xi)(K_1+K_1^2+K_2)+14873104{K_1}^2k^{16}\tau^2$$
$$\sec h^{16}(k\xi)-132757632{K_1}^2k^{18}\tau^2\sec h^{18}(k\xi)+2286144{K_1}^2k^{20}\tau^2\sec h^{20}(k\xi)$$
$$-124k^9\tau\sec h^8(k\xi)\tanh(k\xi)(K_1+K_1^2+K_2)+126k^{11}\tau\sec h^{10}(k\xi)\tanh(k\xi)(K_1+$$
$$K_1^2+K_2)+1098840960{K_1}^2k^{17}\tau^2\sec h^{16}(k\xi)\tanh(k\xi)-690923520{K_1}^2k^{19}\tau^2$$
$$\sec h^{18}(k\xi)\tanh(k\xi)-1365k^{10}\tau\sec h^8(k\xi)\tanh^2(k\xi)(K_1+K_1^2+K_2)-700801$$
$$664{K_1}^2k^{16}\tau^2\sec h^{14}(k\xi)\tanh^2(k\xi)+9693472320{K_1}^2k^{16}\tau^2\sec h^{16}(k\xi)\tanh^2(k$$
$$\xi)-697527936{K_1}^2k^{20}\tau^2\sec h^{18}(k\xi)\tanh^2(k\xi)+384k^9\tau\sec h^6(k\xi)\tanh^3(k\xi)$$
$$(K_1+{K_1}^2+K_2)-2520k^{11}\tau\sec h^8(k\xi)\tanh^3(k\xi)(K_1+{K_1}^2+K_2)-16868980$$
$$800{K_1}^2k^{17}\tau^2\sec h^{14}(k\xi)\tanh^3(k\xi)+19886146560{K_1}^2k^{19}\tau^2\sec h^{16}(k\xi)\tanh^3$$
$$(k\xi)-162570240{K_1}^2k^{21}\tau^2\sec h^{18}(k\xi)\tanh^3(k\xi)+1932k^{10}t\sec h^6(k\xi)\tanh^4$$
$$(k\xi)(K_1+{K_1}^2+K_2)-1008k^{12}\tau\sec h^8(k\xi)\tanh^4(k\xi)(K_1+{K_1}^2+K_2)+360045$$

