# POINCARÉ LIKE INEQUALITIES FOR GENERAL KERNELS

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***ABSTRACT****: In this paper, we establish some new forward and reverse Poincaré like and Dirichlet-Poincaré like inequalities for general kernels with related extreme cases as generalization of results given in [1] and [2]. We give applications of our main results for linear differential operators, Widder's derivatives and other fractional derivatives. At the end, we provide the corresponding discrete analogue of our main results.*

**Keywords**: Poincaré like inequalities; kernel; fractional derivatives; Green's function; linear differential operator

## 1.0 INTRODUCTION

Given a bounded domain $\Omega \subset \mathbb{R}^n$, the Poincaré inequality appear in [3] states that:

$$\|u\|_{L_p(\Omega)} \le C \|\nabla u\|_{L_p(\Omega)}$$

for the function with vanishing mean value over $\Omega$ is a well-known result which holds for $1 \le p \le \infty$ under very general assumptions on $\Omega$, where $\|\nabla u\|_{L_p(\Omega)}$ is defined as the $L_p$ norm of Euclidean norm of $\nabla u$.

This work is motivated by [1,2]. Anastassiou [1] proved some forward and reverse $L_p$ form of Poincaré like inequalities for linear differential operators involving its initial value problem, Green's function and initial condition. Later on Anastassiou [2] established Poincaré and Sobolev like inequalities for Widder derivatives. Our purpose is to give the Poincaré like and Dirichlet-Poincaré like inequalities for general kernels. As applications of our general results we extract the results of [1,2] by taking different kernels. We also provide new Poincaré like inequalities for Riemann-Liouvill's fractional integral, generalized Riemann-Liouvill's fractional derivative, Caputo fractional derivative and Canavati fractional derivative. It is also observed that the Poincaré like inequalities can be obtained from the Hardy-type inequalities given in [4] (see also [5]).

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with $\sigma$-finite measures and we say that a function $y: \Omega_2 \to \mathbb{R}$ belongs to the class $U(f,k)$ if it admits the representation

$$|y(x)| \le \int_{\Omega_2} |k(x,t)|\, |f(t)| d\mu_2(t), \qquad (1)$$

where $f$ is a continuous function on $\Omega_2$ and $k$ is an arbitrary continuous kernel defined on $\Omega_1 \times \Omega_2$. Inequality (1) can be written with equality as:

$$y(x) = \int_{\Omega_2} k(x,t) f(t) d\mu_2(t).$$

For this we have

$$|y(x)| = \left| \int_{\Omega_2} k(x,t) f(t) d\mu_2(t) \right|$$

$$\le \int_{\Omega_2} |k(x,t)|\, |f(t)| d\mu_2(t).$$

Before talking about the Hardy-type inequalities, it is necessary to give the following details:

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with $\sigma-$ finite measures and $A_k$ be an integral operator defined by

$$A_k f(x) := \frac{1}{K(x)} \int_{\Omega_2} k(x,t) f(t) d\mu_2(t), \qquad (2)$$

where $k: \Omega_1 \times \Omega_2 \to \mathbb{R}$ is measurable and non-negative kernel, $f$ is measurable function on $\Omega_2$, and

$$K(x) := \int_{\Omega_2} k(x,t) d\mu_2(t), \quad x \in \Omega_1. \qquad (3)$$

We consider that $K(x) > 0$ a.e. on $\Omega_1$.

The following theorem is give in [4] (see also [5]).

**Theorem 1.1** Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with $\sigma$-finite measures, $u$ be a weight function on $\Omega_1$, $k$ be a non-negative measurable function on $\Omega_1 \times \Omega_2$, and $K$ be defined on $\Omega_2$ by (3). Suppose that the function $x \mapsto u(x) \frac{k(x,t)}{K(x)}$ is integrable on $\Omega_1$ for each fixed $t \in \Omega_2$, and that $v$ is defined on $\Omega_1$ by

$$v(t) := \int_{\Omega_1} \frac{u(x) k(x,t)}{K(x)} d\mu_1(x) < \infty.$$

If $\Phi$ is convex on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\int_{\Omega_1} u(x)\Phi\big(A_k f(x)\big)\, d\mu_1(x) \leq \int_{\Omega_2} v(t)\Phi\big(f(t)\big)\, d\mu_2(t), \tag{4}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$, such that $Imf \subseteq I$, where $A_k$ is defined by (2). First we survey some facts about fractional derivatives needed in this paper.

Let $x \in [a,b]$, $\alpha > 0$, $n = [\alpha]+1$ ($[\cdot]$ is the integer part) and $\Gamma$ is the gamma function $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$. For $f \in L_1[a,b]$ the left sided and right sided Riemann-Liouville fractional integral $I^\alpha_{a_+} f$ and $I^\alpha_{b_-} f$ of order $\alpha$ is defined by

$$I^\alpha_{a_+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt,$$

and

$$I^\alpha_{b_-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt.$$

We denote some properties of the operators $I^\alpha_{a_+} f$ and $I^\alpha_{b_-} f$ of order $\alpha > 0$, see also [6]. The first result yields that the fractional integral operators $I^\alpha_{a_+} f$ and $I^\alpha_{b_-} f$ are bounded in $L_p(a,b)$, $1 \leq p \leq \infty$, that is

$$\| I^\alpha_{a_+} f \|_p \leq K \| f \|_p, \quad \| I^\alpha_{b_-} f \|_p \leq K \| f \|_p, \tag{5}$$

where

$$K = \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}.$$

Inequality (5), that is the result involving the left-sided fractional integral, was proved by G. H. Hardy in one of his first papers, (see [7]). He did not write down the constant, but the calculation of the constant was hidden inside his proof. For more details we refer [8] and the references cited therein.

It is interesting to note that Iqbal et.al. in their paper [8] proved some new inequalities of G. H. Hardy and here we get the similar results as applications of our results for fractional integral and fractional derivatives.

The rest of the paper is organized in the following way: In Section 2, we prove the forward and reverse Poincaré like inequalities for general kernel with related extreme cases. Section 3 covers the applications of our main results for linear differential operators to produce the Poincaré like inequalities given in [1]. In Section 4, we give the results for Widder derivatives and prove some new Dirichlet-Poincaré like inequalities discussed in [2]. Section 5 is dedicated to the applications for generalized Riemann-Liouvlle's fractional derivative, Caputo fractional derivative and Canavati fractional derivative. We conclude this paper by adding the discrete analogue of our main results given in Section 2.

## 2.0 MAIN RESULTS

First let us recall the well known Minkowski's inequality. For details see [9].

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces and let $k$ be a non-negative function on $\Omega_1 \times \Omega_2$ which is integrable with respect to measure $(\mu_1 \times \mu_2)$. If $\nu \geq 1$, then

$$\left[ \int_{\Omega_1} \left( \int_{\Omega_2} k(x,t) d\mu_2(t) \right)^\nu d\mu_1(x) \right]^{\frac{1}{\nu}} \leq \int_{\Omega_2} \left( \int_{\Omega_1} k^\nu(x,t) d\mu_1(x) \right)^{\frac{1}{\nu}} d\mu_2(t).$$

If $0 < \nu < 1$ and

$$(i) \quad \int_{\Omega_1} \left( \int_{\Omega_2} k(x,t) d\mu_2(t) \right)^\nu d\mu_1(x), \quad \int_{\Omega_2} k(x,t) d\mu_2(t) > 0,$$

then the reverse inequality holds.

If $\nu < 0$, the above mentioned assumption (i) and the additional one

$$(ii) \quad \int_{\Omega_1} k^\nu(x,t) d\mu_1(x) > 0 \ \mu_2 - a.e.,$$

then the reverse inequality holds.

Our first main result is given in the following theorem.

