# BOUNDARY CONTROLLABILITY OF CO-OPERATIVE DIRICHLET PARABOLIC SYSTEM WITH DIFFERENT TYPE OF OBSERVATION

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***ABSTRACT***. *In this paper, a controllability problems for co-operative parabolic linear system involving Laplace operator with boundary Dirichlet control and distributed or boundary observations are considered.*

**Key Words**: Optimal control problem, Controllability, solutions of parabolic system, co-operative system.

## 1. INTRODUCTION

Controllability is a mathematical problem, which consists in determining the targets to which one can drive the state of some dynamical system, by means of a control parameter present in the equation. Many physical systems such as quantum systems, fluid mechanical systems, wave propagation, diffusion phenomena, etc. are represented by an infinite number of degrees of freedom, and their evolution follows some partial differential equation. Finding active controls in order to properly influence the dynamics of these systems generate highly involved problems. The control theory for PDEs, and among this theory, controllability problems, is a mathematical description of such situations. Any dynamical system represented by a PDE, and on which an external influence can be described, can be the object of a study from this point of view. In 1978, D.L. Russell [1] made a rather complete survey of the most relevant results that were available in the literature at that time. In that paper, the author described a number of different tools that were developed to address controllability problems, often inspired and related to other subjects concerning partial differential equations: multipliers, moment problems, nonharmonic Fourier series, etc.

Various types of controllability of linear abstract dynamical systems defined in a Banach or Hilbert spaces have been recently extensively explored by several authors (see e.g.[2]-[18]). More recently, J.-L. Lions introduced the so called Hilbert Uniqueness Method (H.U.M.; see [19]).

In this work, we will focus our attention on some special aspects of controllability problems for parabolic system involving Laplace operator with differend type of observation. In order to explain the results we have in mind, it is convenient to consider the abstract form:

Let $V$ and $H$ be two real Hilbert spaces such that $V$ is a dense subspace of $H$. Identifying the dual of $H$ with $H$, we may consider $V \subset H \subset V'$, where the embedding is dense in the following space. Let $A(t)$ ( $t \in ]0,T[$ ) be a family of continuous operators associated with a bilinear forms $\pi(t;.,.)$ defined on $V \times V$ which are satisfied Gårding's inequality

$$\pi(t;y,y) + c_0\|y\|_H^2 \geq c_1\|y\|_V^2, \quad c_0 \geq 0, c_1 > 0, \quad \text{for } y \in V, t \in [0,T]. \tag{1}$$

Then, from [20] and [21], for given $f, y_0$ and $B$ be a bounded linear operator the following abstract systems:

$$\left.\begin{aligned} &\frac{d}{dt}y(t) + A(t)y(t) = f + Bu, t \in ]0,T[, \\ &y(0) = y_0 \end{aligned}\right\} \tag{2}$$

have a unique solution, we denote it by $y(t;u) \in Y$.

We also given an observation equation

$z(u) = Cy(u), \quad C \in L(Y : \mathsf{H}), \quad \mathsf{H}$ being a Hilbert space

**Definition 1** *The system whose state is defined by (2)is said to be controllable if the observation* $z(u)$ *generates a dense (affine) subspace of the space of observations* $\mathsf{H}$.

In the above setting, the equation (2) is typically a partial differential equation, where the influence of $u$ can take multiple different forms: typically, $u$ can be an additional (force) term in the right-hand side of the equation, localized in a part of the domain; it can also appear in the boundary conditions; but other situations can clearly be envisaged (we will describe some of them).

A typical application of a parabolic equation is the heat;

$$\left.\begin{aligned} &\frac{\partial y}{\partial t} = \Delta y + u && \text{in } Q = \Omega \times ]0,T[, \\ &y(x,0) = y_0(x) && \text{in } \Omega, \\ &y(x,t) = u(x,t), && \text{on } \Sigma = \Gamma \times ]0,T[, \end{aligned}\right\} \tag{3}$$

where $\Omega \subset R^N$ is a bounded open domain with smooth boundary $\Gamma$ and $y_0(x)$ is a given function in $L^2(\Omega)$. The results in [20] partly overlap with results in [22] and they were shown that the system (3) (with $u \in L^2(Q)$ ) is controllable.

