AN APPLICATION OF A HYPERGEOMETRIC DISTRIBUTION SERIES ON CERTAIN ANALYTIC FUNCTIONS ASSOCIATEDBWITH INTEGRAL OPERATOR

Qaisar Mehmood [1], **Waqas Nazeer** [2], **Absar Ul Haq** [3]

[1] Department of Mathematics, Lahore Leeds University Lahore, Pakistan

[2] Division of Science and Technology, University of Education, Lahore, Pakistan

[3] Abdus Salam School of Mathmatical Sciences GC University, Lahore 54000, Pakistan

E-mail: qaisar47@hotmail.com, waqas.nazeer@ue.edu.pk, absarulhaq@hotmail.com

ABSTRAC: *The main purpose of this current note is to introduce a Hypergeometric distribution series in associated with integral operator and obtain necessary and sufficient conditions for this integral related series belonging to the classes and* $T(\alpha,\lambda)$ and $C(\alpha,\lambda)$.

Keywords-: Analytic function, Univalent function, hyper geometric distribution.

1. INTRODUCTION

Let A denote the class of functions f of the form

$$f(\text{z}) = \text{z} + \sum_{n=2}^{\infty} a_n z_n, \qquad (1.1)$$

which are analytic in the unit dics $D = \{z : z \in \square \text{ and } |z| \le 1\}$ and satisfy the normalization condition $f(0) = f'(0) - 1$. Further, we denote by S the subclass of A consisting of functions of the form (1.1) which are univelant in D and let T be the class of S consisting of the function of the form

$$f(\text{z}) = \text{z} + \sum_{n=2}^{\infty} |a_n| z_n. \qquad (1.2)$$

Consider $T(\alpha,\lambda)$ be the subclass of T consisting of the functions satisfying the following condition

$$\text{Re}\left\{\frac{zf(\text{z})}{\lambda z f'(\text{z}) + (1-\lambda) f(\text{z})}\right\} > \alpha, \qquad (1.3)$$

for all α $(0 \le \alpha < 1)$, λ $(0 \le \lambda < 1)$ for all $z \in D$.

We also consider $\text{C}(\alpha,\lambda)$ be the subclass of T consisting of the functions satisfying the following condition

$$\text{Re}\left\{\frac{f'(\text{z}) + zf''(\text{z})}{f'(\text{z}) + \lambda f''(\text{z})}\right\} > \alpha, \qquad (1.4)$$

for all α $(0 \le \alpha < 1)$, λ $(0 \le \lambda < 1)$ for all $z \in D$. Form (1.3) and (1.4) we can conclude that

$$f(\text{z}) \in C(\alpha,\lambda) \Leftrightarrow zf'(\text{z}) \in T(\alpha,\lambda). \qquad (1.5)$$

Both $T(\alpha,\lambda)$ and $\text{C}(\alpha,\lambda)$ are extensively studied by Altinates and Owa[1] and certain conditions for hypergeometric function and generalized Bessel functionf for these classes were studied by Mostafa[2] and Porwal and Dixit[3].

It is worthy to note that $T(0,\alpha) \equiv T^*(\alpha)$ be the class starlike functions of order α $(0 \le \alpha < 1)$, and $\text{C}(0,\alpha) \equiv \text{C}(\alpha)$ the class convex functions of order α.

The hyper geometric distribution $f(k,\text{N},m)$ is defined

$$f(k,\text{N},m) = \text{P}(\text{X}=n) = \frac{\binom{k}{n}\binom{N-k}{m-n}}{\binom{N}{m}} \qquad (1.6)$$

Note: Here $n = 0,1,2,\ldots,\min(k,m),\ \ N-k < m-n$ and $f(k,\text{N},m) = 0$ if $n > \min(k,m)$ or $N-k < m-n.$

We introduce a power series whose coffecients are probabiliteis of hyper geometric distribution

$$K(k,N,m,z) = z + \frac{1}{\binom{N}{m}} \sum_{n=2}^{\infty} \binom{k}{n-1}\binom{N-k}{m-n+1} z^n.$$