**Theorem 2.1** Let $y \in U(f,k)$, $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and $\nu \geq p$. Then

$$\left( \int_{\Omega_1} |y(x)|^\nu d\mu_1(x) \right)^{\frac{1}{\nu}} \leq \left( \int_{\Omega_1} \left( \int_{\Omega_2} |k(x,t)|^p d\mu_2(t) \right)^{\frac{\nu}{p}} d\mu_1(x) \right)^{\frac{1}{\nu}} \times \| f \|_{L_q(\Omega_2)}$$

$$\leq \left( \int_{\Omega_2} \left( \int_{\Omega_1} |k(x,t)|^{\nu} \, d\mu_1(x) \right)^{\frac{p}{\nu}} d\mu_2(t) \right)^{\frac{1}{p}}$$

$$\times \| f \|_{L_q(\Omega_2)} . \tag{6}$$

When $\nu = q,$ we obtain

$$\left( \int_{\Omega_1} |y(x)|^{q} \, d\mu_1(x) \right)^{\frac{1}{q}}$$

$$\leq \left( \int_{\Omega_1} \left( \int_{\Omega_2} |k(x,t)|^{p} \, d\mu_2(t) \right)^{\frac{q}{p}} d\mu_1(x) \right)^{\frac{1}{q}}$$

$$\times \| f \|_{L_q(\Omega_2)}$$

$$\leq \left( \int_{\Omega_2} \left( \int_{\Omega_1} |k(x,t)|^{q} \, d\mu_1(x) \right)^{\frac{p}{q}} d\mu_2(t) \right)^{\frac{1}{p}}$$

$$\times \| f \|_{L_q(\Omega_2)} .$$

When $\nu = p = q = 2,$ we obtain

$$\left( \int_{\Omega_1} |y(x)|^{2} \, d\mu_1(x) \right)^{\frac{1}{2}}$$

$$\leq \left( \int_{\Omega_1} \left( \int_{\Omega_2} |k(x,t)|^{2} \, d\mu_2(t) \right) d\mu_1(x) \right)^{\frac{1}{2}}$$

$$\times \| f \|_{L_2(\Omega_2)}$$

$$\leq \left( \int_{\Omega_2} \left( \int_{\Omega_1} |k(x,t)|^{2} \, d\mu_1(x) \right) d\mu_2(t) \right)^{\frac{1}{2}}$$

$$\times \| f \|_{L_2(\Omega_2)} .$$

**Proof.** Since $y \in U(f,k),$ and by using Hölder's inequality, we have

$$|y(x)| \leq \left( \int_{\Omega_2} |k(x,t)|^{p} \, d\mu_2(t) \right)^{\frac{1}{p}}$$

$$\times \left( \int_{\Omega_2} |f(t)|^{q} \, d\mu_2(t) \right)^{\frac{1}{q}}$$

$$= \left( \int_{\Omega_2} |k(x,t)|^{p} \, d\mu_2(t) \right)^{\frac{1}{p}} \| f \|_{L_q(\Omega_2)} . \tag{7}$$

Since $\nu \geq p,$ therefore we can write

$$\left( \int_{\Omega_1} |y(x)|^{\nu} \, d\mu_1(x) \right)^{\frac{1}{\nu}}$$

$$\leq \left( \int_{\Omega_1} \left( \int_{\Omega_2} |k(x,t)|^{p} \, d\mu_2(t) \right)^{\frac{\nu}{p}} d\mu_1(x) \right)^{\frac{1}{\nu}}$$

$$\times \| f \|_{L_q(\Omega_2)} . \tag{8}$$

Applying the integral Minkowski's inequality on right hand side of inequality (8), we obtain (6).
This proves the claim.

**Remark 2.2** If we replace $y$ by $I^{\alpha}_{a_+} f$ and taking

$$k(x,t) = \begin{cases} \frac{1}{\Gamma(\alpha)} (x-t)^{\alpha-1}, & a \leq t \leq x; \\ 0, & x < t \leq b. \end{cases}$$

in first inequality given in $(6)$ we obtain Theorem 2.6 of [8] and use of Minkowski's inequality give its generalization.
In upcoming remark we obtain Poincaré-Like inequality from Hardy-type inequality given in $(4)$.

**Remark 2.3** Take $\Omega_1 = \Omega_2 = (a,b),$ $d\mu_1(x) = dx,$ $d\mu_2(t) = dt$ and $\Phi(x) = x^{\nu},$ $\nu \geq 1$ in inequality $(4),$ we obtain

$$\int_a^b u(x) \left( \frac{1}{K(x)} \int_a^b k(x,t) f(t) dt \right)^{\nu} dx$$

$$\leq \int_a^b v(t) f^{\nu}(t) dt.$$

Particularly choose the weight function $u(x) = K(x),$ we have

$$\int_a^b K^{1-\nu}(x) \left( \int_a^b k(x,t) f(t) dt \right)^{\nu} dx$$

$$\leq \int_a^b v(t) f^{\nu}(t) dt. \tag{9}$$

Inequality (9) gives

$$(K(b)-K(a))^{1-\nu}\int_a^b y^{\nu}(x)dx$$

$$\le \left(v(a)-v(b)\right)\int_a^b f^{\nu}(t)dt.$$

This implies that

$$\|y\|_{\nu}(a,b)\le\left(\frac{v(a)-v(b)}{(K(b)-K(a))^{1-\nu}}\right)^{\frac{1}{\nu}}$$

$$\times\|f\|_{\nu}(a,b),$$

which is Poincaré-like inequality.

**Remark 2.4** By taking $p=1, q=\infty$ in Theorem 2.1, we have

$$\left(\int_{\Omega_1}|y(x)|^{\nu}\,d\mu_1(x)\right)^{\frac{1}{\nu}}$$

$$\le\left(\int_{\Omega_1}\left(\int_{\Omega_2}|k(x,t)|\,d\mu_2(t)\right)^{\nu}d\mu_1(x)\right)^{\frac{1}{\nu}}$$

$$\times\|f\|_{L_\infty(\Omega_2)}$$

$$\le\left(\int_{\Omega_2}\left(\int_{\Omega_1}|k(x,t)|^{\nu}\,d\mu_1(x)\right)^{\frac{1}{\nu}}d\mu_2(t)\right)$$

$$\times\|f\|_{L_\infty(\Omega_2)}.$$

When $\nu=1$, we obtain

$$\int_{\Omega_1}|y(x)|\,d\mu_1(x)$$

$$\le\left(\int_{\Omega_1}\left(\int_{\Omega_2}|k(x,t)|\,d\mu_2(t)\right)d\mu_1(x)\right)$$

$$\times\|f\|_{L_\infty(\Omega_2)}$$

$$\le\left(\int_{\Omega_2}\left(\int_{\Omega_1}|k(x,t)|\,d\mu_1(x)\right)d\mu_2(t)\right)$$

$$\times\|f\|_{L_\infty(\Omega_2)}.$$

The upcoming theorem is direct application of Minkowski's inequality.

**Theorem 2.5** Let $y\in U(f,k)$, and $\nu\ge 1$. Then the following inequality holds:

$$\left(\int_{\Omega_1}|y(x)|^{\nu}\,d\mu_1(x)\right)^{\frac{1}{\nu}}$$

$$\le\int_{\Omega_2}|f(t)|\left(\int_{\Omega_1}|k(x,t)|^{\nu}\,d\mu_1(x)\right)^{\frac{1}{\nu}}d\mu_2(t).$$

**Proof.** Since $y\in U(f,k)$ and applying the general Minkowski's integral inequality for $\nu\ge 1$, we can have

$$\left(\int_{\Omega_1}|y(x)|^{\nu}\,d\mu_1(x)\right)^{\frac{1}{\nu}}$$

$$\le\left(\int_{\Omega_1}\left(\int_{\Omega_2}|k(x,t)\|f(t)|d\mu_2(t)\right)^{\nu}d\mu_1(x)\right)^{\frac{1}{\nu}}$$

$$\le\int_{\Omega_2}\left(\int_{\Omega_1}|k(x,t)|^{\nu}|f(t)|^{\nu}d\mu_1(x)\right)^{\frac{1}{\nu}}d\mu_2(t)$$

$$=\int_{\Omega_2}\left(\int_{\Omega_1}|k(x,t)|^{\nu}d\mu_1(x)\right)^{\frac{1}{\nu}}|f(t)|\,d\mu_2(t).$$

This complete the proof.

We continue by defining Hadamard type fractional integrals. For details see [10, page 114] and [6, page 330].

Let $(a,b), 0\le a<b\le\infty$ be a finite or infinite interval of the half-axis $\mathbb{R}_+$ and $\alpha>0$. The left- and right-sided Hadamard fractional integrals of order $\alpha$ are given by

$$(J_{a+}^{\alpha}f)(x)=\frac{1}{\Gamma(\alpha)}\int_a^x\left(\log\frac{x}{y}\right)^{\alpha-1}\frac{f(y)dy}{y},\ x>a \text{ and}$$

$$(J_{b-}^{\alpha}f)(x)=\frac{1}{\Gamma(\alpha)}\int_x^b\left(\log\frac{y}{x}\right)^{\alpha-1}\frac{f(y)dy}{y},\ x<b,$$

respectively.