In our papers [23]-[29], the above results are extended, the controllability questions related to the time optimal control problem of $n \times n$ co-operative parabolic or hyperbolic systems with distributed or boundary controls was considered.

In this paper, we will consider a boundary controllability problem for the following $n \times n$ Dirichlet co-operative linear parabolic system with different cases of observations (here and everywhere below the vectors are denoted by bold letters and the index $i = 1,2,...,n$ ):

$$\left.\begin{aligned} &\frac{\partial y_i}{\partial t} = (A(t)\mathbf{y})_i && \text{in } Q,\\ &y_i(x,0) = y_{i,0}(x) && \text{in } \Omega,\\ &y_i(x,t) = u_i(x,t), && \text{on } \Sigma \end{aligned}\right\} \quad (4)$$

where $y_{i,0}$, is a given functions , $u_i$ represents a Dirichlet boundary control function defined in $\Sigma$ and $A(t)$ ( $t \in ]0,T[$ ) are a family of $n \times n$ continuous matrix operators,

$$A(t)y = \begin{pmatrix} \Delta + a_1 & a_{12} & & a_{1n} \\ a_{21} & \Delta + a_2 & & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & & \Delta + a_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

with co-operative coefficient functions $a_i, a_{ij}$ satisfying the following conditions:

$$\left.\begin{aligned} &a_i, a_{ij} \text{ are positive functions in } L^\infty(Q),\\ &a_{ij}(x,t) \le \sqrt{a_i(x,t)a_j(x,t)}. \end{aligned}\right\} \quad (5)$$

**2. Solutions of co-operative Dirichlet parabolic systems**

This section is devoted to the analysis of the existence and uniqueness of solutions of system (4). Let $H_0^1(\Omega)$ be the usual Sobolev space( see [30]) of order one which consists of all $\phi \in L^2(\Omega)$ whose distributional derivatives $\frac{\partial \phi}{\partial x_i} \in L^2(\Omega)$ and $\phi|_\Gamma = 0$ with the scalar product $\langle y, \phi \rangle_{H_0^1(\Omega)} = \langle y, \phi \rangle_{L^2(\Omega)} + \langle \nabla y, \nabla \phi \rangle_{L^2(\Omega)}$, where $\nabla = \sum_{k=1}^{N} \frac{\partial}{\partial x_k}$ . We have the following dense embedding form ( see [30]) :

$$H_0^1(\Omega) \subseteq L^2(\Omega) \subseteq H_0^{-1}(\Omega)$$

where $H_0^{-1}(\Omega)$ is the dual of $H_0^1(\Omega)$

For $\mathbf{y} = (y_1, y_2, ..., y_n)^T, \phi = (\phi_1, \phi_2, ..., \phi_n)^T \in (H_0^1(\Omega))^n$ and $t \in ]0,T[$ , let us define a family of continuous bilinear forms $\pi(t;.,.) : (H_0^1(\Omega))^n \times (H_0^1(\Omega))^n \to \Re$ by

$$\begin{aligned} \pi(t;\mathbf{y},\phi) &= \sum_{i=1}^{n} \int_\Omega [(\nabla y_i)(\nabla \phi_i) - a_i(x,t) y_i \phi_i] dx \\ &\quad - \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x,t) y_j \phi_i dx \\ &= \sum_{i=1}^{n} \int_\Omega [(-\Delta y_i) - a_i(x,t) y_i] \phi_i dx - \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x,t) y_j \phi_i dx \\ &= \sum_{i=1}^{n} < -(A(t)\mathbf{y})_i, \phi >_{L^2(\Omega)} \end{aligned} \quad (6)$$

**Lemma 1** *If $\Omega$ is a regular bounded domain in $R^N$, with boundary $\Gamma$, and if $m$ is positive on $\Omega$ and smooth enough ( in particular $m \in L^\infty(\Omega)$ ) then the eigenvalue problem:*

$$\left.\begin{aligned} &-\Delta y = \lambda m(x) y && \text{in } \Omega,\\ &y = 0 && \text{on } \Gamma \end{aligned}\right\}$$

possesses an infinite sequence of positive eigenvalues:

$$0 < \lambda_1(m) < \lambda_2(m) \le \ldots \lambda_k(m) \ldots; \lambda_k(m) \to \infty, \text{as} k \to \infty.$$

Moreover $\lambda_1(m)$ is simple, its associate eigenfunction $e_m$ is positive, and $\lambda_1(m)$ is characterized by:

$$\lambda_1(m) \int_\Omega m y^2 dx \le \int_\Omega |\nabla y|^2 dx \quad (7)$$

*Proof.* See[31].

