ON SUPER (a, d) -EAT LABELING OF SUBDIVIDED TREES

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ABSTRACT: *Enomoto et al. (1998) defined the concept of a super $(a,0)$-edge-antimagic total labeling and proposed the conjecture that every tree is a super $(a,0)$-edge-antimagic total graph. In the favour of this conjecture, the present paper deals with different results on antimagicness of a class of trees, which is called subdivided stars.*

Key Words: Super (a,d)-EAT labeling, Subdivision of star.

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1 INTRODUCTION

All graphs in this paper are finite, undirected and simple. For a graph G, $V(G)$ and $E(G)$ denote the vertex-set and the edge-set, respectively. A (v,e)-graph G is a graph such that $|V(G)|=v$ and $|E(G)|=e$. A general reference for graph-theoretic ideas can be seen in [27]. A *labeling* (or *valuation*) of a graph is a map that carries graph elements to numbers (usually to positive or non-negative integers). In this paper, the domain will be the set of all vertices and edges and such a labeling is called *a total labeling*. Some labelings use the vertex-set only or the edge-set only and we shall call them *vertex-labelings* or *edge-labelings*, respectively.

Definition 1.1. An (s,d)*-edge-antimagic vertex* ((s,d)-EAV) labeling of a (v,e)-graph G is a bijective function $\lambda : V(G) \to \{1,2,\dots,v\}$ such that the set of edge-sums of all edges in G, $\{w(xy)=\lambda(x)+\lambda(y) : xy \in E(G)\}$, forms an arithmetic progression $\{s, s+d, s+2d, \dots, s+(e-1)d\}$, where $s>0$ and $d \geq 0$ are two fixed integers.

Definition 1.2. A bijection $\lambda : V(G) \cup E(G) \to \{1,2,...,v+e\}$ *is called an* (a,d)*-edge-antimagic total* ((a,d)*-EAT*) *labeling of a* (v,e)*-graph* G *if the set of edge-weights* $\{\lambda(x)+\lambda(xy)+\lambda(y) : xy \in V(G)\}$ *forms an arithmetic progression starting from* a *and having common difference* d*, where* $a>0$ *and* $d \geq 0$ *are two fixed integers. A graph that admits an* (a,d)*-EAT labeling is called an* (a,d)*-EAT graph.*

Definition 1.3. *If* λ *is an* (a,d)*-EAT labeling such that* $\lambda(V(G))=\{1,2,...,v\}$ *then* λ *is called a super* (a,d)*-EAT labeling and* G *is known as a super* (a,d)*-EAT graph.*

In definitions 1.2 and 1.3, if $d=0$ then an $(a,0)$-EAT labeling is called an edge-magic total (EMT) labeling and a super $(a,0)$-EAT labeling is called a super edge magic total (SEMT) labeling. Moreover, in general a is called minimum edge-weight but particularly magic constant when $d=0$. The definition of an (a,d)-EAT labeling was introduced by Simanjuntak, Bertault and Miller in [23] as a natural extension of *magic valuation* defined by Kotzig and Rosa [17, 18]. A super (a,d)-EAT labeling is a natural extension of the notion of *super edge-magic labeling* defined by Enomoto, Llado, Nakamigawa and Ringel. Moreover, Enomoto et al. [5] proposed the following conjecture:

Conjecture 1.1. *Every tree admits a super* $(a,0)$*-EAT labeling.*

In the favor of this conjecture, many authors have considered a super $(a,0)$-EAT labeling for different particular classes of trees. Lee and Shah [19] verified this conjecture by a computer search for trees with at most 17 vertices. For different values of d, the results related to a super (a,d)-EAT labeling can be found for w-trees [8], extended w-trees [9, 10], generalized extended w-trees [11, 12], stars [20], subdivided stars [13, 14, 15, 21, 22, 29], path-like trees [2], caterpillars [17, 18, 25], subdivided caterpillar [16], disjoint union of stars and books [6] and wheels, fans and friendship graphs [24], paths and cycles [23] and complete bipartite graphs [1]. For detail studies of a super (a,d)-EAT labeling reader can see [3, 4, 7, 26, 28].