We note that, by ratio test, the radius of convergence of the above series is infinity. Now, we introduce the series

$$K(k,N,m,z) = z - \frac{1}{\binom{N}{m}} \sum_{n=2}^{\infty} \binom{k}{n-1}\binom{N-k}{m-n+1} z^n.$$

Motivated by results on connection between various subclasses of analytic functions by using the hypergeometric function by many author particularly the authors (see[5]-[10]) and generalized Bessel functions (see [12]- [13]), S.Porrwal [4] obtained the necessary and sufficient conditions for a functions $F(m,z)$ defined by using the poisson distribution belong to the class $T(\alpha,\lambda)$ and $C(\alpha,\lambda)$. In this article, we give the analogous conditions an integral operator $H(k,N,m,z)$ defined by the hypergeometric distribution belong to the $T(\alpha,\lambda)$ and $C(\alpha,\lambda)$.

To establish our main results, we will require the following lemmas according to Altintas and Owa [1].

Lemma 1.1. The function $f(z)$ definede by (1.2) in the class $T(\alpha,\lambda)$ if and only if

$$\sum_{n=2}^{\infty}[n-\lambda\alpha n-\alpha+\lambda\alpha]|a_n|\le 1-\alpha$$

Lemma 1.2. The function $f(z)$ definede by (1.2) in the class $C(\alpha,\lambda)$ if and only if

$$\sum_{n=2}^{\infty}n[n-\lambda\alpha n-\alpha+\lambda\alpha]|a_n|\le 1-\alpha$$

2. MAIN RESUTS

Let us define a particular integral operator $H(k,N,m,z)$ as follow

$$H(k,N,m,z)=\int_0^z\frac{K(k,N,m,z)}{t}dt \quad (2.1)$$

Theorem: The function $H(k,N,m,z)$ defined in (2.1) belong to the class $T(\alpha,\lambda)$ if and only if

$$\frac{1}{\binom{N}{m}}\left[k(1-\alpha\lambda)A+\frac{(1-\alpha)}{(k+1)}B\right]\le 1-\alpha,$$

Where

$$A=\sum_{n=0}^{\infty}\frac{1}{n+2}\binom{k-1}{n}\binom{N-k}{m-n-1}$$

and

$$B=\sum_{n=2}^{\infty}\binom{k+1}{n}\binom{N-k}{m-n+1}.$$

Proof. We have defined

$$H(k,N,m,z)=z-\frac{1}{\binom{N}{m}}\sum_{n=2}^{\infty}\binom{k}{n-1}\binom{N-k}{m-n+1}\frac{z_n}{n}$$

according to the lemma (1.1) we shall show that

$$\frac{1}{\binom{N}{m}}\sum_{n=2}^{\infty}[n-\lambda\alpha n-\alpha+\lambda\alpha]\frac{\binom{k}{n-1}\binom{N-k}{m-n+1}}{n}\le 1-\alpha.$$

Now

$$\frac{1}{\binom{N}{m}}\sum_{n=2}^{\infty}[n-\lambda\alpha n-\alpha+\lambda\alpha]\frac{\binom{k}{n-1}\binom{N-k}{m-n+1}}{n}=$$

$$=\frac{1}{\binom{N}{m}}\sum_{n=2}^{\infty}[(n-1)(1-\lambda\alpha)+(1-\alpha)]\frac{\binom{k}{n-1}\binom{N-k}{m-n+1}}{n}$$

$$=\frac{1}{\binom{N}{m}}\left[\sum_{n=2}^{\infty}(1-\lambda\alpha)\frac{k!}{n(k-n+1)!(n-2)!}\binom{N-k}{m-n+1}\right]$$

$$+\frac{1}{\binom{N}{m}}\left[\sum_{n=2}^{\infty}(1-\alpha)\frac{\binom{k}{n-1}\binom{N-k}{m-n+1}}{n}\right]$$

$$=\frac{1}{\binom{N}{m}}\left[\sum_{n=0}^{\infty}(1-\lambda\alpha)\frac{k!}{(n+2)(k-n-1)n!}\binom{N-k}{m-n-1}\right]$$

$$+\frac{1}{\binom{N}{m}}\left[\sum_{n=2}^{\infty}(1-\alpha)\frac{\binom{k}{n-1}\binom{N-k}{m-n+1}}{n}\right]$$

