ANALYSIS OF STABILITY AND ACCURACY FOR FORWARD TIME CENTERED SPACE APPROXIMATION BY USING MODIFIED EQUATION

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Abstract: *In this paper we investigate the quantitative behavior of a wide range of numerical methods for solving linear partial differential equations [PDE's]. In order to study the properties of the numerical solutions, such as accuracy, consistency, and stability, we use the method of modified equation, which is an effective approach. To determine the necessary and sufficient conditions for computing the stability, we use a truncated version of modified equation which helps us in a better way to look into the nature of dispersive as well as dissipative errors. The heat equation with Drichlet Boundary Conditions can serve as a model for heat conduction, soil consolidation, ground water flow etc. Accuracy and Stability of Forward Time Centered Space (FTCS) scheme is checked by using Modified Differential Equation [MDE].*

Key words: Accuracy, Stability, Modified Equation, Dispersive error, Forward Time Center Space Scheme.

1. INTRODUCTION

To analyze the simple linear partial differential equation with the help of modified equation, we consider one dimensional heat equation (Transient diffusion equation) which is parabolic partial differential equation. This equation describes the temperature distribution in a bar as a function of time. For converting this simple PDE into a modified equation, we use finite difference approximations by using the initial value conditions. This is obtained by expanding each term of finite difference approximation into a Taylor series, excluding time derivative, time - space derivatives higher than first order. Terms occurring in this MDE represent a sort of truncation error. These permit the order of stability and accuracy. With this approach, a modified equation, which is an approximating differential equation that is a more accurate model of what is actually solved numerically by the use of given numerical schemes. To explain this scheme we are taking a long thin bar of homogeneous material. The temperature in a long thin bar must be insulated perfectly to maintain the flow of heat horizontally.

2. MATERIAL AND METHODS

2.1 Modified Equation

Modified Equation [1] is used MDE to analyze the accuracy and stability of the solution originally solved by PDE's. This is obtained by expanding each term of finite difference equation with the help of Taylor Series. The general technique of developing modified equation for PDE's is presented by Warming and Hyett [2].

We know that the general Linear PDE [3] is represented as

$$\partial f/\partial t + \mathcal{L}_x(f) = 0$$

Where $\mathcal{L}_x(f)$ is a linear spatial differential operator, f is a function of a spatial variable x. A more specific example of this conviction equation, is

$$\frac{\partial f}{\partial t} + c\left(\frac{\partial f}{\partial x}\right) = 0$$

where "c" is a real constant. The modified equation is used to deal with the numerical solution's behavior.

2.2 Difference Approximation

We are using here forward difference approximations [4]

$$f_t = \frac{f_i^{n+1} - f_i^n}{\Delta t}$$

$$f_{xx} = \frac{f_{i+1}^n - 2f_i^n + f_{i-1}^n}{\Delta x^2}$$

2.3 Forward Time Centered Space Scheme

The partial differential equation of the following form [5] has been used for forward Time Centre Space Scheme

$$\frac{\partial f}{\partial t} = \alpha \frac{\partial^2 f}{\partial x^2} \qquad (1)$$

$$f_t = \alpha f_{xx}$$

where

$$\alpha = \frac{K}{\sigma\rho}$$

K = thermal conductivity,
σ = specific heat,
ρ = density of material of body

Equation (1) describes the temperature distribution in a bar with ends x=0 and x=l as a function of time with boundary conditions f(0,t)=0 and f(l, t)=0 for all t. Equation (1) becomes.

$$\frac{f_i^{n+1} - f_i^n}{\Delta t} = \alpha \frac{f_{i+1}^n - 2f_i^n + f_{i-1}^n}{\Delta x^2} \qquad (2)$$

By using forward difference approximation when dealing with t and central difference approximation when dealing with x at the same point (i,n) with truncation errors $O(\Delta t)$ and $O(\Delta x^2)$ equation (2) may be written in terms of f_i^{n+1}

i.e.

$$f_i^{n+1} = f_i^n + \alpha \frac{\Delta t}{\Delta x^2}(f_{i+1}^n - 2f_i^n + f_{i-1}^n) \qquad (3)$$

or $f_i^{n+1} = f_i^n + d(f_{i+1}^n - 2f_i^n + f_{i-1}^n)$ (4)

and

$$d = \alpha \frac{\Delta t}{\Delta x^2}$$

which is called diffusion constant.

Equation (3) has been discussed within the region shown in figure1.