Definition 1.4. Let $n_i \geq 1$, $1 \leq i \leq r$, and $r \geq 2$. A subdivided star $T(n_1,n_2,...,n_r)$ is a tree obtained by inserting $n_i - 1$ vertices to each of the ith edge of the star $K_{1,r}$. Moreover, suppose that $V(G)=\{c\} \cup \{x_i^{l_i} \mid 1 \leq i \leq r; 1 \leq l_i \leq n_i\}$ is the vertex-set and $E(G)=\{cx_i^1 \mid 1 \leq i \leq r\} \cup \{x_i^{l_i} x_i^{l_i+1} \mid 1 \leq i \leq r; 1 \leq l_i \leq n_i - 1\}$ is the edge-set of the subdivided star $G \cong T(n_1,n_2,...,n_r)$ then $v=\sum_{i=1}^{r} n_i + 1$ and $e=\sum_{i=1}^{r} n_i$.

However, the investigation of the different results related to a super (a,d)-EAT labeling of the subdivided star $T(n_1,n_2,n_3,...,n_r)$ for $n_1 \neq n_2 \neq n_2,..., \neq n_r$ is still open. In this paper, for $d \in \{0,1,2\}$, we formulate a super (a,d)-EAT labeling on the subclasses of subdivided stars denoted by $T(kn,kn,kn,kn,2kn,n_6,...,n_r)$ and $T(kn,kn,2n,2n+2,n_5,...,n_r)$ under certain conditions.

2 Basic Results

In this section, we present some basic results which will be used frequently in the main results.

Ngurah et al. [21] found lower and upper bounds of the minimum edge-weight a for a subclass of the subdivided stars, which is stated as follows:

Lemma 2.1. If $T(n_1,n_2,n_3)$ is a super $(a,0)$-EAT graph, then $\frac{1}{2l}(5l^2+3l+6) \le a \le \frac{1}{2l}(5l^2+11l-6)$, where $l=\sum_{i=1}^{3} n_i$.

The lower and upper bounds of the minimum edge-weight a for another subclass of subdivided stats established by Salman et al. [22] are given below:

Lemma 2.2. If $T(\underbrace{n,n,...,n}_{n-times})$ is a super $(a,0)$-EAT graph, then $\frac{1}{2l}(5l^2+(9-2n)l+n^2-n) \le a \le \frac{1}{2l}(5l^2+(2n+5)l+n-n^2)$ where $l=n^2$.

Moreover, the following lemma presents the lower and upper bound of the minimum edge-weight a for the most generalized subclass of subdivided stars proved by Javaid and Akhlaq [13, 15]:

Lemma 2.3. If $T(n_1,n_2,n_3,...,n_r)$ has a super (a,d)-EAT labeling, the

$\frac{1}{2l}(5l^2+r^2-2lr+9l-r-(l-1)ld) \le a \le$,

$\frac{1}{2l}(5l^2-r^2+2lr+5l+r-(l-1)ld)$

where $l=\sum_{i=1}^{r} n_i$ and $d\in\{0,1,2,3\}$.

Bača and Miller [3] state a necessary condition far a graph to be super (a,d)-EAT, which provides an upper bound on the parameter d. Let a (v,e)-graph G be a super (a,d)-EAT. The minimum possible edge-weight is at least $v+4$. The maximum possible edge-weight is no more than $3v+e-1$. Thus $a+(e-1)d \le 3v+e-1$ or $d \le \frac{2v+e-5}{e-1}$. For any subdivided star, where $v=e+1$, it follows that $d \le 3$.

Let us consider the following proposition which we will use frequently in the main results.

Proposition 2.1. [2] If a (v,e)-graph G has a (s,d)-EAV labeling then

(i) G has a super $(s+v+1,d+1)$-EAT labeling,

(ii) G has a super $(s+v+e,d-1)$-EAT labeling.

3 Super (a,d) - EAT labeling of subdivided stars

In this section, we prove the main results related to a super (a,d)-EAT labeling on more generalized subclasses of subdivided stars for $d\in\{0,1,2\}$.

Theorem 3.1. For any odd $n\ge 3$ and $r\ge 6$, $G\cong T(n,n,n,n,2n,n_6,...,n_r)$ admits a super $(a,0)$-EAT labeling with $a=2v+s-1$ and a super $(a,2)$-EAT labeling with $a=v+s+1$, where $v=|V(G)|$ and $s=(3n+4)+\sum_{m=6}^{r}[2^{m-5}n-m+6]$ and $n_p=2^{p-4}n-2p+11$ for $6\le p\le r$.