Let $\alpha>0, 1\le p\le\infty$ and $0\le a<b\le\infty$. Then the operators $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ are bounded in $L_p(a,b)$ as follows:

$$\|J_{a+}^{\alpha}f\|_p\le K_1\|f\|_p, \text{and} \|J_{b-}^{\alpha}f\|_p\le K_2\|f\|_p,$$

(10) where $K_1=\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^{\log(b/a)}t^{\alpha-1}e^{\frac{t}{p}}dt$

and $K_2=\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^{\log(b/a)}t^{\alpha-1}e^{-\frac{t}{p}}dt.$

Love proved the following theorem in [11] by using Minkowski's and Hölder's inequality.

**Theorem 2.6** If $s\ge r\ge 1$, $0\le a<b\le\infty$, are real, $w(x)$ is decreasing and positive in $(a,b)$, $f(x)$ and $k(x,y)$ are measurable and non-negative on $(a,b)$,

$k(x,y)$ is homogenous of degree $-1$,

$$Af(x)=\int_a^x k(x,y)\,f(y)dy,$$

and

$$\| f \|_r=\left(\int_a^b f^r(x)x^{\delta-1}w(x)dx\right)^{\frac{1}{r}}, \tag{11}$$

then

$$\| Af \|_r \leq C \| f \|_s,$$

where

$$C=\int_{\frac{a}{b}}^{1} k(1,t)t^{-\frac{\delta}{r}}\left(\int_a^{bt} x^{\delta-1}w(x)dx\right)^{\frac{1}{r}-\frac{1}{s}} dt.$$

Here $\frac{a}{b}$ is to mean $0$ if $a=0$ or $b=\infty$ or both; and $bt$ is to mean $\infty$ if $b=\infty.$

The upcoming theorem is the application of Love's result for Hadamard-type fractional integral.

**Theorem 2.7** If $s\geq r\geq 1,\ 0\leq a<b\leq\infty,$ are real, $w(x)$ is decreasing and positive in $(a,b),$ $J_{a+}^{\alpha}f$ denotes the left-sided Hadamard type fractional integral and $\| f \|_r$ is defined by (11) then

$$\| J_{a+}^{\alpha}f \|_r \leq C_1 \| f \|_s, \tag{12}$$

where

$$C_1=\int_{\frac{a}{b}}^{1}\log^{\alpha-1}\left(\frac{1}{t}\right)t^{-\frac{\delta}{r}-1}\left(\int_a^{bt}x^{\delta-1}w(x)dx\right)^{\frac{1}{r}-\frac{1}{s}}dt.$$

**Proof.** Applying Theorem 2.6 and replace general kernel $k(x,t)$ by particular kernel defined by

$$k(x,t)=\begin{cases}\dfrac{(\log x-\log t)^{\alpha-1}}{t\Gamma(\alpha)}, & a\leq t\leq x;\\ 0, & x<t\leq b.\end{cases} \tag{13}$$

and $Af$ by $J_{a+}^{\alpha}f,$ we get the inequality (12).

Next we give the generalization of Love's result for general kernel.

**Theorem 2.8** Let $r\geq 1,\ 0\leq a<b\leq\infty,$ are real, $w(x)$ is decreasing and positive in $(a,b),$ $f(x)$ and $k(x,y)$ are measurable and non-negative on $(a,b)$

$$y(x)=\int_a^x k(x,y)\,f(y)dy,$$

$$\| y \|_r \leq$$

then $$\int_{\frac{a}{b}}^{1} t^{-\frac{\gamma}{r}}\left(\int_{\frac{a}{t}}^{b}x^r k^r(x,xt)\,f^r(xt)(xt)^{\gamma-1}w(xt)tdx\right)^{\frac{1}{r}}dt.$$

**Proof.** Since we have

$$y(x)=\int_a^x k(x,y)\,f(y)dy,$$

Taking $y=tx,$ as $a\leq y\leq x,$ then $\frac{a}{x}\leq t\leq 1.$ Using Minkowski's inequality and decreasing property of $w(x)$ we have

$$\| y \|_r=\left(\int_a^b\left(\int_{\frac{a}{x}}^{1}xk(x,xt)\,f(xt)dt\right)^r x^{\gamma-1}w(x)dx\right)^{\frac{1}{r}}$$

$$\leq\int_{\frac{a}{x}}^{1}\left(\int_a^b[xk(x,xt)]^r\,f^r(xt)x^{\gamma-1}w(x)dx\right)^{\frac{1}{r}}dt$$

$$=\int_{\frac{a}{b}}^{1}t^{-\frac{\gamma}{r}}\left(\int_{\frac{a}{t}}^{b}[xk(x,xt)]^r\,f^r(xt)(xt)^{\gamma-1}w(x)tdx\right)^{\frac{1}{r}}dt.$$ This complete the proof.

**Corollary 2.9** If we replace $xt$ by $y$ and general kernel $k(x,t)$ by particular homogenous kernel $H(x,t)$ of degree $-1$, we get the inequality (7) of [11].

In upcoming remark we give application of our general result for Hadamard fractional integral.

**Remark 2.10** If we take

$$\Omega_1=\Omega_2=(a,b),d\mu_1(x)=dx,d\mu_2(t)=dt,$$

and particular kernel $k(x,y)$ defined by (13) in

Theorem 2.5, we have

$$\| J_{a+}^{\alpha}f \|_{L_\nu(a,b)}$$

$$\leq\frac{a^{1-\nu}}{\Gamma(\alpha)}\int_a^b|\,f(t)|\left(\int_0^{\log(\frac{b}{a})}z^{(\alpha-1)\nu}e^z dz\right)^{\frac{1}{\nu}}dt,$$

or

$$\| J_{a+}^{\alpha} f \|_{L_\nu}(a,b)$$

$$\leq \frac{a^{1-\nu}}{\Gamma(\alpha)}\left(\int_0^{\log(\frac{b}{a})} z^{(\alpha-1)\nu} e^z dz\right)^{\frac{1}{\nu}} \| f \|_{L_1}(a,b).$$

Particularly for $\nu = 1,$ we get

$$\| J_{a+}^{\alpha} f \|_{L_1}(a,b) \leq \left(\frac{1}{\Gamma(\alpha)}\int_0^{\log(\frac{b}{a})} z^{\alpha-1} e^z dz\right)$$

$$\times \| f \|_{L_1}(a,b). \qquad (14)$$

It is interesting to note that if we take $p = 1$ in (10) we get (14).

**Theorem 2.11** Let $s \geq r > 0$ and $k(x,y)$ be defined by (13). Then the following inequality holds:

$$\left[\int_0^\infty\left(\int_0^\infty k(x,y) f^r(y)dy\right)^{\frac{s}{r}} dx\right]^{\frac{1}{s}} \qquad (15)$$

$$\leq \left(\int_0^\infty k(1,t)t^{-\frac{r}{s}}dt\right)^{\frac{1}{r}}\left(\int_0^\infty f^s(u)du\right)^{\frac{1}{s}}.$$

**Proof.** Since $k(x,y)$ is defined by (13) and substitute $y = tx$ we have $k(x,y) = x^{-1}k(1,t).$ Consider the left side of the inequality (15) with above substitution and applying Minkowski's inequality we obtained

$$\left[\int_0^\infty\left(\int_0^\infty k(x,y) f^r(y)dy\right)^{\frac{s}{r}} dx\right]^{\frac{1}{s}}$$

$$= \left(\int_0^\infty\left(\int_0^\infty k(1,t) f^r(tx)dt\right)^{\frac{s}{r}} dx\right)^{\frac{1}{s}}$$

$$\leq \left(\int_0^\infty\left(\int_0^\infty k^{\frac{s}{r}}(1,t) f^s(tx)dx\right)^{\frac{r}{s}} dt\right)^{\frac{1}{r}}$$

$$= \left(\int_0^\infty k(1,t)\left(\int_0^\infty f^s(tx)dx\right)^{\frac{r}{s}} dt\right)^{\frac{1}{r}}. \qquad (16)$$

Now replace $u = tx$ and $dx = t^{-1}du$ in right hand side of the inequality (16) we obtain inequality (15).