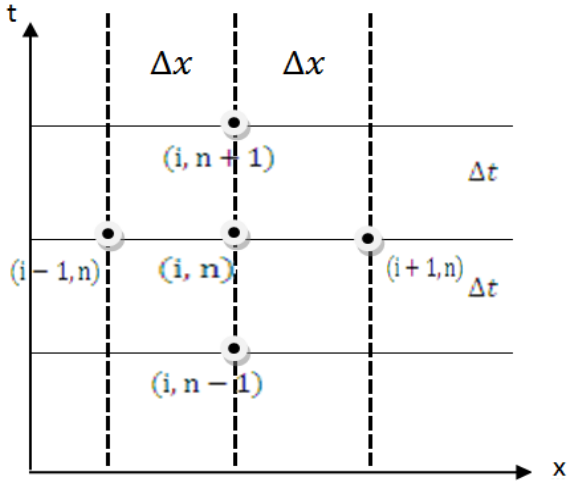

Figure-1 Grid points under discussion

For the solution of problems with this method, we need both boundary conditions and initial conditions. This method is also called an explicit method.

2.4 Analysis of FTCS Approximation using Modified Equation

The modified equation for the FTCS approximation for $f_t = \alpha f_{xx}$ is

$$f_t = \alpha f_{xx} + \left(\frac{1}{12} \propto \Delta x^2 - \frac{1}{2} \propto^2 \Delta t\right) f_{xxxx} + \left(\frac{1}{360} \propto \Delta t^4 - \frac{1}{12} \propto^2 \Delta t \Delta x^2 + \frac{1}{3} \propto^3 \Delta t^2\right) f_{xxxxxx} + \cdots \quad (5)$$

As the leading term in equation (5) is an even derivative, the solution of the equation $f_t = \alpha f_{xx}$ for the FTCS approximation becomes

$$f_i^{n+1} = f_i^{n} - \frac{\alpha \Delta t}{\Delta x^2}\left(f_{i+1}^n - 2 f_i^{n} + f_{i-1}^n\right)$$

which predominately exhibits dissipative error [6]. The lowest-order even derivate on the right-hand side of the modified equation (5) is 2. Therefore for the modified equation (5), the stability condition is $C\,(2 = 2l) > 0$, which implies that $\alpha > 0$. However this yields no useful information since this parameter is chosen to be positive, and it is a coefficient in the original equation. However, if we implement the more general stability condition given by

$$\left(8\frac{\Delta t}{\Delta x^4}\right)\left[\frac{\Delta x^2}{3} C(2l) - \Delta t C^2(2l) - C(4l)\right] > 0 \; if \;\; l = 1$$

we obtain

$$\left(8\frac{\Delta t}{\Delta x^4}\right)\left[\frac{\Delta x^2}{3} \propto - \Delta t \propto^2 - \left(\frac{1}{12} \propto \Delta x^2 - \frac{1}{2} \propto^2 \Delta t\right)\right] > 0$$

which implies

$$4r\left(\frac{1}{2} - r\right) > 0$$

Hence, the necessary and sufficient condition for stability is $r < 1/2$, which is the well-known stability condition. In the above expression r is $\alpha \frac{\Delta t}{\Delta x^2}$.

2.5 Consistency

In equation (5) as $\Delta t \to 0$ and $\Delta x \to 0$, then equation (5) approaches equation

$$f_t = \alpha f_{xx}.$$

Consequently, equation

$$f_i^{n+1} = f_i^{n} - \frac{\propto \Delta t}{\Delta x^2}\left(f_{i+1}^n - 2 f_i^{n} + f_{i-1}^n\right)$$

is a consistent approximation of the equation $f_t = \alpha f_{xx}$.

2.6 Order of Accuracy

From the following equation

$$f_t = \alpha f_{xx} + \left(\frac{1}{12}\alpha \Delta x^2 - \frac{1}{2}\alpha^2 \Delta t\right) f_{xxxx} + \left(\frac{1}{360}\alpha \Delta t^4 - \frac{1}{12}\alpha^2 \Delta t \Delta x^2 + \frac{1}{3}\alpha^3 \Delta t^2\right) f_{xxxxxx} + \cdots$$

The truncation error is $O(\Delta t) + O(\Delta x^2)$. This means that the FTCS approximation

$$f_i^{n+1} = f_i^{n} + \frac{\alpha \Delta t}{\Delta x^2}\left(f_{i+1}^n - 2 f_i^{n} + f_{i-1}^n\right)$$

of PDE is first order accurate in Δt and 2nd order accurate in Δx.

