SOLUTION OF HIGHER ORDER NONLINEAR BOUNDARY VALUE PROBLEMS

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ABSTRACT: *In this paper, we present efficient numerical algorithms for the approximate solution of non-linear higher order boundary value problems. Algorithms are, based on Adomian decomposition. Also, the Laplace Transformation with Adomian decomposition technique is proposed to solve the problems. Three examples are given to illustrate the performance of each technique.*

Keywords: Initial Value Problems, Adomian Decomposition Method, Laplace Transformation, Non-linear Higher Order Boundary Value Problems.

1. INTRODUCTION

By the Adomian decomposition method (ADM) there is offered a genuine technique of boundary value problems (BVPs) for nonlinear ordinary and partial differential equations [1-2]. In addition ADM can give estimated analytic solution without applying the Green function thought, which significantly makes possible systematic estimation and numeric computation [3-5].

Nonlinear algebraic equations in the unresolved coefficients are frequently concerned, which enlarge the complication of the calculation [6-8]. For solving nonlinear algebraic equations for a two-point BVPs for second-order nonlinear differential equations, the famous authors Adomian and Rach introduce the double decomposition method in organize to keep away and the different transform inverse linear operator introduce by Jang and Ebaid. Agarwal and Wazwaz solved BVPs solved higher order BVPs first time with the help of ADM [9-12].

2. MATERIAL AND METHODS

Consider the nonlinear differential equation of second order

$$Lu = Nu + g(x), \quad a \le x \le b \tag{1}$$

By the Dirichlet boundary form

$$u(a) = \alpha \, , u(b) = \beta$$

where as $L(.) = \frac{d^2}{dx^2}(.)$ Nu is a systematic nonlinear operator and g (x) is the scheme participation.

The inverse linear differential operator is

$$L^{-1}(.) = \int_a^x \int_\xi^x (.)dxdx \tag{2}$$

$$L^{-1}Lu = u(x) - u(a) - u'(\xi)(x-a)$$

where$\xi \epsilon [a,b]$ is a fix value.

Using the operator L^{-1} to both sides of eq. (1) give up

$$u(x) - u(a) - u'(\xi)(x-a) = L^{-1}Nu + L^{-1}g(x) \tag{3}$$

Let x=b in eq. (3) and solve for $u'(\xi)$, then

$$u'(\xi) = \frac{1}{b-a}[u(b) - u(a) - [L^{-1}Nu]_{x=b} - [L^{-1}g(x)]_{x=b}] \tag{4}$$

Where $L^{-1}(.)]_{x=b} = \int_a^b \int_\xi^x (.)dxdx$

Substituting eq. (4) into eq. (3) gives

$$u(x) = u(a) + \frac{u(b)-u(a)}{b-a}(x-a) + L^{-1}g - \frac{x-a}{b-a}[L^{-1}g]_{x=b} + L^{-1}Nu - \frac{x-a}{b-a}[L^{-1}Nu]_{x=b} \tag{5}$$

Consequently the right hand side of eq. (5) does not hold the uncertain coefficient $u_0(x)$.

Next, we verify the solution and the nonlinearity

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \; Nu = \sum_{n=0}^{\infty} A_n \tag{6}$$

Where $A_n = A_n(u_0(x), u_1(x), \cdots u_n(x))$ are the Adomianpolynomials

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n} N(\textstyle\sum_{n=0}^{\infty} u_k \lambda^k)_{\lambda=0} \; , \; n \ge 0 \tag{7}$$

The Adomian polynomials for nonlinearity Nu = f(u)

$$A_0 = f(u_0)$$

$$A_1 = f'(u_0)u_1$$

$$A_2 = f'(u_0)u_2 + f''(u_0)\frac{u_1^2}{2!}$$

$$A_3 = f'(u_0)u_3 + f''(u_0)u_1u_2 + f'''(u_0)\frac{u_1^3}{3!}$$

$$A_4 = f'(u_o)u_4 + f''(u_0)\left(\frac{u_2^2}{2!} + u_1u_3\right) + f'''(u_0)\frac{u_1^2u_2}{2!} + f^{(4)}(u_0)\frac{u_1^4}{4!}$$