$$\begin{aligned}
&5232K_1^2k^{16}\tau^2\sec h^{12}(k\xi)\tanh^4(k\xi)-75130884864K_1^2k^{18}\tau^2\sec h^{14}(k\xi)\tanh^4\\
&(k\xi)+13190542848K_1^2k^{20}\tau^2\sec h^{16}(k\xi)\tanh^4(k\xi)-120k^9\tau\sec h^4(k\xi)\tanh^5\\
&(k\xi)(K_1+K_1^2+K_2)+1512k^{11}\tau\sec h^6(k\xi)\tanh^5(k\xi)(K_1+K_1^2+K_2)+4927249\\
&1520K_1^2k^{17}\tau^2\sec h^{12}(k\xi)\tanh^5(k\xi)-96433876K_1^2k^{19}\tau^2\sec h^{14}(k\xi)\tanh^5(k\xi)\\
&+3186376704K_1^2k^{21}\tau^2\sec h^{16}(k\xi)\tanh^5(k\xi)-252k^{10}\tau\sec h^4(k\xi)\tanh^6(k\xi)(K_1\\
&+K_1^2+K_2)-4368350080K_1^2k^{16}\tau^2\sec h^{10}(k\xi)\tanh^6(k\xi)+139227871104K_1^2k^{18}\\
&\tau^2\sec h^{12}(k\xi)\tanh^6(k\xi)-45092920320K_1^2k^{20}\tau^2\sec h^{14}(k\xi)\tanh^6(k\xi)+26011\\
&2384K_1^2k^{22}\tau^2\sec h^{16}(k\xi)\tanh^6(k\xi)+2k^9\tau\sec h^2(k\xi)\tanh^7(k\xi)(K_1+K_1^2+K_2)\\
&-39040061760K_1^2k^{17}\tau^2\sec h^{10}(k\xi)\tanh^7(k\xi)+119211499008K_1^2k^{21}\tau^2\sec h^{12}(k\\
&\xi)\tanh^7(k\xi)-8258568192K_1^2k^{21}\tau^2\sec h^{14}(k\xi)\tanh^7(k\xi)+1423223840K_1^2k^{16}\tau^2\\
&\sec h^8(k\xi)\tanh^8(k\xi)-73492957440K_1^2k^{18}\tau^2\sec h^{10}(k\xi)\tanh^8(k\xi)+362409707\\
&52K_1^2k^{20}\tau^2\sec h^{12}(k\xi)\tanh^8(k\xi)-520224768K_1^2k^{22}\tau^2\sec h^{14}(k\xi)\tanh^8(k\xi)+85\\
&95336960K_1^2k^{17}\tau^2\sec h^8(k\xi)\tanh^9(kx)-39787776000K_1^2k^{19}\tau^2\sec h^{10}(k\xi)\tanh^9\\
&(k\xi)+3511517184K_1^2k^{21}\tau^2\sec h^{12}(k\xi)\tanh^9(k\xi)-107454336K_1^2k^{16}\tau^2\sec\\
&h^6(k\xi)\tanh^{10}(k\xi)+10322565120K_1^2k^{18}\tau^2\sec h^8(k\xi)\tanh^{10}(k\xi)-6055741\\
&440K_1^2k^{20}\tau^2\sec h^{10}(k\xi)\tanh^{10}(k\xi)-420053760K_1^2k^{17}\tau^2\sec h^6(k\xi)\tanh^{11}(k\\
&\xi)+2780467200K_1^2k^{19}\tau^2\sec h^8(k\xi)\tanh^{11}(k\xi)+1047616K_1^2k^{16}\tau^2\sec h^4(k\\
&\xi)\tanh^{12}(k\xi)-259338240K_1^2k^{18}\tau^2\sec h^6(k\xi)\tanh^{12}(k\xi)+2096640K_1^2k^{17}\tau^2\\
&\sec h^4(k\xi)\tanh^{13}(k\xi)-128K_1^2k^{16}\tau^2\sec h^2(k\xi)\tanh^{14}(k\xi)).
\end{aligned}\quad(3.25)$$

Add (3.17) to (3.25), we obtained solution in the following form

$$\tilde{w}(\xi,\tau;K_1,K_2)=w_0(\xi,\tau)+w_1(\xi,\tau;K_1)+w_2(\xi,\tau;K_1,K_2).\quad(3.13)$$

Using least square method to compute the constant values K_1 and K_2 which are as below

$$K_1=-1.084259592197655,\quad K_2=0.8831984610365042.\quad(3.14)$$

2nd order approximate solution of OHAM is evaluated by putting constants values K_1 and K_2.

Table 2(a): Comparison of Absolute errors of HPM and OHAM with exact solution for $\tau=0.1$.

ξ	Absolute error HPM	Absolute error OHAM
0.1	9.68087×10^{-5}	3.24071×10^{-9}
0.2	1.93593×10^{-4}	1.26255×10^{-9}
0.3	2.90358×10^{-4}	5.77130×10^{-9}
0.4	3.87106×10^{-4}	1.02658×10^{-8}
0.5	4.83840×10^{-4}	1.47269×10^{-8}

Table 2(b): Comparison of Absolute errors of HPM and OHAM with exact solution for $\tau=0.3$.

ξ	Absolute error HPM	Absolute error OHAM
0.1	9.63425×10^{-5}	1.65768×10^{-8}
0.2	1.92540×10^{-4}	3.16429×10^{-8}
0.3	2.88597×10^{-4}	4.64205×10^{-8}
0.4	3.84516×10^{-4}	6.08325×10^{-8}
0.5	4.80300×10^{-4}	7.48078×10^{-8}

Table 2(c): Comparison of Absolute errors of HPM and OHAM with exact solution for $\tau = 0.5$.

