AN APPLICATION OF A HYPERGEOMETRIC DISTRIBUTION SERIES ON CERTAIN ANALYTIC FUNCTIONS

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ABSTRACT: *The main purpose of this current note is to introduce a Hypergeometric distribution series and obtain necessary and sufficient conditions for this series belonging to the classes* $T(\alpha,\lambda)$ and $C(\alpha,\lambda)$.

Keywords-: *Analytic function, Univalent function, hyper geometric distribution.*

1. INTRODUCTION

Let A represent the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z_n, \qquad (1.1)$$

which are analytic in the unit dics $D = \{z : z \in \mathbb{C} \text{ and } |z| \le 1\}$ and satisfy the normalization condition $f(0) = f'(0) - 1$. Further, we denote by S the subclass of A consisting of functions of the form (1.1) which are univelant in D and further let T be the class of S consisting of the functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} |a_n| z_n. \qquad (1.2)$$

Consider $T(\alpha,\lambda)$ be the subclass of T consisting of the functions satisfying the following condition

$$\operatorname{Re}\left\{\frac{zf(z)}{\lambda zf'(z) + (1-\lambda)f(z)}\right\} > \alpha, \qquad (1.3)$$

for all α $(0 \le \alpha < 1)$, λ $(0 \le \lambda < 1)$ for all $z \in D$.

We also suppose that $C(\alpha,\lambda)$ be the subclass of T consisting of the functions satisfying the following condition

$$\operatorname{Re}\left\{\frac{f'(z) + zf''(z)}{f'(z) + \lambda f''(z)}\right\} > \alpha, \qquad (1.4)$$

for all α $(0 \le \alpha < 1)$, λ $(0 \le \lambda < 1)$ for all $z \in D$.

Form (1.3) and (1.4) we can easly conclude that

$$f(z) \in C(\alpha,\lambda) \Leftrightarrow zf'(z) \in T(\alpha,\lambda). \qquad (1.5)$$

Both $T(\alpha,\lambda)$ and $C(\alpha,\lambda)$ are extensively studied by Altinates and Owa[1] and certain conditions for hypergeometric function and generalized Bessel functionf for these classes were studied by Mostafa[2] and Porwal and Dixit[3].

It is virtuous to note that $T(0,\alpha) \equiv T^*(\alpha)$ be the class starlike functions of order α $(0 \le \alpha < 1)$, and $C(0,\alpha) \equiv C(\alpha)$ be the class convex functions of order α.

The hypergeometric distribution $f(k, N, m)$ is defined

$$f(k, N, m) = P(X = n) = \frac{\binom{k}{n}\binom{N-k}{m-n}}{\binom{N}{m}} \qquad (1.6)$$

Note: Here $n = 0,1,2,\ldots,\min(k,m)$, $N-k < m-n$ and $f(k,N,m) = 0$ if $n > \min(k,m)$ or $N-k < m-n$.

Now, we establish a power series whose coffecients are probabiliteis of hypergeometric distribution

$$K(k,N,m,z) = z + \frac{1}{\binom{N}{m}} \sum_{n=2}^{\infty} \binom{k}{n-1}\binom{N-k}{m-n+1} z^n.$$

We note that, by ratio test, the radius of convergence of the above series is infinity. Now, we introduce the series

$$K(k,N,m,z) = z - \frac{1}{\binom{N}{m}} \sum_{n=2}^{\infty} \binom{k}{n-1}\binom{N-k}{m-n+1} z^n.$$

Motivated by results on connection between various subclasses of analytic functions by using the hypergeometric function by many authors particularly the authors (see[5-10]), S.Porrwal [4] obtained the necessary and sufficient conditions for a function $F(m,z)$ defined by using the poisson distribution belong to the class $T(\alpha,\lambda)$ and $C(\alpha,\lambda)$. In this article, we give the analogous conditions for $K(k,N,m,z)$ defied by the hypergeometric distribution belong to the $T(\alpha,\lambda)$ and $C(\alpha,\lambda)$.