Now, let

$$\lambda_1(a_i) \ge n-1, \quad i = 1,2,......,n. \quad (8)$$

**Lemma 2** *If (5) and (8) hold then, the bilinear form (6) satisfy the Gårding inequality*

$$\pi(t;\mathbf{y},\mathbf{y}) + c_0 \|\mathbf{y}\|^2_{\left(L^2(\Omega)\right)^n} \ge c_1 \|\mathbf{y}\|^2_{\left(H^1(\Omega)\right)^n}, \quad c_0, c_1 > 0. \quad (9)$$

*Proof.* In fact

$$\begin{aligned} \pi(t;\mathbf{y},\mathbf{y}) &= \sum_{i=1}^{n} \int_\Omega \left[ |\nabla y_i|^2 - a_i(x,t) y_i^2 \right] dx \\ &\quad - \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x,t) y_i y_j dx \\ &\ge \sum_{i=1}^{n} \int_\Omega \left[ |\nabla y_i|^2 - a_i(x,t) y_i^2 \right] dx \\ &\quad - 2 \sum_{i>j}^{n} \int_\Omega \sqrt{a_i(x,t) a_j(x,t)} y_i y_j dx. \end{aligned}$$

By Cauchy Schwarz inequality and (7),we obtain

$$\pi(t;\mathbf{y},\mathbf{y}) \geq \sum_{i=1}^{n}\left(1-\frac{1}{\lambda_1(a_i)}\right)\int_\Omega |\nabla y_i|^2 dx - \sum_{i=1}^{n}\int_\Omega a_i y_i^2 dx$$

$$-2\sum_{i>j}\int_\Omega\left(\frac{1}{\sqrt{\lambda_1(a_i)\lambda_1(a_j)}}\right)\left(\int_\Omega |\nabla y_i|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla y_j|^2 dx\right)^{\frac{1}{2}}$$

$$\geq \sum_{i=1}^{n}\left(\frac{\lambda_1(a_i)-n+1}{\lambda_1(a_i)}\right)\int_\Omega |\nabla y_i|^2 dx - \sum_{i=1}^{n}\int_\Omega a_i y_i^2 dx.$$

Finally, from (8) we have (9).

Under the above lemma ( Lemma 2) and using the results of Lions [20] and Lions and Magenes [21] we can prove the following theorems:

Now the solution of (4) is defined by transposition ( see [20-21]): **Theorem 1** There exists a uniqe solution $\mathbf{y} \in L^2(Q)$ for system (4) such that

$$\int_Q [-\frac{\partial \phi_i}{\partial t} - (A^*(t)\phi)_i] y_i(u) dxdt = \int_\Sigma u_i \frac{\partial \phi_i}{\partial \nu} d\Gamma dt \quad (10)$$

for all $\phi_i \in H^{2,1}(Q), \quad \phi_i(x,T)=0, \quad \phi_\Sigma = 0$

where $A^*(t)$ is the adjoint of $A(t)$ and the sobolev space $H^{r,s}(Q), r,s \geq 0$ (see [21]) is defined by

$$H^{r,s}(Q) = H^0\left(0,T;H^r(\Omega)\right) \cap H^s(0,T;H^0(\Omega))$$

$H^s(0,T;X)$ denotes the sobolev space of order $s$ of functions defined on $[0,T]$ and taking values in $X$.

### 3 Controllability problems

In this section, let $\mathbf{y}(\mathbf{u})$ denote to the unique solution of (4) . we take the three cases of observations:

### 3.1 Distributed observation

Let the observations be given by

$$y_i(\mathbf{u}) \in L^2(Q) \quad (11)$$

**Theorem 2** *Assume that (5) and (8) hold, then the system (4) with control* $\mathbf{u} \in (L^2(\Sigma))^n$ *and observation (11) is controllable.*