Such as $Nu = f(x,u,u')$ where derivatives are parameterized by k,

$$A_n = \sum_{k=0}^{\infty} f^{(k)}(u_0)c(k,n), \quad n \ge 1$$

From eq. (5)

$$u_0 = u(a) + \frac{u(b)-u(a)}{b-a}(x-a) + L^{-1}g - \frac{x-a}{b-a}[L^{-1}g]_{x=b} \tag{8}$$

$$u_{n+1} = L^{-1}A_n - \frac{x-a}{b-a}[L^{-1}A_n]_{x=b}\, , \quad n \ge 0 \tag{9}$$

Or explicitly

$$u_{n+1} = \int_a^x \int_\xi^x A_n dxdx - \frac{x-a}{b-a}[\int_a^x \int_\xi^x A_n dxdx]_{x=b}\, , n \ge 0$$

We highlight to the right hand side of eq. (8) and (9) are self-determining of n, which contain n, will cancel.

$$h^{[1]}(x) = \int h(x)dx\, , \quad h^{[2]}(x) = \int h^{[1]}(x)dx$$

Where the right hand side signifies pure integration

$$L^{-1}h(x) - \frac{x-a}{b-a}[L^{-1}h(x)]_{x=b} = \int_a^x \int_\xi^x h(x)dxdx - \frac{x-a}{b-a}[\int_a^x \int_\xi^x h(x)dxdx]_{x=b}$$

$$= h^{[2]}(x) - h^{[1]}(a) - (x-a)h^{[1]}(\xi) - \frac{x-a}{b-a}[h^{[2]}(x) - h^{[2]}(a) - (x-a)h^{[1]}(\xi)]_{x=b}$$

$$= h^{[2]}(x) - h^{[2]}(a) - \frac{x-a}{b-a}[h^{[2]}(b) - [h^{[2]}(b) - h^{[2]}(a)]$$

Therefore the components

$$u_0 = c$$

$$u_1 = -c + u(a) + \frac{u(b)-u(a)}{b-a}(x-a) + L^{-1}g - \frac{x-a}{b-a}[L^{-1}g]_{x=b} + L^{-1}A_0 \quad - \frac{x-a}{b-a}[L^{-1}A_0]_{x=b}$$

$$u_{n+2} = L^{-1}A_{n+1} - \frac{x-a}{b-a}[L^{-1}A_{n+1}]_{x=b}, \; n \ge 0$$

where the constant c. In the subsequent we regard as a nonlinear differential equation

$$Lu = Nu + g(x) \tag{10}$$

Subject to the mixed set of Neumann and Dirichlet boundary conditions

$u(x_0)=\alpha_0, u'(x_1)=\alpha_1, u'(x_2)=\alpha_2 , x_1\neq x_2$

Where $L=\frac{d^3}{dx^3}$ linear differential operator to be reversed, *Nu* is a systematic nonlinear operator and g (x) is the system contribution. The field of x in eq. (10) is $min\{x_0,x_1,x_2\}\leq x\leq \max\{x_0,x_1,x_2\}$

the inverse linear operator is defined as

$L^{-1}(.)=\int_{x_0}^{x}\int_{x_1}^{x}\int_{\xi}^{x}(.)dxdxdx$

Where ξ is a given value in the particular interval, next we include

$L^{-1}Lu=u(x)-u(x_0)-u'(x_1)(x-x_0)-\frac{1}{2}u''(\xi)[(x-x_1)^2-(x_0-x_1)^2]$

Relate the inverse operator L^{-1} of eq. (10) give way

$$u(x)-u(x_0)-u'(x_1)(x-x_0)-\frac{1}{2}u''(\xi)[(x-x_1)^2-(x_0-x_1)^2]=L^{-1}[Nu+g] \quad (11)$$

Differentiate eq. (11), then let $x=x_2$ and solve for$u''(\xi)$, hence

$$u''(\xi)=\frac{u'(x_2)-u'(x_1)}{x_2-x_1}-\frac{1}{x_2-x_1}\int_{x_1}^{x_2}\int_{\xi}^{x}[Nu+g]dxdx \quad (12)$$

putting eq. (12) into eq. (11) give in

$$u(x)=u(x_0)+u'(x_1)(x-x_0)+\frac{1}{2}[(x-x_1)^2-(x_0-x_1)^2]\frac{u'(x_2)-u'(x_1)}{x_2-x_1}+L^{-1}g-\frac{1}{2}\frac{(x-x_1)^2-(x_0-x_1)^2}{x_2-x_1}\int_{x_1}^{x_2}\int_{\xi}^{x}gdxdx+L^{-1}Nu-\frac{1}{2}\frac{(x-x_1)^2-(x_0-x_1)^2}{x_2-x_1}\int_{x_1}^{x_2}\int_{\xi}^{x}Nudxdx \quad (13)$$

Thus in eq. (13), the boundary values $u(x_0), u'(x_1) and\ u'(x_2)$ are included and the undetermined coefficient was replaced.