$$=\frac{1}{\binom{N}{m}}\left[k\sum_{n=0}^{\infty}(1-\lambda\alpha)\frac{(k-1)!}{(n+2)(k-n-1)n!}\binom{N-k}{m-n-1}\right]$$

$$+\frac{1}{\binom{N}{m}}\left[\sum_{n=2}^{\infty}(1-\alpha)\frac{\binom{k}{n-1}\binom{N-k}{m-n+1}}{n}\right]$$

$$=\frac{1}{\binom{N}{m}}\left[k(1-\lambda\alpha)\sum_{n=0}^{\infty}\frac{(k-1)!}{(n+2)(k-n-1)n!}\binom{N-k}{m-n-1}\right]$$

$$+\frac{1}{\binom{N}{m}}\left[\frac{(1-\alpha)}{(k+1)}\sum_{n=2}^{\infty}\binom{k+1}{n}\binom{N-k}{m-n+1}\right]$$

$$=\frac{1}{\binom{N}{m}}\left[\frac{k(1-\lambda\alpha)}{n+2}\sum_{n=0}^{\infty}\binom{k-1}{n}\binom{N-k}{m-n-1}\right]$$

$$+\frac{1}{\binom{N}{m}}\left[\frac{(1-\alpha)}{(k+1)}\sum_{n=2}^{\infty}\binom{k+1}{n}\binom{N-k}{m-n+1}\right]$$

$$=\frac{1}{\binom{N}{m}}\left[k(1-\alpha\lambda)A+\frac{(1-\alpha)}{(k+1)}B\right]\le 1-\alpha.$$

Where

$$A=\sum_{n=0}^{\infty}\frac{1}{n+2}\binom{k-1}{n}\binom{N-k}{m-n-1}$$

and

$$B=\sum_{n=2}^{\infty}\binom{k+1}{n}\binom{N-k}{m-n+1}.$$

Theorem: The function $H(k,N,m,z)$ is defined in (2.1) is in the class $C(\alpha,\lambda)$ if and only if

$$\frac{1}{\binom{N}{m}}\left[k(1-\alpha\lambda)A+(1-\alpha)B\right]\le 1-\alpha,$$

where

$$A=\sum_{n=0}^{\infty}\binom{k-1}{n}\binom{N-k}{m-n-1}$$

and

$$B=\sum_{n=1}^{\infty}\binom{k}{n}\binom{N-k}{m-n}.$$

Proof. We have defined

$$H(k,N,m,z)=z-\frac{1}{\binom{N}{m}}\sum_{n=2}^{\infty}\binom{k}{n-1}\binom{N-k}{m-n+1}\frac{z_n}{n}.$$

according to the lemma (1.2) it is sufficient to show that

$$\frac{1}{\binom{N}{m}}\sum_{n=2}^{\infty}n[n-\lambda\alpha n-\alpha-\lambda\alpha]\frac{\binom{k}{n-1}\binom{N-k}{m-n+1}}{n}\le 1-\alpha.$$

Now,

$$\frac{1}{\binom{N}{m}}\sum_{n=2}^{\infty}n[n-\lambda\alpha n-\alpha+\lambda\alpha]\frac{\binom{k}{n-1}\binom{N-k}{m-n+1}}{n}=$$

$$=\frac{1}{\binom{N}{m}}\sum_{n=2}^{\infty}[(n-1)(1-\lambda\alpha)+(1-\alpha)]\binom{k}{n-1}\binom{N-k}{m-n+1}$$

$$=\frac{1}{\binom{N}{m}}\left[\sum_{n=2}^{\infty}(1-\lambda\alpha)\frac{k!}{(k-n+1)!(n-2)!}\binom{N-k}{m-n+1}\right]$$

$$+\frac{1}{\binom{N}{m}}\left[\sum_{n=2}^{\infty}(1-\alpha)\binom{k}{n-1}\binom{N-k}{m-n+1}\right]$$

$$=\frac{1}{\binom{N}{m}}\left[\sum_{n=0}^{\infty}(1-\lambda\alpha)\frac{k!}{(k-n-1)!n!}\binom{N-k}{m-n-1}\right]$$