*Proof.* let us first remark that by translation we may always reduce the problem of controllability to the case were the system (4) with $y_{i,0}=0.$ We can show quit easily that (4) is controllable in $(L^2(Q))^n$ if and only if the observation generates a dense subspace of the space $(L^2(Q))^n$. By the Hahn-Banach theorem, this will be the case if

$$\int_Q \psi_i y_i(\mathbf{u}) dxdt = 0, \quad \psi_i \in L^2(Q), \quad (12)$$

for all $\mathbf{u} \in (L^2(\Sigma))^n$ implies that $\psi_i = 0, i=1,2,...,n.$

We introduce $\xi = (\xi_1,\xi_2,...,\xi_n)^T$ as the solution of the following system

$$\left.\begin{array}{ll} -\dfrac{\partial \xi_i}{\partial t} - \left(A^*(t)\xi\right)_i = \psi_i & \text{in} Q, \\ \xi_i(T) = 0 & \text{in}\Omega, \\ \xi_i = 0. & \text{on}\Sigma. \end{array}\right\} \quad (13)$$

The existence of a unique solution for system (13) can be proved using Theorem1, with an obvious change of variables.

Multiply the first equation in (13) by $y_i(\mathbf{u})$ and using Green formula, we obtain the following identity:

$$\int_Q \psi_i y_i(\mathbf{u}) dxdt = \int_Q\left[-\frac{\partial \xi_i}{\partial t} - \left(A^*(t)\xi\right)_i\right] y_i(\mathbf{u}) dxdt$$

$$= \int \xi_i\left[\frac{\partial}{\partial t} y_i(t;\mathbf{u}) + \left(A(t)y(t;\mathbf{u})\right)_i\right] dxdt - \int_\Sigma \frac{\partial \xi_i}{\partial \nu} u_i d\Gamma dt$$

$$= -\int_\Sigma \frac{\partial \xi_i}{\partial \nu} u_i d\Gamma d$$

and so, if (12) holds, then

$$\int_\Sigma \frac{\partial \xi_i}{\partial \nu} u_i d\Gamma d = 0 \quad \forall u_i \in L^2(\Sigma)$$

hence $\dfrac{\partial \xi_i}{\partial \nu} = 0,$ on $\Sigma$, The Cauchy data of $\xi_i$ on $\Sigma$, being zero, we conclude (see [32]) $\xi_i = 0.$ and hence $\psi_i = 0.$

### 3.2 Final state observation

Let the observation be given by

$$y(T;u) \in H_0^{-1}(\Omega) \quad (14)$$

Since $y_i(\mathbf{u}) \in L^2(Q)$ (and is defined by (10)) and $\Delta y_i(\mathbf{u}) \in L^2(0,T;H^{-2}(\Omega)),$ we deduce from (4) that $\dfrac{d}{dt} y_i(\mathbf{u}) \in L^2(0,T;H^{-2}(\Omega)),$ from which we may deduce (cf. [21] Chapter 1) that $t \to y_i(t;\mathbf{u})$ is continuous function from $[0,T] \to H_0^{-1}(\Omega)$ . Hence (14) has meaning and the observation is in $H_0^{-1}(\Omega)$

**Theorem 3** *Assume that (5) and (8) hold, then the system (4) with control* $\mathbf{u} \in (L^2(\Sigma))^n$ *and observation (14) is controllable.*

*Proof.* We can reduce the problem of controllability to the case were the system (4) with $y_{i,0}=0.$ To show the system is controllable let $\psi_i \in H^1(\Omega)$ such that

$\left\langle y_i(\mathbf{u}),\psi_i\right\rangle = 0, \forall \mathbf{u} \in (L^2(\Sigma))^n$ (15) Where the bracket denotes duality between $H^{-1}(\Omega)$ and

$H^1(\Omega)$. We introduce $\xi = (\xi_1, \xi_2, ..., \xi_n)^T$ as the solution of the following system

$$\left.\begin{array}{ll} -\dfrac{\partial \xi_i}{\partial t} - \left(A^*(t)\xi\right)_i = 0 & \text{in} Q, \\ \xi_i(T) = \psi_i & \text{in} \Omega, \\ \xi_i = 0. & \text{on} \Sigma. \end{array}\right\} \quad (16)$$

The existence of a unique solution for system (16) can be proved using Theorem1, with an obvious change of variables.