$u(x)=\sum_{n=0}^{\infty}u_n(x), Nu=\sum_{n=0}^{\infty}A_n$,Where $A_n=A_n(u_0(x),u_1(x),\dots,u_n(x))$ are the Adomian polynomials. Starting eq. (13), the components are

$$u_0=u(x_0)+u'(x_1)(x-x_0)+\frac{1}{2}[(x-x_1)^2-(x_0-x_1)^2]\frac{u'(x_2)-u'(x_1)}{x_2-x_1}+L^{-1}g-\frac{1}{2}\frac{(x-x_1)^2-(x_0-x_1)^2}{x_2-x_1}\int_{x_1}^{x_2}\int_{\xi}^{x}gdxdx+L^{-1}Nu-\frac{1}{2}\frac{(x-x_1)^2-(x_0-x_1)^2}{x_2-x_1}\int_{x_1}^{x_2}\int_{\xi}^{x}Nudxdx$$

$$u_{n+1}=L^{-1}A_n-\frac{1}{2}\frac{(x-x_1)^2-(x_0-x_1)^2}{x_2-x_1}\int_{x_1}^{x_2}\int_{\xi}^{x}A_ndxdx,\quad n\geq 0$$

Or by the recursion method

$$u_0=c$$

$$u_1=-c+u(x_0)+u'(x_1)(x-x_1)+\frac{1}{2}[(x-x_1)^2-(x_0-x_1)^2]\frac{u'(x_2)-u'(x_1)}{x_2-x_1}+L^{-1}g-\frac{1}{2}\frac{(x-x_1)^2-(x_0-x_1)^2}{x_2-x_1}\int_{x_1}^{x_2}\int_{\xi}^{x}gdxdx+L^{-1}A_0-\frac{1}{2}\frac{(x-x_1)^2-(x_0-x_1)^2}{x_2-x_1}\int_{x_1}^{x_2}\int_{\xi}^{x}A_0dxdx$$

$$u_{n+1}=L^{-1}A_n-\frac{1}{2}\frac{(x-x_1)^2-(x_0-x_1)^2}{x_2-x_1}\int_{x_1}^{x_2}\int_{\xi}^{x}A_ndxdx$$

In follow, the n-term approximation

$$\varphi_n(x)=\sum_{k=0}^{\infty}u_k$$

For a general k[th]-order

$$\frac{d^k}{dx^k}u=Nu+g(x),\ k\geq 2$$

Subject to k boundary conditions

$$u^{(p_0)}(x_0)=\alpha_0, u^{(p_1)}(x_1)=\alpha_1,\dots u^{(p_{k-1})}(x_{k-1})=\alpha_{k-1}$$

then we obtain the inverse linearoperator like

$$L^{-1}(.)=\int_{x_0}^{x}\int_{x_1}^{x}\int_{x_2}^{x}\cdots\int_{x_{k-1}}^{x}(.)dx^k$$

In this case $p_1=0, p_2=p_3=\cdots p_{k-1}=2, k\geq 3.$ the inverse linear operator as

$$L^{-1}(.)=\int_{x_0}^{x}\int_{\xi}^{x}\int_{x_2}^{x}\int_{\xi}^{x}\int_{\xi}^{x}\cdots\int_{\xi}^{x}(.)dx^k \quad (14)$$

the boundary values u (x_0) and $u''(x_2)$ and where n is a given value in the specific interval. The replacing $x_0\ with\ x_1 and/or x_2 with\ x_j, for\ 3\leq j\leq k-1$, in eq. (14)

$$u-u(x_0)-u'(\xi)(x-x_0)-u''(x_2)\frac{(x-\xi)^2-(x_0-\xi)^2}{2}-\sum_{l=0}^{k-4}u^{(l+3)}(\xi)[\frac{(x-\xi)^{l+3}-(x_0-\xi)^{l+3}}{(l+3)!}-\frac{(x_2-\xi)^{l+1}}{(l+1)!}\frac{(x-\xi)^2-(x_0-\xi)^2}{2}]=L^{-1}Nu+L^{-1}g \quad (15)$$

Where there are k-2 undetermined coefficients $u'(\xi),u^{(3)}(\xi),u^{(4)}(\xi),\cdots,u^{(k-1)}(\xi)$, which know how to be solve by the residual k-2 boundary values $u(x_1),u''(x_3),u''(x_4),\cdots,u''(x_{k-1})$ through eq. (15).