Now we continue with reverse Poincaré like inequalities.

**Theorem 2.12** Let $y \in U(f,k),$ and let $0 < p < 1, q < 0 : \frac{1}{p} + \frac{1}{q} = 1,\quad 0 < \nu < p.$ Suppose that $k(x,t) \geq 0$ for $a \leq t \leq b$ and $f$ is of fixed sign and nowhere zero. Then

$$\left(\int_{\Omega_1} | y(x)|^\nu \, d\mu_1(x)\right)^{\frac{1}{\nu}}$$

$$\geq \left(\int_{\Omega_1}\left(\int_{\Omega_2} k^p(x,t)d\mu_2(t)\right)^{\frac{\nu}{p}} d\mu_1(x)\right)^{\frac{1}{\nu}}$$

$$\times \| f \|_{L_q(\Omega_2)}$$

$$\geq \left(\int_{\Omega_2}\left(\int_{\Omega_1} k^\nu(x,t)d\mu_1(x)\right)^{\frac{p}{\nu}} d\mu_2(t)\right)^{\frac{1}{p}}$$

$$\times \| f \|_{L_q(\Omega_2)}. \qquad (17)$$

When $\nu = q,$ we obtain

$$\left(\int_{\Omega_1} | y(x)|^q \, d\mu_1(x)\right)^{\frac{1}{q}}$$

$$\geq \left(\int_{\Omega_1}\left(\int_{\Omega_2} k^p(x,t)d\mu_2(t)\right)^{\frac{q}{p}} d\mu_1(x)\right)^{\frac{1}{q}}$$

$$\times \| f \|_{L_q(\Omega_2)}$$

$$\geq \left(\int_{\Omega_2}\left(\int_{\Omega_1} k^q(x,t)d\mu_1(x)\right)^{\frac{p}{q}} d\mu_2(t)\right)^{\frac{1}{p}}$$

$$\times \| f \|_{L_q(\Omega_2)}.$$

When $\nu = p = q = 2,$ we obtain

$$\left(\int_{\Omega_1} | y(x)|^2 \, d\mu_1(x)\right)^{\frac{1}{2}}$$

$$\geq \left( \int_{\Omega_1} \left( \int_{\Omega_2} k^2(x,t)\,d\mu_2(t) \right) d\mu_1(x) \right)^{\frac{1}{2}}$$
$$\times \|f\|_{L_2(\Omega_2)}$$

$$\geq \left( \int_{\Omega_2} \left( \int_{\Omega_1} k^2(x,t)\,d\mu_1(x) \right) d\mu_2(t) \right)^{\frac{1}{2}}$$
$$\times \|f\|_{L_2(\Omega_2)} .$$

**Proof.** Since $y \in U(f,k)$, $f$ is of fixed sign and using reverse Hölder's inequality, we get

$$|y(x)| = \int_{\Omega_2} k(x,t) |f(t)| dt$$

$$\geq \left( \int_{\Omega_2} k^p(x,t)dt \right)^{\frac{1}{p}} \left( \int_{\Omega_2} |f(t)|^q \, dt \right)^{\frac{1}{q}}$$

$$= \left( \int_{\Omega_2} k^p(x,t)dt \right)^{\frac{1}{p}} \|f\|_{L_q(a,b)} . \qquad (18)$$

Since $\nu < p,$ we have

$$\left( \int_{\Omega_1} |y(x)|^{\nu} \, d\mu_1(x) \right)^{\frac{1}{\nu}}$$

$$\geq \left( \int_{\Omega_1} \left( \int_{\Omega_2} k^p(x,t)\,d\mu_2(t) \right)^{\frac{\nu}{p}} d\mu_1(x) \right)^{\frac{1}{\nu}}$$
$$\times \|f\|_{L_q(\Omega_2)} .$$

Applying reverse Minkowski's inequality we obtain inequality (17).

## 3.0 POINCARÉ LIKE INEQUALITIES FOR LINEAR DIFFERENTIAL OPERATOR

Let $[a,b] \subset \mathrm{R}$, $a_i(x), i = 1,\dots,n-1 (n \in \mathrm{N})$, and $h(x)$ be continuous functions on $[a,b]$, and let

$$L = D^n + a_{n-1}(x)D^{n-1} + \dots + a_0(x),$$

be a fixed linear differential operator on $C^n[a,b]$. Let $y_1(x),\dots,y_n(x)$ be a set of linearly independent solution to $Ly = 0$ and the associated Green's function for $L$ is

$$H(x,t) := \frac{\begin{vmatrix} y_1(t) & \cdot & \cdot & \cdot & y_n(t) \\ y_1'(t) & \cdot & \cdot & \cdot & y_n'(t) \\ \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ y_1^{(n-2)}(t) & & & \cdot & y_n^{(n-2)}(t) \\ y_1(x) & \cdot & \cdot & \cdot & y_n(x) \end{vmatrix}}{\begin{vmatrix} y_1(t) & \cdot & \cdot & \cdot & y_n(t) \\ y_1'(t) & \cdot & \cdot & \cdot & y_n'(t) \\ \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ y_1^{(n-2)}(t) & & & \cdot & y_n^{(n-2)}(t) \\ y_1(t) & \cdot & \cdot & \cdot & y_n(t) \end{vmatrix}}$$

which is continuous function on $[a,b]^2$. Consider fixed $a$, then

$$y(x) = \int_a^b H(x,t)h(t)dt,$$

is the unique solution to the initial value problem

$Ly = h, \quad y^{(i)}(a) = 0, \quad i = 0,1,\dots,n-1.$ Now we present the Poincaré like inequality for linear differential operators and we will show that the results in this section generalizes the results of [1].

**Theorem 3.1** Let $y \in U(h,H)$, $p,q > 1 : \dfrac{1}{p} + \dfrac{1}{q} = 1;$ $\nu \geq p.$ Then

$$\left( \int_a^b |y(x)|^{\nu} \, dx \right)^{\frac{1}{\nu}}$$

$$\leq \left( \int_a^b \left( \int_a^b |H(x,t)|^p \, dt \right)^{\frac{\nu}{p}} dx \right)^{\frac{1}{\nu}} \|h\|_{L_q(a,b)}$$

$$\leq \left( \int_a^b \left( \int_a^b |H(x,t)|^{\nu} \, dx \right)^{\frac{p}{\nu}} dt \right)^{\frac{1}{p}} \|h\|_{L_q(a,b)} . \qquad (19)$$

When $\nu = q,$ we obtain

$$\left(\int_a^b |y(x)|^q\,dx\right)^{\frac{1}{q}}$$

$$\leq \left(\int_a^b\left(\int_a^b |H(x,t)|^p\,dt\right)^{\frac{q}{p}}dx\right)^{\frac{1}{q}} \|h\|_{L_q(a,b)}$$

$$\leq \left(\int_a^b\left(\int_a^b |H(x,t)|^q\,dx\right)^{\frac{p}{q}}dt\right)^{\frac{1}{p}} \|h\|_{L_q(a,b)}.$$

When $\nu = p = q = 2,$ we obtain

$$\left(\int_a^b |y(x)|^2\,dx\right)^{\frac{1}{2}}$$

$$\leq \left(\int_a^b\left(\int_a^b |H(x,t)|^2\,dt\right)dx\right)^{\frac{1}{2}} \|h\|_{L_2(a,b)}$$

$$\leq \left(\int_a^b\left(\int_a^b |H(x,t)|^2\,dx\right)dt\right)^{\frac{1}{2}} \|h\|_{L_2(a,b)}.$$

**Proof.** Applying Theorem 2.1 with $\Omega_1 = \Omega_2 = (a,b),$ $d\mu_1(x) = dx, d\mu_2(t) = dt$ and replace general kernel $k(x,t)$ with particular the Green's function $H(x,t),$ and $f$ by $h$ we get the inequality (19). In the upcoming remark we give the related extreme cases of Theorem 3.1.