Proof. If $v=|V(G)|$ and $e=|E(G)|$ then

$$v=(6n+1)+\sum_{m=6}^{r}[2^{m-4}n-2m+11]$$

and

$$e=v-1.$$

Now, we define $\lambda: V(G)\to\{1,2,...,v\}$ as follows:

$$\lambda(c)=(4n+2)+\sum_{m=6}^{r}[2^{m-5}n-m+6].$$

For odd $1\le l_i\le n_i$, where $i=1,2,3,4,5$ and $6\le i\le r$:

$$\lambda(u)=\begin{cases} \frac{l_1+1}{2}, & \text{for } u=x_1^{l_1},\\ n+1-\frac{l_2-1}{2}, & \text{for } u=x_2^{l_2},\\ (n+2)+\frac{l_3-1}{2}, & \text{for } u=x_3^{l_3},\\ 2(n+1)-\frac{l_4-1}{2}, & \text{for } u=x_4^{l_4}.\\ (3n+2)-\frac{l_5-1}{2}, & \text{for } u=x_5^{l_5}. \end{cases}$$

$$\lambda(x_i^{l_i})=(3n+2)+\sum_{m=6}^{i}[2^{m-5}n-m+6]-\frac{l_i-1}{2}$$

respectively.

For even $1\le l_i\le n_i$ and $\alpha=(3n+2)+\sum_{m=6}^{r}[2^{m-6}n+1]$, where $i=1,2,3,4,5$ and $6\le i\le r$:

$$\lambda(u)=\begin{cases}(\alpha+1)+\frac{l_1-2}{2}, & \text{for } u=x_1^{l_1},\\ (\alpha+n-1)-\frac{l_2-2}{2}, & \text{for } u=x_2^{l_2},\\ (\alpha+n+1)+\frac{l_3-2}{2}, & \text{for } u=x_3^{l_3},\\ (\alpha+2n-1)-\frac{l_4-2}{2}, & \text{for } u=x_4^{l_4},\\ (\alpha+3n-1)-\frac{l_5-2}{2}, & \text{for } u=x_5^{l_5}.\end{cases}$$

and

$$\lambda(x_i^{l_i})=(\alpha+3n-1)+\sum_{m=6}^{i}[2^{m-4}(3n)-2m+11]-\frac{l_i-2}{2}$$

respectively.

The set of all edge-sums generated by the above formula forms a consecutive integer sequence $s=\alpha+2,\alpha+3,\cdots,\alpha+1+e$. Therefore, by Proposition 2.1, λ can be extended to a super $(a,0)$-EAT labeling with $a=v+e+s=2v+(3n+3)+\sum_{m=6}^{r}[2^{m-5}n-m+6]$ and to a super $(a,2)$-EAT labeling with $a=v+1+s=v+(3n+5)+\sum_{m=6}^{r}[2^{m-5}n-m+6]$.

Theorem 3.2. For any odd $n\geq 3$ and $r\geq 6$, $G\cong T(n,n,n,n,2n,n_6,...,n_r)$ admits a super $(a,1)$-EAT labeling with $a=s+\frac{3v}{2}$ if v is even, where $v=|V(G)|$, $s=(3n+4)+\sum_{m=6}^{r}[2^{m-5}n-m+6]$ and $n_p=2^{p-4}n-2p+11$ for $6\leq p\leq r$.

Proof. Let us consider $v=|V(G)|$, $e=|E(G)|$ and the set of vertex-labels $\lambda(V(G))$ are defined as in Theorem 3.1. It follows that the edge-sums of all edges of G constitute an arithmetic sequence $s=\alpha+2,\alpha+3,\cdots,\alpha+1+e$ with common difference 1, where $\alpha=(3n+2)+\sum_{m=6}^{r}[2^{m-5}n-m+6]$. We denote it by $A=\{a_i;1\leq i\leq e\}$. Consequently the set of edge-labels is $\lambda(E(G))=\{b_j;1\leq j\leq e\}$, where $b_j=v+j$. Define the set of edge-weights as

$$C=\{a_{2i-1}+b_{e-i+1};1\leq i\leq\frac{e+1}{2}\}\cup\{a_{2j}+b_{\frac{e-1}{2}-j+1};1\leq j\leq\frac{e+1}{2}-1\}.$$

It is easy to see that C constitutes an arithmetic sequence with $d=1$ and $a=s+\frac{3(v)}{2}=(12n+\frac{11}{2})+\frac{1}{2}\sum_{m=6}^{r}[2^{m-2}n-8m+45]$. Since all vertices receive the smallest labels, λ is a super $(a,1)$-EAT labeling.