ξ	Absolute error HPM	Absolute error OHAM
0.1	9.51987×10^{-5}	7.14596×10^{-8}
0.2	1.90135×10^{-4}	9.91635×10^{-8}
0.3	2.84813×10^{-4}	1.25879×10^{-7}
0.4	3.79236×10^{-4}	1.51446×10^{-7}
0.5	4.73405×10^{-4}	1.75725×10^{-7}

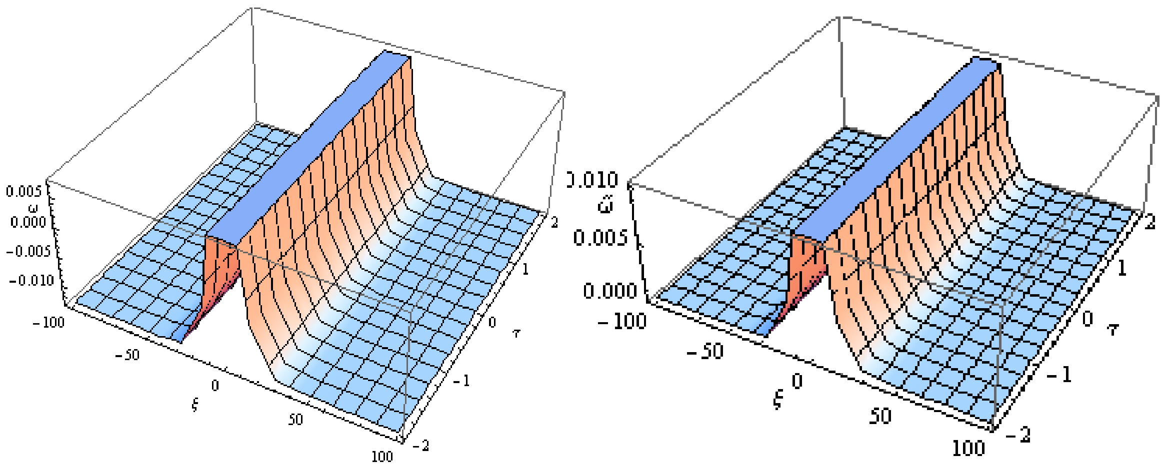

Fig 2(a): The graphs of exact solution in comparison with numerical results for $\tilde{\omega}(\xi,\tau)$ through OHAM of model 2.

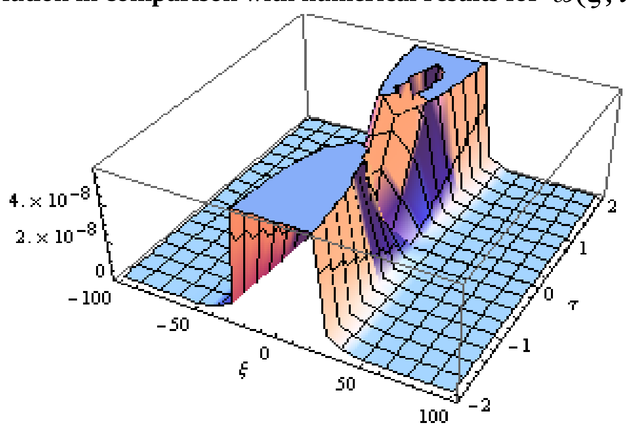

Fig 2(b): Graph of absolute error of OHAM with exact solution of model 2.

CONCLUSION

7th order Kdv and SK equations have been selected for the application of our suggested algorithm OHAM.Table 1(a)-1(c) and table 2(a)-2(c) have been illustrated for the comparison between OHAM and HPM. It is observed that OHAM provide remarkable results for different values of ξ *and* τ, the convergence rate of OHAM toward exact solution is too fast as compared to HPM. The analysis of Fig. 1(a) and Fig. 2(a) demonstrate OHAM as significant tool for computing semi analytic solutions of high nonlinear equations.

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