To charactiese our main results, we will require the following lemmas according to Altintas and Owa [1]

Lemma 1.1. The function $f(z)$ definede by (1.2) in the class $T(\alpha,\lambda)$ if and only if

$$\sum_{n=2}^{\infty} [n - \lambda\alpha n - \alpha + \lambda\alpha] |a_n| \le 1 - \alpha$$

Lemma 1.2. The function $f(z)$ definede by (1.2) in the class $C(\alpha,\lambda)$ if and only if

$$\sum_{n=2}^{\infty} n[n-\lambda\alpha n-\alpha+\lambda\alpha]|a_n| \le 1-\alpha$$

2. MAIN RESUTS

Theorem: The function $K(k,N,m,z)$ is in the class $T(\alpha,\lambda)$ if and only if

$$\frac{1}{\binom{N}{m}}\left[k(1-\alpha\lambda)A+(1-\alpha)B\right] \le 1-\alpha,$$

Where

$$A=\sum_{n=0}^{\infty}\binom{k-1}{n}\binom{N-k}{m-n-1}$$

and

$$B=\sum_{n=1}^{\infty}\binom{k}{n}\binom{N-k}{m-n}.$$

Proof. Since we have defined

$$K(k,N,m,z)=z-\frac{1}{\binom{N}{m}}\sum_{n=2}^{\infty}\binom{k}{n-1}\binom{N-k}{m-n+1}z_n,$$

according to the lemma (1.1), it is enough to show that

$$\frac{1}{\binom{N}{m}}\sum_{n=2}^{\infty}[n-\lambda\alpha n-\alpha+\lambda\alpha]\binom{k}{n-1}\binom{N-k}{m-n+1} \le 1-\alpha.$$

Now

$$\frac{1}{\binom{N}{m}}\sum_{n=2}^{\infty}[n-\lambda\alpha n-\alpha+\lambda\alpha]\binom{k}{n-1}\binom{N-k}{m-n+1}=$$

$$=\frac{1}{\binom{N}{m}}\sum_{n=2}^{\infty}[(n-1)(1-\lambda\alpha)+(1-\alpha)]\binom{k}{n-1}\binom{N-k}{m-n+1}$$

$$=\frac{1}{\binom{N}{m}}\left[\sum_{n=2}^{\infty}(1-\lambda\alpha)\frac{k!}{(k-n+1)!(n-2)!}\binom{N-k}{m-n+1}\right]$$

$$+\frac{1}{\binom{N}{m}}\left[\sum_{n=2}^{\infty}(1-\alpha)\binom{k}{n-1}\binom{N-k}{m-n+1}\right]$$

$$=\frac{1}{\binom{N}{m}}\left[\sum_{n=0}^{\infty}(1-\lambda\alpha)\frac{k!}{(k-n-1)n!}\binom{N-k}{m-n-1}\right]$$

$$+\frac{1}{\binom{N}{m}}\left[\sum_{n=2}^{\infty}(1-\alpha)\binom{k}{n-1}\binom{N-k}{m-n+1}\right]$$

$$=\frac{1}{\binom{N}{m}}\left[k\sum_{n=0}^{\infty}(1-\lambda\alpha)\frac{(k-1)!}{(k-n-1)n!}\binom{N-k}{m-n-1}\right]$$

$$+\frac{1}{\binom{N}{m}}\left[\sum_{n=2}^{\infty}(1-\alpha)\binom{k}{n-1}\binom{N-k}{m-n+1}\right]$$

$$=\frac{1}{\binom{N}{m}}\left[k(1-\lambda\alpha)\sum_{n=0}^{\infty}\frac{(k-1)!}{(k-n-1)n!}\binom{N-k}{m-n-1}\right]$$

$$+\frac{1}{\binom{N}{m}}\left[(1-\alpha)\sum_{n=1}^{\infty}\binom{k}{n}\binom{N-k}{m-n}\right]$$

$$=\frac{1}{\binom{N}{m}}\left[k(1-\lambda\alpha)\sum_{n=0}^{\infty}\binom{k-1}{n}\binom{N-k}{m-n-1}\right]$$

$$+\frac{1}{\binom{N}{m}}\left[(1-\alpha)\sum_{n=2}^{\infty}\binom{k}{n}\binom{N-k}{m-n}\right]$$

$$=\frac{1}{\binom{N}{m}}\left[k(1-\alpha\lambda)A+(1-\alpha)B\right] \le 1-\alpha.$$

Where

$$A=\sum_{n=0}^{\infty}\binom{k-1}{n}\binom{N-k}{m-n-1}$$

and

$$B=\sum_{n=1}^{\infty}\binom{k}{n}\binom{N-k}{m-n}.$$

Theorem: The function $K(k,N,m,z)$ is in the $C(\alpha,\lambda)$ if and only if

$$\frac{1}{\binom{N}{m}}\left[k(k-1)(1-\alpha\lambda)A+k(3-2\alpha\lambda-\alpha)B+(1-\alpha)C\right]$$

$$\le 1-\alpha,$$

where

$$A=\sum_{n=0}^{\infty}\binom{k-2}{n}\binom{N-k}{m-n-2}\ ,\ B=\sum_{n=0}^{\infty}\binom{k-1}{n}\binom{N-k}{m-n-1}\ \text{and}$$

$$C=\sum_{n=1}^{\infty}\binom{k}{n}\binom{N-k}{m-n}$$

Proof. We have defined

$$K(k,N,m,z)=z-\frac{1}{\binom{N}{m}}\sum_{n=2}^{\infty}\binom{k}{n-1}\binom{N-k}{m-n+1}z_n$$

according to the lemma (1.2), we shall show that

$$\frac{1}{\binom{N}{m}}\sum_{n=2}^{\infty}n[n-\lambda\alpha n-\alpha+\lambda\alpha]\binom{k}{n-1}\binom{N-k}{m-n+1}\le 1-\alpha.$$