Theorem 3.3. For any odd $n\geq 3$ and $r\geq 6$, $G\cong T(3n,3n,3n,3n,6n,n_6,...,n_r)$ admits a super $(a,0)$-EAT labeling with $a=2v+s-1$ and a super $(a,2)$-EAT labeling with $a=v+s+1$ where $v=|V(G)|$, $s=(9n+4)+\sum_{m=6}^{r}[2^{m-5}(3n)-m+6]$ and $n_p=2^{p-4}(3n)-2p+11$ for $6\leq p\leq r$.

Proof. If $v=|V(G)|$ and $e=|E(G)|$ then

$$v=(18n+1)+\sum_{m=6}^{r}[2^{m-4}(3n)-2m+11]$$

and

$e=v-1.$

Now, we define $\lambda:V(G)\rightarrow\{1,2,...,v\}$ as follows:

$$\lambda(c)=(12n+2)+\sum_{m=6}^{r}[2^{m-5}(3n)-m+6].$$

For odd $1\leq l_i\leq n_i$, where $i=1,2,3,4,5$ and $6\leq i\leq r$:

$$\lambda(u)=\begin{cases}\frac{l_1+1}{2}, & \text{for } u=x_1^{l_1},\\ 3n+1-\frac{l_2+1}{2}, & \text{for } u=x_2^{l_2},\\ (3n+2)+\frac{l_3-1}{2}, & \text{for } u=x_3^{l_3},\\ (6n+2)-\frac{l_4-1}{2}, & \text{for } u=x_4^{l_4},\\ 9n+2-\frac{l_5-1}{2}, & \text{for } u=x_5^{l_5}.\end{cases}$$

$$\lambda(x_i^{l_i})=(9n+2)+\sum_{m=6}^{i}[2^{m-5}(3n)-m+6]-\frac{l_i-1}{2}$$

respectively.

For even $1\leq l_i\leq n_i$ and

$$\alpha=(9n+2)+\sum_{m=6}^{r}[2^{m-5}(3n)-m+6],$$

Where $i=1,2,3,4,5$

and $6 \le i \le r$:

$$\lambda(u) = \begin{cases} (\alpha+1)+\frac{l_1-2}{2}, & \text{for } u = x_1^{l_1}, \\ (\alpha+3n-1)-\frac{l_2-2}{2}, & \text{for } u = x_2^{l_2}, \\ (\alpha+3n+1)+\frac{l_3-2}{2}, & \text{for } u = x_3^{l_3}, \\ (\alpha+6n-1)-\frac{l_4-2}{2}, & \text{for } u = x_4^{l_4}. \\ (\alpha+9n-1)-\frac{l_5-2}{2}, & \text{for } u = x_5^{l_5}. \end{cases}$$

and $\lambda(x_i^{l_i}) = (\alpha+9n-1)+\sum_{m=6}^{i}[2^{m-4}(3n)-2m+11]-\frac{l_i-2}{2}$.

raespectively.

The set of all edge-sums generated by the above formula forms a consecutive integer sequence $s=\alpha+2,\alpha+3,\cdots,\alpha+1+e$. Therefore, by Proposition 2.1, λ can be extended to a super $(a,0)$-EAT labeling with $a=v+e+s=2v+(9n+3)+\sum_{m=6}^{r}[2^{m-5}(3n)-m+6]$ and to a super $(a,2)$-EAT total labeling with $a=v+1+s=v+(9n+5)+\sum_{m=6}^{r}[2^{m-5}(3n)-m+6]$.

Theorem 3.4. For any odd $n \ge 3$, and $r \ge 6$, $G \cong T(3n,3n,3n,3n,2n,n_6,...,n_r)$ admits a super $(a,1)$-EAT labeling with $a = s+\frac{3v}{2}$ if v is even, where $v=|V(G)|$, $s=(9n+4)+\sum_{m=6}^{r}[2^{m-5}(3n)-m+6]$ and $n_p = 2^{p-4}(3n)-p+5$ for $6 \le p \le r$.