3. NUMERICAL ILLUSTRATION

Example 1

Consider the following BVP with product nonlinearity

$$u^{(4)}(x)+u(x)u'(x)-4x^7-24=0,\quad 0\leq x\leq 1$$

$$u(0)=0,\qquad u'''(0.25)=6, u''(0.5)=3, u(1)=1$$

Solution: The exact solution for this is $u^*(x)=x^4$ **.**we have

$L^{-1}(.)=\int_0^x\int_0^x\int_{0.5}^x\int_{0.25}^x(.)dxdxdxdx$

Then for the boundary values at x = 0, 0.25 and 0.5,

$\int_0^x\int_0^x\int_{0.5}^x\int_{0.25}^x u^{(4)}(x)dxdxdxdx=\int_0^x\int_0^x\int_{0.5}^x[u'''(x)-u'''(0.25)]dxdxdx$

$$u(x)-u(0)-xu'(0)-x^3=u(x)-u'(0)x-x^3$$

$$L^{-1}u^{(4)}(x)=u(x)-u'(0)x-x^3$$

In addition

$\int_0^x\int_0^x\int_{0.5}^x\int_{0.25}^x[4x^7+24]dxdxdxdx=\int_0^x\int_0^x\int_{0.5}^x[\frac{4x^8}{8}+24x]_{0.25}^x dxdxdx$

$$L^{-1}(4x^7+24)=\frac{x^{11}}{1980}+x^4-\frac{786433}{786432}x^3-\frac{1}{18432}x^2 \quad (16)$$

with the boundary value at x=1

$$u'(0)=\frac{-117157}{259522560}+[L^{-1}uu']_{x=1} \quad (17)$$

Substituting eq. (17) into (16), we have

$$u(x)=\frac{-117157x}{259522560}-\frac{247x^2}{4718592}-\frac{x^3}{786432}+x^4+\frac{x^{11}}{1980}-L^{-1}u(x)u'(x)+x[L^{-1}u(x)u'(x)]_{x=1}$$

The solution and the nonlinearity $u=\sum_{n=0}^{\infty}u_n,\ \ uu'=\sum_{n=0}^{\infty}A_n$,respectively, where the Adomian polynomials also are

$A_0=u_0u_0'$

$A_1=u_0u_1'+u_1u_0'$

$A_2=u_0u_2'+u_1u_1'+u_2u_0'$

$A_3=u_0u_3'+u_1u_2'+u_2u_1'+u_3u_0'\cdots$

By the modified recursion method

$u_0=\frac{-117157x}{259522560}-\frac{247x^2}{4718592}-\frac{x^3}{786432}+x^4+\frac{x^{11}}{1980}$

$u_n=-L^{-1}A_{n-1}+x[L^{-1}A_{n-1}]_{x=1}\ \ n=1,2,\cdots$

$u_1=-L^{-1}A_0+x[L^{-1}A_0]_{x=1},\quad n=1,2,$

$=-L^{-1}u_0u_0'+x[L^{-1}u_0u_0']_{x=1}$

We record $u_1 and\ u_2$ as follows,

$u_1=-9.241884516\times10^{-12}x^{25}-1.031557404\times10^{-7}x^{18}+1.574031684\times10^{-13}x^{17}+7.86828457\times10^{-12}x^{16}+$

$8.351515442 \times 10^{-11}x^{15} - 0.0005050505051x^{11} + 1.766063549 \times 10^{-9}x^{10} + 1.038613547 \times 10^{-7}x^{9} + 1.343549849 \times 10^{-6}x^{8} - 9.257546841 \times 10^{-12}x^{7} - 1.96922983 \times 10^{-10}x^{6} - 1.698263226 \times 10^{-9}x^{5} + 1.197071763 \times 10^{-6}x^{3} + 0.00005184309329x^{2} + 0.0004506661406x,$