Multiply the first equation in (16) by $y_i(\mathbf{u})$ and using Green formula, we obtain the following identity:

$$0 = \int_Q \left[ -\frac{\partial \xi_i}{\partial t} - \left(A^*(t)\xi\right)_i \right] y_i(\mathbf{u}) dxdt.$$

$$= \int \xi_i \left[ \frac{\partial}{\partial t} y_i(t;\mathbf{u}) + \left(A(t) y(t;\mathbf{u})\right)_i \right] dxdt$$

$$+ \int_\Sigma u_i \frac{\partial \xi}{\partial \nu} d\Gamma dt - \langle y_i(\mathbf{u}), \psi_i \rangle$$

$$= \int_\Sigma u_i \frac{\partial \xi}{\partial \nu} d\Gamma dt - \langle y_i(\mathbf{u}), \psi_i \rangle$$

and so, if (15) holds, then

$$\int_\Sigma \frac{\partial \xi_i}{\partial \nu} u_i d\Gamma d = 0 \quad \forall u_i \in L^2(\Sigma)$$

hence $\dfrac{\partial \xi_i}{\partial \nu} = 0$, on $\Sigma$, The Cauchy data of $\xi_i$ on $\Sigma$, being zero, we conclude (see [32]) $\xi_i = 0$. and hence $\psi_i = 0$.

### . 3.3 Boundary observation

In this section, let $\mathbf{y}(\mathbf{u})$ denote to the unique solution of (4), corresponding to a given control $\mathbf{u} \in (L^2(\Sigma))^n$

Let the observation be given by

$$\frac{\partial y_i(\mathbf{u})}{\partial \nu} \in H^{-1}(\Sigma) \quad (17)$$

**Theorem 4** *Assume that (5) and (8) hold, then the system (4) with control* $\mathbf{u} \in (L^2(\Sigma))^n$ *and observation (17) is controllable.*

*Proof.* We can reduce the problem of controllability to the case were the system (4) with $y_{i,0} = 0$. To show the system is controllable let $\psi_i \in H^1(\Sigma)$ such that

$$\left\langle \frac{\partial y_i(\mathbf{u})}{\partial \nu}, \psi_i \right\rangle = 0, \forall \mathbf{u} \in (L^2(\Sigma))^n \quad (18)$$

Where the bracket denotes duality between $H^{-1}(\Sigma)$ and $H^1(\Sigma)$. We introduce $\xi = (\xi_1, \xi_2, ..., \xi_n)^T$ as the solution of the following system

$$\left.\begin{array}{ll} -\dfrac{\partial \xi_i}{\partial t} - \left(A^*(t)\xi\right)_i = 0 & \text{in} Q, \\ \xi_i(T) = 0 & \text{in} \Omega, \\ \xi_i = \psi_i. & \text{on} \Sigma. \end{array}\right\} \quad (19)$$

Multiply the first equation in (18) by $y_i(\mathbf{u})$ and using Green formula, we obtain the following identity:

$$0 = \int_\Sigma u_i \frac{\partial \xi}{\partial \nu} d\Gamma dt - \left\langle \frac{\partial y_i(\mathbf{u})}{\partial \nu}, \psi_i \right\rangle$$

and so, if (15) holds, then

$$\int_\Sigma \frac{\partial \xi_i}{\partial \nu} u_i d\Gamma d = 0 \quad \forall u_i \in L^2(\Sigma)$$

Hence $\psi_i = 0$.

## 4 CONCLUSION

In this study, we have proved the controllability to a special co-operative parabolic systems with Neumann conditions, with different cases of observation. Most of the results we described in this paper apply, without any change on the results, to more general parabolic systems involving the following second order operator :

$$L(x,.) = \sum_{i,j=1}^{n} b_{ij}(x,.) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x,.) \frac{\partial}{\partial x_j} + b_0(x,.)$$

with sufficiently smooth coefficients (in particular, $b_{ij}, b_j, b_0 \in L^\infty(Q), b_j, b_0 > 0$ ) and under the Legendre-Hadamard ellipticity condition

$$\sum_{i,j=1}^{n} \eta_i \eta_j \geq \sigma \sum_{i=1}^{n} \eta_i \quad \forall (x,t) \in Q,$$

for all $\eta_i \in \Re$ and some constant $\sigma > 0$.

In this case, we replace the first eigenvalue of the Laplace operator by the first eigenvalue of the operator $L$ (see [31]).

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