Proof. Let us consider $v=|V(G)|$, $e=|E(G)|$ and the set of vertex-labels $\lambda(V(G))$ are defined as in Theorem 3.3. It follows that edge-sums of all edges of G constitute an arithmetic sequence $s=\alpha+2,\alpha+3,\cdots,\alpha+1+e$ with common difference 1, where $\alpha=(9n+2)+\sum_{m=6}^{r}[2^{m-5}3n-m+6]$. We denote it by $A=\{a_i;1\le i\le e\}$. Consider the set of edge-labels $\lambda(E(G))=\{b_j;1\le j\le e\}$, where $b_j=v+j$. Define the set of edge-weights as $C=\{a_{2i-1}+b_{e-i+1};1\le i\le \frac{e+1}{2}\}\cup\{a_{2j}+b_{\frac{e-1}{2}-j+1};1\le j\le \frac{e+1}{2}-1\}$. It is easy to see that C constitutes an arithmetic sequence with $d=1$ and $a=s+\frac{3(v)}{2}=(36n+\frac{11}{2})+\frac{1}{2}\sum_{m=6}^{r}[2^{m-3}6n-8m+45]$. Since all vertices receive the smallest labels, λ is a super $(a,1)$-EAT labeling.

Theorem 3.5. For any odd $n \ge 3$, $r \ge 6$ and odd $k \ge 1$, $G \cong T(kn,kn,kn,kn,2kn,n_6,...,n_r)$ admits a super $(a,0)$-EAT labeling with $a=2v+s-1$ and a super $(a,2)$-EAT labeling with $a=v+s+1$, where $v=|V(G)|$, $s=(3kn+4)+\sum_{m=6}^{r}[2^{m-5}kn-m+6]$ and $n_p=2^{p-4}kn-2p+11$ for $6 \le p \le 5$.

Proof. If $v=|V(G)|$ and $e=|E(G)|$ then

$$v=(6kn+1)+\sum_{m=6}^{r}[2^{m-4}kn-2m+11]$$

and

$e=v-1$.

Now, we define $\lambda: V(G)\to\{1,2,...,v\}$ as follows:

$$\lambda(c)=(4kn+2)+\sum_{m=6}^{r}[2^{m-5}kn-m+6].$$

For odd $1\le l_i\le n_i$, where $i=1,2,3,4,5$ and $6\le i\le r$:

$$\lambda(u) = \begin{cases} \frac{l_1+1}{2}, & \text{for } u = x_1^{l_1}, \\ kn+1-\frac{l_2-1}{2}, & \text{for } u = x_2^{l_2}, \\ (kn+2)+\frac{l_3-1}{2}, & \text{for } u = x_3^{l_3}, \\ 2(kn+1)-\frac{l_4-1}{2}, & \text{for } u = x_4^{l_4}, \\ (3kn+2)-\frac{l_5-1}{2}, & \text{for } u = x_5^{l_5}. \end{cases}$$

$\lambda(x_i^{l_i})=(3kn+2)+\sum_{m=6}^{i}[2^{m-5}kn-m+6]-\frac{l_i-1}{2}$.

Respectively.

For even $1\le l_i\le n_i$, and $\alpha=(3kn+2)+\sum_{m=6}^{r}[2^{m-6}2kn-(m-6)]$, where $i=1,2,3,4,5$ and $6\le i\le r$:

$$\lambda(u)=\begin{cases}(\alpha+1)+\frac{l_1-2}{2}, & \text{for } u=x_1^{l_1},\\ (\alpha+kn-1)-\frac{l_2-2}{2}, & \text{for } u=x_2^{l_2},\\ (\alpha+kn+1)+\frac{l_3-2}{2}, & \text{for } u=x_3^{l_3},\\ (\alpha+2kn-1)-\frac{l_4-2}{2}, & \text{for } iu=x_4^{l_4}.\\ (\alpha+3kn-1)-\frac{l_5-2}{2}, & \text{for } u=x_5^{l_5}.\end{cases}$$

and

$$\lambda(x_i^{l_i})=[\alpha+(3kn-1)+\sum_{m=6}^{i}[2^{m-6}kn-2m+11]-\frac{l_i-2}{2}.$$

respectively.