Now

$$\frac{1}{\binom{N}{m}}\sum_{n=2}^{\infty}n[n-\lambda\alpha n-\alpha+\lambda\alpha]\binom{k}{n-1}\binom{N-k}{m-n+1}=$$

$$=\frac{1}{\binom{N}{m}}\sum_{n=2}^{\infty}[(n-1)(n-2)(1-\lambda\alpha)]\binom{k}{n-1}\binom{N-k}{m-n+1}$$

$$+\frac{1}{\binom{N}{m}}\sum_{n=2}^{\infty}(n-1)(3-2\lambda\alpha-\alpha)]\binom{k}{n-1}\binom{N-k}{m-n+1}$$

$$+\frac{1}{\binom{N}{m}}\sum_{n=2}^{\infty}(1-\alpha)\binom{k}{n-1}\binom{N-k}{m-n+1}$$

$$=\frac{1}{\binom{N}{m}}\left[\sum_{n=3}^{\infty}\frac{(1-\lambda\alpha)k!}{(k-n+1)!(n-3)!}\binom{N-k}{m-n+1}\right]$$

$$+\frac{1}{\binom{N}{m}}\left[\sum_{n=3}^{\infty}\frac{(3-2\lambda\alpha-\alpha)k!}{(k-n+1)!(n-2)!}\binom{N-k}{m-n+1}\right]$$

$$+\frac{1}{\binom{N}{m}}\left[\sum_{n=2}^{\infty}(1-\alpha)\binom{k}{n-1}\binom{N-k}{m-n+1}\right]$$

$$=\frac{1}{\binom{N}{m}}\left[\sum_{n=0}^{\infty}(1-\lambda\alpha)\frac{k!}{(k-n-2)!n!}\binom{N-k}{m-n-2}\right]$$

$$+\frac{1}{\binom{N}{m}}\left[\sum_{n=0}^{\infty}(3-2\lambda\alpha-\alpha)\frac{k!}{(k-n-1)!n!}\binom{N-k}{m-n-1}\right]$$

$$+\frac{1}{\binom{N}{m}}\left[\sum_{n=1}^{\infty}(1-\alpha)\binom{k}{n}\binom{N-k}{m-n}\right]$$

$$=\frac{1}{\binom{N}{m}}\left[\sum_{n=0}^{\infty}k(k-1)(1-\lambda\alpha)\frac{(k-2)!}{(k-n-2)n!}\binom{N-k}{m-n-2}\right]$$

$$+\frac{1}{\binom{N}{m}}\left[k(3-2\lambda\alpha-\alpha)\sum_{n=0}^{\infty}\binom{k-1}{n}\binom{N-k}{m-n-1}\right]$$

$$+\frac{1}{\binom{N}{m}}\left[\sum_{n=1}^{\infty}(1-\alpha)\binom{k}{n}\binom{N-k}{m-n}\right]$$

$$=\frac{1}{\binom{N}{m}}\left[k(k-1)(1-\lambda\alpha)\sum_{n=0}^{\infty}\binom{k-2}{n}\binom{N-k}{m-n-2}\right]$$

$$+\frac{1}{\binom{N}{m}}\left[(k-1)(3-2\lambda\alpha-\alpha)\sum_{n=0}^{\infty}\binom{k-1}{n}\binom{N-k}{m-n-1}\right]$$

$$+\frac{1}{\binom{N}{m}}\left[(1-\alpha)\sum_{n=1}^{\infty}\binom{k}{n}\binom{N-k}{m-n}\right]$$

$$=\frac{1}{\binom{N}{m}}\left[k(k-1)(1-\alpha\lambda)A+k(3-2\alpha\lambda-\alpha)B+(1-\alpha)C\right]$$

$$\le 1-\alpha,$$

where

$$A=\sum_{n=0}^{\infty}\binom{k-2}{n}\binom{N-k}{m-n-2}\ ,B=\sum_{n=0}^{\infty}\binom{k-1}{n}\binom{N-k}{m-n-1}\ \text{and}$$

$$C=\sum_{n=1}^{\infty}\binom{k}{n}\binom{N-k}{m-n}$$

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