**Remark 3.2** In Remark 2.4 if we replace $\Omega_1 = \Omega_2 = (a,b),$ $d\mu_1(x) = dx,$ $d\mu_2(t) = dt,$ general kernel $k(x,t) = H(x,t)$ and $f = h$ we get the following inequalities:

$$\left(\int_a^b |y(x)|^\nu\,dx\right)^{\frac{1}{\nu}}$$

$$\leq \left(\int_a^b\left(\int_a^b |H(x,t)|\,dt\right)^{\nu}dx\right)^{\frac{1}{\nu}} \|h\|_{L_\infty(a,b)}$$

$$\leq \left(\int_a^b\left(\int_a^b |H(x,t)|^\nu\,dx\right)^{\frac{1}{\nu}}dt\right) \|h\|_{L_\infty(a,b)}.$$

When $\nu = 1,$ we obtain

$$\int_a^b |y(x)|\,dx$$

$$\leq \left(\int_a^b\left(\int_a^b |H(x,t)|\,dt\right)dx\right) \|h\|_{L_\infty(a,b)}$$

$$\leq \left(\int_a^b\left(\int_a^b |H(x,t)|\,dx\right)dt\right) \|h\|_{L_\infty(a,b)}.$$

The upcoming theorem is direct application of Minkowski's inequality for linear differential operator.

**Theorem 3.3** Let $y \in U(h,H),$ and $\nu > 1.$ Then the following inequality holds:

$$\left(\int_a^b |y(x)|^\nu\,dx\right)^{\frac{1}{\nu}}$$

$$\leq \int_a^b\left(\int_a^b |H(x,t)|^\nu\,dx\right)^{\frac{1}{\nu}} |h(t)|\,dt. \tag{20}$$

**Proof.** Applying Theorem 2.5 with $\Omega_1 = \Omega_2 = (a,b),$ $d\mu_1(x) = dx, d\mu_2(y) = dy$ and replace general kernel $k(x,t)$ with particular Green's function $H(x,t),$ and $f$ by $h$ we get the inequality (20). Now we continue with reverse Poincaré like inequalities.

**Theorem 3.4** Let $0 < p < 1, q < 0 : \frac{1}{p} + \frac{1}{q} = 1;$ $0 < \nu < p.$ Suppose that $H(x,t) \geq 0$ for $a \leq t \leq b$ and $h$ is of fixed sign and nowhere zero.

$$\left(\int_a^b |y(x)|^\nu\,dx\right)^{\frac{1}{\nu}}$$

Then

$$\geq \left(\int_a^b\left(\int_a^b H^p(x,t)dx\right)^{\frac{\nu}{p}}dt\right)^{\frac{1}{\nu}} \|h\|_{L_q(a,b)}$$

$$\geq \left(\int_a^b\left(\int_a^b H^\nu(x,t)dt\right)^{\frac{p}{\nu}}dx\right)^{\frac{1}{p}} \|h\|_{L_q(a,b)}. \tag{21}$$

When $\nu = q,$ we obtain

$$\left(\int_a^b |y(x)|^q\, dx\right)^{\frac{1}{q}}$$

$$\geq \left(\int_a^b \left(\int_a^b H^p(x,t)dx\right)^{\frac{q}{p}} dt\right)^{\frac{1}{q}} \|h\|_{L_q(a,b)}$$

$$\geq \left(\int_a^b \left(\int_a^b H^q(x,t)dx\right)^{\frac{p}{q}} dt\right)^{\frac{1}{p}} \|h\|_{L_q(a,b)}.$$

When $\nu = p = q = 2,$ we obtain

$$\left(\int_a^b |y(x)|^2\, dx\right)^{\frac{1}{2}}$$

$$\geq \left(\int_a^b \left(\int_a^b H^2(x,t)dx\right) dt\right)^{\frac{1}{2}} \|h\|_{L_2(a,b)}$$

$$\geq \left(\int_a^b \left(\int_a^b H^2(x,t)dx\right) dt\right)^{\frac{1}{2}} \|h\|_{L_2(a,b)}.$$

**Proof.** By applying Theorem 2.12 and replace $\Omega_1 = \Omega_2 = (a,b), d\mu_1(x) = dx, d\mu_2(t) = dt,$ general kernel $k(x,t) = H(x,t)$ and $f = h$ we get the inequality (21).

**Remark 3.5** If we replace $k(x,t)$ by $H(x,t)$ in inequality (7) and (18), we get Theorem [12, Theorem 17.1] and [12, Theorem 17.5] respectively. Moreover use of Minkowski's inequality give us the generalizations of the results given in [12, Chapter 17].

## 4.0 WEIGHTED DIRICHLET-POINCARÉ LIKE INEQUALITIES

Now we give the application for Widder derivatives to produce forward and reverse Dirichlet-Poincaré like inequalities. First it is necessary to give some important details about Widder derivatives (see[13]). Let $f, u_0, u_1, \ldots, u_n \in C^{n+1}[a,b], n \geq 0,$ and the Wronskians

$$W_i(x) := W[u_0(x), u_1(x), \ldots, u_i(x)]$$

$$= \begin{vmatrix} u_0(x) & \cdot & \cdot & \cdot & u_i(x) \\ u_0'(x) & \cdot & \cdot & \cdot & u_i'(x) \\ \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ u_0^{(i)}(x) & \cdot & \cdot & \cdot & u_i^{(i)}(x) \end{vmatrix},$$

$i = 0,1,\ldots,n.$ Here $W_0(x) = u_0(x).$ Assume $W_i(x) > 0$ over $[a,b], i = 0,1,\ldots,n.$ For $i \geq 0,$ the differential operator of order $i$ (Widder derivative):

$$L_i f(x) := \frac{W[u_0(x), u_1(x), \ldots, u_{i-1}(x), f(x)]}{W_{i-1}(x)},$$

$$i = 1,\ldots,n+1; L_0 f(x) = f(x)$$

for all $x \in [a,b].$ Consider also

$$g_i(x,t) := \frac{1}{W_i(t)} \begin{vmatrix} u_0(t) & \cdot & \cdot & \cdot & u_i(t) \\ u_0'(t) & \cdot & \cdot & \cdot & u_i'(t) \\ \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ u_0(x) & \cdot & \cdot & \cdot & u_i(x) \end{vmatrix},$$

$$i = 1,2,...,n; \; g_0(x,t) := \frac{u_0(x)}{u_0(t)}$$

for all $x, t \in [a,b].$

**Example 4.1** [13]. Sets of the form $\{u_0, u_1, u_2, \ldots, u_n\}$ are

$\{1, x, x^2, \ldots, x^n\},$

$\{1, \sin x, \cos x, -\sin 2x, \cos 2x, \ldots,$

$(-1)^{n-1}\sin nx, (-1)^{n-1}\cos nx\},$

etc.

We also mention the generalized Widder-Talylor's formula, see [13] (see also [12]).

**Theorem4.2** Let the functions $f, u_0, u_1, \ldots, u_n \in C^{n+1}[a,b],$ and the Wronkians $W_0(x), W_1(x), \ldots, W_n(x) > 0$ on $[a,b], x \in [a,b].$ Then for $t \in [a,b]$ we have

$$f(x) = f(t)\frac{u_0(x)}{u_0(t)} + L_1 f(t) g_1(x,t) + ...$$

$$+ L_n f(t) g_n(x,t) + R_n(x),$$

where $R_n(x) := \int_t^x g_n(x,s) L_{n+1} f(s) ds.$

For example (see [13]) one could take $u_0(x) = c > 0.$ If $u_i(x) = x^i, i = 0,1,\ldots,n,$ defined on [a,b], then

$$L_i f(t) = f^{(i)}(t) \; and \; g_i(x,t) = \frac{(x-t)^i}{i!}, \quad t \in [a,b].$$

We need the following corollary.

**Corollary 4.3** By additionally assuming for fixed $a \in [a,b]$ that $L_i f(a) = 0, i = 0,1,...,n,$ we get that

$$f(x) := \int_a^x g_n(x,t) L_{n+1} f(t) dt \quad for\, all\, x \in [a,b].$$ Now

we prove some Dirichlet-Poincaré like inequalities as consequence of our main results for the Widder derivative in upcoming theorem and we extract the results of [2].