The set of all edge-sums generated by the above formula forms a consecutive integer sequence $s=\alpha+2,\alpha+3,\cdots,\alpha+1+e$. Therefore, by Proposition 2.1, λ can be extended to a super $(a,0)$-EAT labeling with $a=v+e+s=2v+(3kn+3)+\sum_{m=6}^{r}[2^{m-5}kn-(m-6)]$ and to a super $(a,2)$-EAT labeling with $a=v+1+s=v+(3kn+5)+\sum_{m=6}^{r}[2^{m-6}2kn-(m-6)]$

.**Theorem 3.6.** For any odd $n\geq 3$, $r\geq 5$ and odd $k\geq 1$, $G\cong T(kn,kn,kn,kn,2n,n_6,...,n_r)$ admits a super $(a,1)$-EAT labeling with $a=s+\frac{3v}{2}$ if v is even, where $v=|V(G)|$, $s=(3kn+4)+\sum_{m=6}^{r}[2^{m-5}kn-m+6)]$ and $n_p=2^{p-4}kn-2p+11$.

Proof. Let us consider $v=|V(G)|$, $e=|E(G)|$ and the set of vertex-labels $\lambda(V(G))$ are defined as in Theorem 3.5. It follows that the edge-sums of all edges of G constitute an arithmetic sequence $s=\alpha+2,\alpha+3,\cdots,\alpha+1+e$ with common difference 1, where $\alpha=(3kk+2)+\sum_{m=6}^{r}[2^{m-6}2Kn-(m-6)]$. We denote it by $A=\{a_i;1\leq i\leq e\}$. Consequently, the set of edge-labels is $\lambda(E(G))=\{b_j;1\leq j\leq e\}$, where $b_j=v+j$. Define the set of edge-weights as

$$C=\{a_{2i-1}+b_{e-i+1};1\leq i\leq\frac{e+1}{2}\}\cup\{a_{2j}+b_{\frac{e-1}{2}-j+1};1\leq j\leq\frac{e+1}{2}-1\}.$$

It is easy to see that C constitutes an arithmetic sequence with $d=1$ and $a=s+\frac{3(v)}{2}=[12nk+\frac{11}{2}]+\frac{1}{2}\sum_{m=6}^{r}[2^{m-2}kn-8m+45]$. Since all vertices receive the smallest labels, λ is a super $(a,1)$-EAT labeling.

Theorem 3.7. For any odd $n\geq 3$, $r\geq 6$ and odd $k\geq 1$, $G\cong T(kn,kn,2n,2n+2,4n+3,n_6,...,n_r)$ admits a super $(a,0)$-EAT labeling with $a=2v+s-1$ and a super $(a,2)$-EAT labeling with $a=v+s+1$, where $v=|V(G)|$, $s=[(k+4)n+6]+\sum_{m=6}^{r}[2^{m-6}(4n+2)+1]$ and $n_p=2^{p-5}(4n+2)+1$ for $6\leq p\leq r$.

Proof. If $v=|V(G)|$ and $e=|E(G)|$ then

$$v=[(2k+8)n+6]+\sum_{m=6}^{r}[2^{m-5}(4n+2)+1]$$

and

$$e=v-1.$$

Now, we define $\lambda:V(G)\to\{1,2,...,v\}$ as follows:

$$\lambda(c)=[(2k+4)n+4]+\sum_{m=6}^{r}[2^{m-6}(4n+2)+1].$$

For odd $1\leq l_i\leq n_i$, where $i=1,2,3,4,5$ and $6\leq i\leq r$:

$$\lambda(u)=\begin{cases}\frac{l_1+1}{2}, & \text{for } u=x_1^{l_1},\\ kn+1-\frac{l_2-1}{2}, & \text{for } u=x_2^{l_2},\\ (kn+2)+\frac{l_3-1}{2}, & \text{for } u=x_3^{l_3},\\ (k+2)n+2-\frac{l_4-1}{2}, & \text{for } u=x_4^{l_4},\\ (k+2)n+4-\frac{l_5-1}{2}, & \text{for } u=x_5^{l_5}.\end{cases}$$

$$\lambda(x_i^{l_i})=[(k+4)n+4]+\sum_{m=6}^{i}[2^{m-6}(4n+2)+1]-\frac{l_i-1}{2}.$$