$$+\frac{1}{\binom{N}{m}}\left[\sum_{n=1}^{\infty}(1-\alpha)\binom{k}{n}\binom{N-k}{m-n}\right]$$

$$=\frac{1}{\binom{N}{m}}\left[\sum_{n=0}^{\infty}k(1-\lambda\alpha)\frac{(k-1)!}{(k-n-1)n!}\binom{N-k}{m-n-1}\right]$$

$$+\frac{1}{\binom{N}{m}}\left[\sum_{n=1}^{\infty}(1-\alpha)\binom{k}{n}\binom{N-k}{m-n}\right]$$

$$=\frac{1}{\binom{N}{m}}\left[k(1-\lambda\alpha)\sum_{n=0}^{\infty}\binom{k-1}{n}\binom{N-k}{m-n-1}\right]$$

$$+\frac{1}{\binom{N}{m}}\left[(1-\alpha)\sum_{n=1}^{\infty}\binom{k}{n}\binom{N-k}{m-n}\right]$$

$$=\frac{1}{\binom{N}{m}}\left[k(1-\alpha\lambda)A+(1-\alpha)B\right]\le 1-\alpha.$$

where

$$A=\sum_{n=0}^{\infty}\binom{k-1}{n}\binom{N-k}{m-n-1}$$

and

$$B=\sum_{n=1}^{\infty}\binom{k}{n}\binom{N-k}{m-n}.$$

REFERENCES

[1] O. Altintas and S. Owa, *"On subclasses of univalent functions with negative Coefficients"*, Pusan Kyongnam Mathematical Journal, vol. 4, pp. 41-56, 1988.

[2] A. O. Mostafa, "*A study on starlike and convex properties for hypergeometric Functions"*, Journal of Inequalities in Pure and Applied Mathematics, vol. 10, no. 3, article 87, pp. 1-16, 2009.

[3] S. Porwal and K. K. Dixit, "*An application of generalized Bessel functions oncertain analytic functions"*, Acta Universitatis Matthiae Belii. Series Mathematics, pp. 51-57, 2013.

[4] S. Porwal, "*An Application of a Poisson Distribution Series on Certain Analytic",*Functions Journal of Complex Analysis, Volume 2014, Article ID 984135.

[5] H. Silverman, "*Univalent functions with negative coefficients"*, Proceedings of the American Mathematical Society, vol. 51, pp. 109-116, 1975.

[6] B. C. Carlson and D. B. Shaffer, *Starlike and prestarlike hypergeometric functions*, SIAM Journal on Mathematical Analysis, vol. 15, no. 4, pp. 737-745, 1984.

[7] E. P. Merkes and W. T. Scott, *Starlike hypergeometric functions*, Proceedingsof the American Mathematical Society, vol. 12, pp. 885-888, 1961.

[8] S. Porwal and K. K. Dixit, *An application of certain convolution operator involving hypergeometric functions*, Journal of Rajasthan Academy of Physical Sciences, vol. 9, no. 2, pp. 173 -186, 2010.

[9] A. K. Sharma, S.Porwal, and K.K.Dixit, "*Classmappings properties of convolu-tions involving* *certain univalent functions associated with hypergeometric func-tions"*, Electronic Journal of Mathematical Analysis and Applications, vol. 1, no. 2, pp. 326- 333, 2013.

[10] A. Gangadharan, T. N. Shanmugam, and H. M. Srivastava, *"Generalized hyper- geometric functions associated with uniformly convex functions"*, Computers and Mathematics with Applications, vol. 44, no. 12, pp. 1515-1526, 2002.

[**11]** A. Baricz, "*Generalized Bessel Functions of the First Kind*, vol. 1994 of Lecture Notes in Mathematics, Springer, Berlin, Germany, 2010.

[12] S. R. Mondal and A. Swaminathan, "*Geometric properties of generalized Bessel Functions"*, Bulletin of the Malaysian Mathematical Sciences Society, vol. 35, no.1, pp. 179-194, 2012.

[13] S. Porwal, "*Mapping properties of generalized Bessel functions on some sub-Classes of univalent functions"*, Analele Universitatii Oradea Fasc. Matematica.