2.7 Von-Neumann Stability Analysis for the FTCS Scheme

The FTCS approximation of Equation $f_t = \alpha f_{xx}$ is

$$f_i^{n+1} = f_i^{n} + \frac{\alpha \Delta t}{\Delta x^2}\left(f_{i+1}^n - 2 f_i^{n} + f_{i-1}^n\right)$$

To find G, using $f_i^n = G^n e^{lkx}$ into FTCS approximation we have

where I= $\sqrt{-1}$, K is real wave number, G= G(K) is called the amplification factor, in general a complex constant then we have

$$G^{n+1} e^{IKx} = G^n e^{IKx} + \frac{\alpha \Delta t}{\Delta x^2}\left(G^n e^{IK(x+\Delta x)} - 2G^n e^{IKx} + G^n e^{IK(x-\Delta x)}\right)$$

$$G^n . G . e^{IKx} = G^n e^{IKx}\left[1 + \frac{\alpha \Delta t}{\Delta x^2}\left(e^{IK\Delta x} - 2 + \bar{e}^{IK\Delta x}\right)\right]$$

$$G = 1 + \frac{\alpha \Delta t}{\Delta x^2}\left(e^{I\beta} - 2 + \bar{e}^{I\beta}\right)$$

$$G = 1 + \frac{\alpha \Delta t}{\Delta x^2}\left(2 Cos\beta - 2\right)$$

$$G = 1 + 2\frac{\alpha \Delta t}{\Delta x^2}\left(Cos\beta - 1\right)$$

$$G = 1 + 2r\left(Cos\beta - 1\right)$$

$$r = \frac{\alpha \Delta t}{\Delta x^2}$$

For stability

$$|G| \le 1$$

i.e.

$$-1 \le 1 + 2r(\cos\beta - 1) \le 1$$

The upper limit is always satisfied for r ≥1

Because $\left(Cos\beta - 1\right)$ varies between -2 and 0 as β ranges from -∞ to ∞. The lower limit

$$-1 \le 1 + 2r(\cos\beta - 1)$$

implies

$$r \le \frac{1}{(1 - \cos\beta)}$$

The minimum value of r corresponds to the maximum value of (1 – Cos β). As β ranges from -∞ to ∞. (1 – Cos β) varies between 0 and 2. Consequently, the minimum value of r is ½. Thus $|G| \le 1$ if

$$0 \le r \le \frac{1}{2}$$

Consequently, the FTCS approximation

$$f_i^{\,n+1} = f_i^{\,n} + \frac{\alpha \Delta t}{\Delta x^2}\left(f_{i+1}^{n} - 2 f_i^{\,n} + f_{i-1}^{n} \right)$$

of the equation is conditionally stable. Since the amplification factor of the FTCS scheme has no imaginary part, it has no phase shift. In order to find the exact amplification (decay) factor, we substitute the elemental solution.

$$f = e^{-\alpha k^2 t} e^{Ikx}$$

in the following equation

$$G_e = \frac{f(t+\Delta t)}{f(t)}$$

which reduces to

$$G_e = e^{-\alpha k^2 \Delta t}$$
$$= e^{-r\beta^2}$$

3. RESULTS AND DISCUSSION

The amplitude of the exact solution decreases by the factor $e^{r\beta^2}$ during one time step, assuming no boundary condition influence. The amplification factor is plotted in figure 2 for two values of r and is compared with the exact amplification factor of the solution. The diamond signs are the graph of G for r = 1/6, the solid line is the graph of $|G_e|$ for r = 1/6, the plus signs are the graph of G for r=1/2 and the dashed line is the graph of $|G_e|$ for r = ½. In this figure 2, we observe that the FTCS is highly dissipative for large value of β where $r = \frac{1}{2}$. As expected, the amplification factor agrees closer with the exact decay when $r = \frac{1}{6}$.

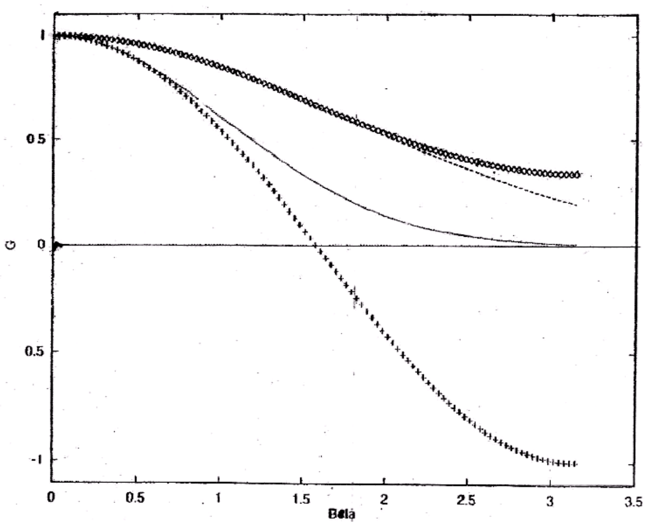

Figure-2 Amplification factor modulus for FTCS scheme

4. CONCLUSION

MDE's in specific problems are more convenient for discussing the solution behavior, including physical interpretation, i.e. accuracy, stability and consistency. Many ordinary and higher order boundary value problems have been analyzed with the help of modified equation. The appropriate solution converges rapidly to accurate solution. So we say that MDE's are more beneficial for future use.

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