For even $1\leq l_i\leq n_i$, and $\alpha=[(k+4)n+4]+\sum_{m=6}^{r}[2^{m-6}(4n+2)+1]$ For $i=1,2,3,4,5$ and $6\leq i\leq r$:

$$\lambda(u)=\begin{cases}(\alpha+1)+\frac{l_1-2}{2}, & \text{for } u=x_1^{l_1},\\ (\alpha+kn-1)-\frac{l_2-2}{2}, & \text{for } u=x_2^{l_2},\\ (\alpha+kn+1)+\frac{l_3-2}{2}, & \text{for } u=x_3^{l_3},\\ (\alpha+(k+2)n-1)-\frac{l_4-2}{2}, & \text{for } u=x_4^{l_4}.\\ (\alpha+(k+4)n+2)-\frac{l_5-2}{2}, & \text{for } u=x_5^{l_5}.\end{cases}$$

and

$$\lambda(x_i^{l_i})=[\alpha+[k+4)n+2]+\sum_{m=6}^{i}[2^{m-6}(4n+2)+1]-\frac{l_i-2}{2}$$

respectively.

The set of all edge-sums generated by the above formula forms a consecutive integer sequence $s=\alpha+2,\alpha+3,\cdots,\alpha+1+e$. Therefore, by Proposition 2.1, λ can be extended to a super $(a,0)$-EAT labeling with $a=v+e+s=2v+[(k+4)n+5]+\sum_{m=6}^{r}[2^{m-6}(4n+2)+1]$ and to a super $(a,2)$-EAT labeling wit $a=v+1+s=v+[(k+4)n+7]+\sum_{m=6}^{r}[2^{m-6}(4n+2)+1]$

Theorem 3.8. For any odd $n\geq 3$, $r\geq 5$ and odd $k\geq 1$, $G\cong T(kn,kn,2n,2n+2,n_5,...,n_r)$ admits a super $(a,1)$-EAT labeling with $a=s+\frac{3v}{2}$ if v is even, where $v=|V(G)|$, $s=[(k+4)n+6]+\sum_{m=6}^{r}[2^{m-6}(4n+2)+1]$ and $n_p=2^{p-5}(4n+2)+1$ for $6\leq p\leq r$.

Proof. Let us consider $v=|V(G)|$, $e=|E(G)|$ and the set of vertex-labels $\lambda(V(G))$ are defined as in Theorem 3.7. It follows that the edge-sums of all edges of G constitute an arithmetic sequence $s=\alpha+2,\alpha+3,\cdots,\alpha+1+e$ with common difference 1, where $\alpha=(k+4)n+4+\sum_{m=6}^{r}[2^{m-6}(4n+2)+1]$. We denote it by $A=\{a_i;1\leq i\leq e\}$. Consequently, the set of edge-labels is $\lambda(E(G))=\{b_j;1\leq j\leq e\}$. Define the set of edge-weights as It is easy to see that C constitutes an arithmeticsequence

$$C=\{a_{2i-1}+b_{e-i+1};1\leq i\leq \frac{e+1}{2}\}\cup$$
$$\{a_{2j}+b_{\frac{e-1}{2}-j+1};1\leq j\leq \frac{e+1}{2}-1\}.$$

with $d=1$ and $a=s+\frac{3(v)}{2}=[4(k+4)n+15]+\frac{1}{2}\sum_{m=6}^{r}[2^{m-3}(4n+2)+5]$. Since all vertices receive the smallest labels, λ is a super $(a,1)$-EAT labeling.

4 CONCLUSION

In this paper, we have shown that the following subclasses of subdivided stars admit a super (a,d)-EAT labeling for $d\in\{0,1,2\}$:

- $T(kn,kn,kn,kn,2kn,n_6,...,n_r)$, where $n\geq 3$ odd, $k\geq 1$ odd, $r\geq 6$ and $n_p=2^{p-4}kn-2p+11$ for $6\leq p\leq r$.
- $T(kn,kn,2n,2n+2,4n+3,n_6,...,n_r)$, where $n\geq 3$ odd, $k\geq 1$ odd, $r\geq 6$ and $n_p=2^{p-5}(4n+2)+1$ for $6\leq p\leq r$.

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