**Theorem 4.4** Let $f, u_0, u_1, ..., u_n \in C^{n+1}[a,b],\ n \in \mathbb{Z}_+,$ $W_0, W_1, ..., W_n > 0$ on $[a,b]$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1,$ and $\nu \geq p.$ Then

$$\left( \int_a^b | f(x) |^{\nu} \, dx \right)^{\frac{1}{\nu}}$$

$$\leq \left( \int_a^b \left( \int_a^b | g_n(x,t) |^p \, dt \right)^{\frac{\nu}{p}} dx \right)^{\frac{1}{\nu}} \| L_{n+1} f \|_{L_q(a,b)}$$

$$\leq \left( \int_a^b \left( \int_a^b | g_n(x,t) |^{\nu} \, dx \right)^{\frac{p}{\nu}} dt \right)^{\frac{1}{p}} \| L_{n+1} f \|_{L_q(a,b)} . \quad (22)$$

When $\nu = q,$ we obtain

$$\left( \int_a^b | f(x) |^{q} \, dx \right)^{\frac{1}{q}}$$

$$\leq \left( \int_a^b \left( \int_a^b | g_n(x,t) |^p \, dt \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}} \| L_{n+1} f \|_{L_q(a,b)}$$

$$\leq \left( \int_a^b \left( \int_a^b | k(x,t) |^{q} \, dx \right)^{\frac{p}{q}} dt \right)^{\frac{1}{p}} \| f \|_{L_q(a,b)} .$$

When $\nu = p = q = 2,$ we obtain

$$\left( \int_a^b | f(x) |^{2} \, dx \right)^{\frac{1}{2}}$$

$$\leq \left( \int_a^b \left( \int_a^b | g_n(x,t) |^p \, dt \right) dx \right)^{\frac{1}{2}} \| L_{n+1} f \|_{L_2(a,b)}$$

$$\leq \left( \int_a^b \left( \int_a^b | g_n(x,t) |^2 \, dx \right) dt \right)^{\frac{1}{2}} \| L_{n+1} f \|_{L_2(a,b)} .$$

**Proof.** By applying Theorem 2.1 with $\Omega_1 = \Omega_2 = (a,b),$ $d\mu_1(x) = dx,\ d\mu_2(t) = dt$ and replacing general kernel $k(x,t)$ with particular kernel $g_n(x,t),\ y = f$ and $f = L_{n+1} f,$ we get the inequality (22). In the upcoming remark we give the related extreme cases.

**Remark 4.5** By taking $p = 1$ and $q = \infty$ in inequality (22) we get

$$\left( \int_a^b | f(x) |^{\nu} \, dx \right)^{\frac{1}{\nu}}$$

$$\leq \left( \int_a^b \left( \int_a^b | g_n(x,t) | \, dt \right)^{\nu} dx \right)^{\frac{1}{\nu}} \| L_{n+1} f \|_{L_\infty(a,b)}$$

$$\leq \left( \int_a^b \left( \int_a^b | g_n(x,t) |^{\nu} \, dx \right)^{\frac{1}{\nu}} dt \right) \| L_{n+1} f \|_{L_\infty(a,b)} .$$

When $\nu = 1,$ we obtain

$$\int_a^b | f(x) | \, dx$$

$$\leq \left( \int_a^b \left( \int_a^b | g_n(x,t) | \, dt \right) dx \right) \| L_{n+1} f \|_{L_\infty(a,b)}$$

$$\leq \left( \int_a^b \left( \int_a^b | g_n(x,t) | \, dx \right) dt \right) \| L_{n+1} f \|_{L_\infty(a,b)} .$$

The upcoming theorem is direct application of Minkowski's inequality for Widder derivatives.

**Theorem 4.6** Let $f, u_0, u_1, ..., u_n \in C^{n+1}[a,b],\ n \in \mathbb{Z}_+,$ $W_0, W_1, ..., W_n > 0$ on $[a,b]$ and $\nu > 1.$ Then the following inequality holds:

$$\left( \int_a^b | f(x) |^{\nu} \, dx \right)^{\frac{1}{\nu}} \leq \int_a^b \left( \int_a^b | g_n(x,t) |^{\nu} \, dx \right)^{\frac{1}{\nu}} | L_{n+1} f(t) | \, dt. \quad (23)$$

**Proof.** By applying Theorem 2.5 with $\Omega_1 = \Omega_2 = (a,b),$ $d\mu_1(x) = dx, d\mu_2(t) = dt$ and replacing general kernel $k(x,t)$ with particular kernel $g_n(x,t),\ y = f$ and $f = L_{n+1} f,$ we get the inequality (23). Now we continue with reverse Dirichlet-Poincaré like inequalities.

**Theorem 4.7** Let $f, u_0, u_1, ..., u_n \in C^{n+1}[a,b],$ $W_0, W_1, ..., W_n > 0$ on $[a,b]$. Let $0 < p < 1, q < 0 : \frac{1}{p} + \frac{1}{q} = 1,$ and $0 < \nu < p.$ Further suppose $L_{n+1} f$ is of fixed sign and nowhere zero on $[a,b]$. Then

$$\left(\int_a^b |f(x)|^\nu dx\right)^{\frac{1}{\nu}}$$

$$\geq \left(\int_a^b \left(\int_a^b |g_n(x,t)|^p dx\right)^{\frac{\nu}{p}} dt\right)^{\frac{1}{\nu}} \| L_{n+1} f \|_{L_q(a,b)}$$

$$\geq \left(\int_a^b \left(\int_a^b |g_n(x,t)|^\nu dt\right)^{\frac{p}{\nu}} dx\right)^{\frac{1}{p}} \| L_{n+1} f \|_{L_q(a,b)} . \quad (24)$$

When $\nu = q,$ we obtain

$$\left(\int_a^b |f(x)|^q dx\right)^{\frac{1}{q}}$$

$$\geq \left(\int_a^b \left(\int_a^b |g_n(x,t)|^p dx\right)^{\frac{q}{p}} dt\right)^{\frac{1}{q}} \| L_{n+1} f \|_{L_q(a,b)}$$

$$\geq \left(\int_a^b \left(\int_a^b |g_n(x,t)|^q dt\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}} \| L_{n+1} f \|_{L_q(a,b)} .$$

When $\nu = p = q = 2,$ we obtain

$$\left(\int_a^b |f(x)|^2 dx\right)^{\frac{1}{2}}$$

$$\geq \left(\int_a^b \left(\int_a^b |g_n(x,t)|^2 dx\right) dt\right)^{\frac{1}{2}} \| L_{n+1} f \|_{L_2(a,b)}$$

$$\geq \left(\int_a^b \left(\int_a^b |g_n(x,t)|^2 dt\right) dx\right)^{\frac{1}{2}} \| L_{n+1} f \|_{L_2(a,b)} .$$

**Proof.** Applying Theorem 2.12 with $\Omega_1 = \Omega_2 = (a,b),$ $d\mu_1(x) = dx, d\mu_2(t) = dt$ and replace general kernel $k(x,t) = g_n(x,t),$ $y = f$ and $f = L_{n+1} f$ we get the inequality (24).

Here we provide the example by taking particular kernel in Theorem 4.4.

**Example 4.8** If we take $u_0(x) = c > 0$ and $u_n(x) = x^n, n = 0,1,2,...,n$ defined on $[a,b],$ then $L_n f(x) = f^{(n)}(x)$ and $g_n(x,t) = \frac{(x-t)^n}{n!}, t \in [a,b],$ and the inequality (22) becomes

$$\left(\int_a^b |f(x)|^\nu dx\right)^{\frac{1}{\nu}}$$

$$\leq \frac{(b-a)^{n+\frac{1}{p}+\frac{1}{\nu}}}{n!(np+1)^{\frac{1}{p}}((np+1)\frac{\nu}{p}+1)^{\frac{1}{\nu}}} \| f^{(n+1)} \|_{L_q(a,b)}$$

$$\leq \frac{(b-a)^{n+\frac{1}{p}+\frac{1}{\nu}}}{n!(n\nu+1)^{\frac{1}{\nu}}((n\nu+1)\frac{p}{\nu}+1)^{\frac{1}{p}}} \| f^{(n+1)} \|_{L_q(a,b)} .$$

**Remark 4.9** Similar examples can be given for all other results given in Section 4 by taking $g_n(x,t) = \frac{(x-t)^n}{n!}, t \in [a,b],$ but due to lack of space we omit the details.

**Remark 4.10** If we replace $k(x,t)$ by $g_n(x,t)$ in inequality (7) and (18), we get [12, Theorem 18.8] and [12, Theorem 18.10] respectively. Moreover use of Minkowski's inequality gives us the generalizations of the Dirichlet-Poincaré like inequalities given in [12, Chapter 18].

## 5.0 APPLICATIONS FOR FRACTIONAL DERIVATIVES

In upcoming applications of general results for fractional derivatives we construct inequalities of G. H. Hardy. Such type inequalities are widely discussed in [8]

**Theorem 5.1** Let $p,q > 1: \frac{1}{p} + \frac{1}{q} = 1,$ and $\nu \geq p.$ Let $I_{a^+}^\alpha f$ denotes the left sided Riemann-Liouvill's fractional integral. Then

$$\| I_{a^+}^\alpha f \|_{L_\nu(a,b)}$$

$$\leq \frac{(b-a)^{\alpha-1+\frac{1}{\nu}+\frac{1}{p}}}{\Gamma(\alpha)[p(\alpha-1)+1]^{\frac{1}{p}}[\nu(\alpha-1)+\frac{\nu}{p}+1]^{\frac{1}{\nu}}} \quad (25)$$

$$\times \| f \|_{L_q(a,b)} .$$

When $\nu = q,$ we can have

$$\| I_{a^+}^\alpha f \|_{L_q(a,b)}$$

$$\leq \frac{(b-a)^\alpha}{\Gamma(\alpha)[p(\alpha-1)+1]^{\frac{1}{p}}(q\alpha)^{\frac{1}{q}}} \| f \|_{L_q(a,b)} .$$

**Proof.** Applying Theorem 2.1 with $\Omega_1 = \Omega_2 = (a,b),$ $d\mu_1(x) = dx, d\mu_2(t) = dt$ and the kernel

$$k(x,t)=\begin{cases}\frac{1}{\Gamma(\alpha)}(x-t)^{\alpha-1}, & a\le t\le x;\\ 0, & x<t\le b.\end{cases}$$

and replace $y$ by $I^{\alpha}_{a^+}f$, we get the inequality (25).

For $f:[a,b]\to \mathbb{R}$ the Riemann-Liouville fractional derivative $D^{\alpha}f$ of order $\alpha$ is defined by

$$D^{\alpha}f(x)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x (x-t)^{n-\alpha-1}f(t)dt=\frac{d^n}{dx^n}J^{n-\alpha}f(x).$$

In addition, we stipulate $J^0 f:=f=:D^0 f$ and $J^{-\alpha}f:=D^{\alpha}f$ if $\alpha>0$.

Next, define $n$ as

$$n=[\alpha]+1,\ for\ \alpha\notin \mathbb{N}_0,\qquad n=\alpha,\ for\ \alpha\in\mathbb{N}_0. \tag{26}$$

For $n$ given by (26) and $f\in AC^n[a,b]$ the Caputo fractional derivative ${}^{C}D^{\alpha}f$ of order $\alpha$ is defined by

$${}^{C}D^{\alpha}f(x)=\frac{1}{\Gamma(n-\alpha)}\int_a^x (x-t)^{n-\alpha-1}f^{(n)}(t)dt=J^{n-\alpha}f^{(n)}(x).$$

A third fractional derivative, the Canavati fractional derivative ${}^{\bar{C}}D^{\alpha}f$ of order $\alpha$, is defined for $f\in C^{\alpha}[a,b]=\left\{f\in C^{n-1}[a,b]: J^{n-\alpha}f^{(n-1)}\in C^1[a,b]\right\}$ by

$${}^{\bar{C}}D^{\alpha}f(x)=\frac{1}{\Gamma(n-\alpha)}\frac{d}{dx}\int_a^x (x-t)^{n-\alpha-1}f^{(n-1)}(t)dt=\frac{d}{dx}J^{n-\alpha}f^{(n-1)}(x).$$

If $\alpha\in\mathbb{N}$ then $D^{\alpha}f={}^{C}D^{\alpha}f={}^{\bar{C}}D^{\alpha}f=f^{(\alpha)}$, the ordinary $\alpha$-order derivatives. The next theorem is composition identity for the Riemann-Liouville fractional derivatives. For details see [14, Theorem 4].

**Theorem 5.2** Let $\alpha>\beta\ge 0$, $n=[\alpha]+1$, $m=[\beta]+1$ and let $f\in AC^n[a,b]$ be such that $D^{\alpha}f, D^{\beta}f\in L_1[a,b]$.

i. If $\alpha-\beta\notin\mathbb{N}$ and $f$ is such that $D^{\alpha-k}f(a)=0$ for $k=1,\dots,n$ and $D^{\beta-k}f(a)=0$ for $k=1,\dots,m$, then

$$D^{\beta}f(x)=\frac{1}{\Gamma(\alpha-\beta)}\int_a^x (x-t)^{\alpha-\beta-1}D^{\alpha}f(t)dt,\quad x\in[a,b]. \tag{27}$$

ii. If $\alpha-\beta=l\in\mathbb{N}$ and $f$ is such that $D^{\alpha-k}f(a)=0$ for $k=1,\dots,l$, then (27) holds.

**Corollary 5.3** [14, Corollary 1] Let $\alpha>\beta\ge 0$, $n=[\alpha]+1$, $m=[\beta]+1$. Composition identity (27) is valid if one of the following conditions holds:

i. $f\in J^{\alpha}\left(L_1[a,b]\right)=\left\{f: f=J^{\alpha}\varphi, \varphi\in L_1[a,b]\right\}$.

ii. $J^{n-\alpha}f\in AC^n[a,b]$ and $D^{\alpha-k}f(a)=0$ for $k=1,\dots n$.

iii. $D^{\alpha-1}f\in AC[a,b]$, $D^{\alpha-k}f\in C[a,b]$ and $D^{\alpha-k}f(a)=0$ for $k=1,\dots n$.

iv. $f\in AC^n[a,b]$, $D^{\alpha}f, D^{\beta}f\in L_1[a,b]$, $\alpha-\beta\notin\mathbb{N}$, $D^{\alpha-k}f(a)=0$ for $k=1,\dots,n$ and $D^{\beta-k}f(a)=0$ for $k=1,\dots,m$.

v. $f\in AC^n[a,b]$, $D^{\alpha}f, D^{\beta}f\in L_1[a,b]$, $\alpha-\beta=l\in\mathbb{N}$, $D^{\alpha-k}f(a)=0$ for $k=1,\dots,l$.

vi. $f\in AC^n[a,b]$, $D^{\alpha}f, D^{\beta}f\in L_1[a,b]$ and $f^{(k)}(a)=0$ for $k=0,\dots,n-2$.

vii. $f\in AC^n[a,b]$, $D^{\alpha}f, D^{\beta}f\in L_1[a,b]$, $\alpha\notin\mathbb{N}$ and $D^{\alpha-1}f$ is bounded in a neighborhood of $t=a$.

Our first application is for Generalized Riemann-Liouville fractional derivative is given in upcoming theorem.

**Theorem 5.4** Let $p,q>1: \frac{1}{p}+\frac{1}{q}=1,$ and $\nu\ge p.$ Let the assumptions in the Corollary 5.3 be satisfied. Then

$$\|D^{\beta}f\|_{L_{\nu}(a,b)}\le\frac{(b-a)^{\alpha-\beta-1+\frac{1}{\nu}+\frac{1}{p}}}{\Gamma(\alpha-\beta)[p(\alpha-\beta-1)+1]^{\frac{1}{p}}[\nu(\alpha-\beta-1)+\frac{\nu}{p}+1]^{\frac{1}{\nu}}}\times\|D^{\alpha}f\|_{L_q(a,b)}.$$

When $\nu=q,$ we can have

$$\| D^{\beta} f \|_{L_q(a,b)}$$

$$\leq \frac{(b-a)^{\alpha-\beta}}{\Gamma(\alpha-\beta)[p(\alpha-\beta-1)+1]^{\frac{1}{p}}(q(\alpha-\beta))^{\frac{1}{q}}}$$

$$\times \| D^{\alpha} f \|_{L_q(a,b)} .$$

**Proof.** Similar to proof of Theorem 5.1. The upcoming theorem is composition identity for the Caputo fractional derivatives. For details see [15, Theorem 2.1].

**Theorem 5.5** Let $\alpha > \beta \geq 0$ with $n$ and $m$ are defined by $(26)$. Let $f \in AC^n[a,b]$ be such that $f^{(i)}(a)=0$ for $i = m, m+1, \ldots, n-1$. Let ${}^C D^{\alpha} f, {}^C D^{\beta} f \in L_1[a,b]$. Then

$${}^C D^{\beta} f(x)$$

$$= \frac{1}{\Gamma(\alpha-\beta)} \int_a^x (x-t)^{\alpha-\beta-1}\, {}^C D^{\alpha} f(t) dt, \quad x \in [a,b].$$ Here

we give the applications of general result for Caputo fractional derivatives.

**Theorem 5.6** Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1,$ and $\nu \geq p.$ Let the assumptions in the Theorem 5.5 be satisfied. Then

$$\| {}^C D^{\beta} f \|_{L_\nu(a,b)}$$

$$\leq \frac{(b-a)^{\alpha-\beta-1+\frac{1}{\nu}+\frac{1}{p}}}{\Gamma(\alpha-\beta)[p(\alpha-\beta-1)+1]^{\frac{1}{p}}[\nu(\alpha-\beta-1)+\frac{\nu}{p}+1]^{\frac{1}{\nu}}}$$ When

$$\times \| {}^C D^{\alpha} f \|_{L_q(a,b)} . \tag{28}$$

$\nu = q,$ we can have

$$\| {}^C D^{\beta} f \|_{L_q(a,b)}$$

$$\leq \frac{(b-a)^{\alpha-\beta}}{\Gamma(\alpha-\beta)[p(\alpha-\beta-1)+1]^{\frac{1}{p}}(q(\alpha-\beta))^{\frac{1}{q}}}$$

$$\times \| {}^C D^{\alpha} f \|_{L_q(a,b)} .$$

**Proof.** Similar to proof of Theorem 5.1.

The following theorem gives the conditions in the composition rule for Canavati fractional derivatives. For details see [16, Theorem 2.1].

**Theorem 5.7** Let $\gamma > \delta > 0$, $n = [\gamma]+1$, $m = [\delta]+1$. Let $f \in C^{\gamma}[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m-1, m, \ldots, n-2$. Then $f \in C^{\delta}[a,b]$ and

$${}^{\bar{C}} D^{\delta} f(x) = \frac{1}{\Gamma(\gamma-\delta)} \int_a^x (x-t)^{\gamma-\delta-1}\, {}^{\bar{C}} D^{\gamma} f(t) dt,$$ Now

$$x \in [a,b].$$

we give the corresponding connection of our results for Canavati fractional derivatives.

**Theorem 5.8** Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1,$ and $\nu \geq p.$ Let the assumptions in the Theorem 5.7 be satisfied. Then

$$\| {}^{\bar{C}} D^{\delta} f \|_{L_\nu(a,b)}$$

$$\leq \frac{(b-a)^{\gamma-\delta-1+\frac{1}{\nu}+\frac{1}{p}}}{\Gamma(\gamma-\delta)[p(\gamma-\delta-1)+1]^{\frac{1}{p}}[\nu(\gamma-\delta-1)+\frac{\nu}{p}+1]^{\frac{1}{\nu}}}$$ When

$$\times \| {}^{\bar{C}} D^{\gamma} f \|_{L_q(a,b)} .$$

$\nu = q,$ we can have

$$\| {}^{\bar{C}} D^{\delta} f \|_{L_q(a,b)}$$

$$\leq \frac{(b-a)^{\gamma-\delta}}{\Gamma(\gamma-\delta)[p(\gamma-\delta-1)+1]^{\frac{1}{p}}(q(\gamma-\delta))^{\frac{1}{q}}}$$

$$\times \| {}^{\bar{C}} D^{\gamma} f \|_{L_q(a,b)} .$$

**Proof.** Similar to proof of Theorem 5.1.

## 6.0 DISCRETE ANALOGUES TO MAIN RESULTS

This section deals with discrete analogues as a consequence of our general results given in Section 2.

**Theorem 6.1** For any real numbers $k_{mn}$ and $b_n$, we can write $|a_m| \leq \sum_{n=0}^{m-1} |k_{mn}| |b_n|.$ Then for any constants $p, q > 1$, such that $\frac{1}{p} + \frac{1}{q} = 1,$ $\nu \geq p,$ and integers $m, n = 0, 1, 2 \cdots, \alpha - 1$, then following inequality holds:

$$\left( \sum_{m=0}^{n-1} |a_m|^{\nu} \right)^{\frac{1}{\nu}} \leq \left( \sum_{m=0}^{n-1} \left( \sum_{n=0}^{m-1} |k_{mn}|^{p} \right)^{\frac{\nu}{p}} \right)^{\frac{1}{\nu}} \| b_n \|_q$$

$$\leq \left( \sum_{n=0}^{m-1} \left( \sum_{m=0}^{n-1} |k_{mn}|^{\nu} \right)^{\frac{p}{\nu}} \right)^{\frac{1}{p}} \| b_n \|_q . \tag{29}$$

**Proof.** For $n = 0, 1, 2, \cdots, m-1,$ applying Hölder's inequality for $\{p, q\},$ we get

$$|a_m| \leq \left( \sum_{n=0}^{m-1} |k_{mn}|^{p} \right)^{\frac{1}{p}} \left( \sum_{n=0}^{m-1} |b_n|^{q} \right)^{\frac{1}{q}}$$

$$\leq \left( \sum_{n=0}^{m-1} |k_{mn}|^{p} \right)^{\frac{1}{p}} \| b_n \|_q .$$

This can also be written as:

$$|a_m|^{\nu} \le \left( \sum_{n=0}^{m-1} |k_{mn}|^p \right)^{\frac{\nu}{p}} \|b_n\|_q^{\nu} .$$

This implies that

$$\left( \sum_{m=0}^{n-1} |a_m|^{\nu} \right)^{\frac{1}{\nu}} \le \left( \sum_{m=0}^{n-1} \left( \sum_{n=0}^{m-1} |k_{mn}|^p \right)^{\frac{\nu}{p}} \right)^{\frac{1}{\nu}} \|b_n\|_q .$$

Applying Minkowski's inequality, we get inequality (29). The extreme case of Theorem 6.1 is given in the following remark.

**Remark 6.2** For any real numbers $k_{mn}$ and $b_n$, then for any constants $p=1, q=\infty$, we have

$$\left( \sum_{m=0}^{n-1} |a_m|^{\nu} \right)^{\frac{1}{\nu}} \le \left( \sum_{m=0}^{n-1} \left( \sum_{n=0}^{m-1} |k_{mn}| \right)^{\nu} \right)^{\frac{1}{\nu}} \|b_n\|_{\infty}$$

$$\le \left( \sum_{n=0}^{m-1} \left( \sum_{m=0}^{n-1} |k_{mn}|^{\nu} \right)^{\frac{1}{\nu}} \right) \|b_n\|_{\infty} .$$

**Theorem 6.3** *For any* $k_{mn} > 0,\ b_n$ *be real number and is of fixed sign, such that*

$|a_m| = \sum_{n=0}^{m-1} k_{mn} |b_n|.$ Then for any constants $0 < p < 1, q < 0 : \frac{1}{p} + \frac{1}{q} = 1, 0 < \nu < p,$ we have

$$\left( \sum_{m=0}^{n-1} |a_m|^{\nu} \right)^{\frac{1}{\nu}} \ge \left( \sum_{m=0}^{n-1} \left( \sum_{n=0}^{m-1} k_{mn}^p \right)^{\frac{\nu}{p}} \right)^{\frac{1}{\nu}} \|b_n\|_q$$

$$\ge \left( \sum_{n=0}^{m-1} \left( \sum_{m=0}^{n-1} k_{mn}^{\nu} \right)^{\frac{p}{\nu}} \right)^{\frac{1}{p}} \|b_n\|_q . \qquad (30)$$

**Proof.** For any $k_{mn} > 0$, we have $|a_m| = \sum_{n=0}^{m-1} k_{mn} |b_n|$.

Applying reverse Hölder's inequality for $\{p, q\}$, we get

$$|a_m| \ge \left( \sum_{n=0}^{m-1} k_{mn}^p \right)^{\frac{1}{p}} \|b_n\|_q .$$

Since $\nu > 0$, we can write

$$\left( \sum_{m=0}^{n-1} |a_m|^{\nu} \right)^{\frac{1}{\nu}} \ge \left( \sum_{m=0}^{n-1} \left( \sum_{n=0}^{m-1} k_{mn}^p \right)^{\frac{\nu}{p}} \right)^{\frac{1}{\nu}} \|b_n\|_q .$$

Applying Minkowski's inequality, we get inequality (30).

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