u_2

$=-3.38332271178819 \times 10^{-21}x^{31} - 1.8299088362943092 \times 10^{-19}x^{30} - 2.114188144007494 \times 10^{-18}x^{29} + 2.595882268568207 \times 10^{-11}x^{25} - 9.721045476235172 \times 10^{-17}x^{24} - 6.185141837504404 \times 10^{-15}x^{23} - 8.751553594634062 \times 10^{-14}x^{22} + 6.59731186730981 \times 10^{-19}x^{21} + 1.5698793638472888 \times 10^{-17}x^{20} + 1.5400931909423285 \times 10^{-16}x^{19} + 1.0315574041064238 \times 10^{-7}x^{18} - 7.384436702019146 \times 10^{-13}x^{17} - 4.657207413215791 \times 10^{-11}x^{16} - 6.590311932698167 \times 10^{-10}x^{15} + 7.875438264780266 \times 10^{-15}x^{14} + 1.8306469783644867 \times 10^{-13}x^{13} + 1.7460492288019172 \times 10^{-12}x^{12} - 1.6814917783267445 \times 10^{-17}x^{11} - 1.66259991822622368 \times 10^{-9}x^{10} - 1.0286327883752899 \times 10^{-7}x^{9} - 1.3412678929520298 \times 10^{-6}x^{8} + 1.8224924175027626 \times 10^{-11}x^{7} + 3.9161191589406249 \times 10^{-10}x^{6} + 3.3907580800253106 \times 10^{-9}x^{5} + 7.322810785423231 \times 10^{-8}x^{3} + 5.013353855662456 \times 10^{-7}x^{2} + 7.6495524731979151 \times 10^{-7}x$

we plot the error functions $E_n(x) = \varphi_n(x) - u^*(x)$ *for* $n = 1$ *and* 2 , respectively.

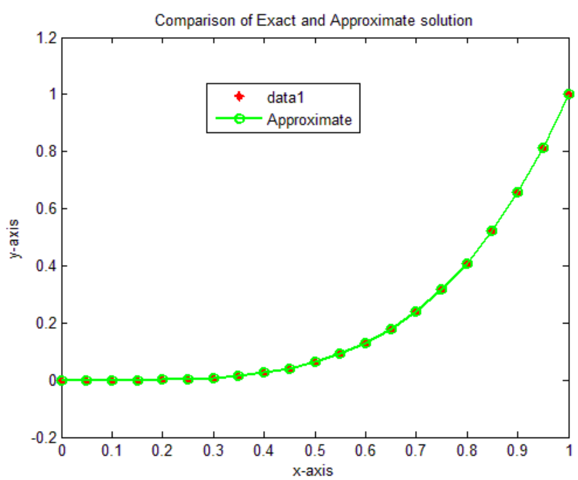

Figure 1: Graph of Example 1

Example 2

Let the nonlinear differential equation of fourth-order with two-point BVP through an exponential nonlinearity

$$u^{(4)}(x) = -6e^{-4u(x)}, \quad 0 \le x \le 4-e$$

$$u(0) = 1\,, u''(0) = \frac{-1}{e^2}\,,\; u(4-e) = \ln(4), \quad u''(4-e) = \frac{-1}{16}$$

Solution:

The exact solution is $u^*(x) = \ln(e + x)$

We take

$$L^{-1}(.) = \int_0^x \int_0^x \int_0^x \int_0^x (.)dxdxdxdx$$

Then for boundary values at x = 0

$$L^{-1}u^{(4)}(x) = -6L^{-1}e^{-4u}$$

$$u(x) - u(0) - xu'(0) - \frac{x^2}{2}u''(0) - \frac{x^3}{6}u'''(0) = -6L^{-1}e^{-4u}$$

$$u(x) - 1 - u'(0)x + \frac{x^2}{2e^2} - \frac{x^3}{6}u'''(0) = -6L^{-1}e^{-4u} \qquad (18)$$

Let x=4-e in eq. (18), we obtain

$$(4-e)u'(0) + \frac{(4-e)^3}{6}u'''(0) = \ln(4) - 1 + \frac{(4-e)^2}{2e^2} + 6[L^{-1}e^{-4u}]_{x=4-e} \qquad (19)$$

Distinguish eq. (19) two time, then let $x = 4 - e$, we get

$$(4-e)u'''(0) = \frac{1}{e^2} - \frac{1}{16} + 6\int_0^{4-e}\int_0^x e^{-4u}dxdx \qquad (20)$$

Solving for $u'(0)$ *and* $u'''(0)$ from eqns. (19) and (20) yields

$$(4-e)u'(0) + \frac{(4-e)^3}{6}u'''(0) = \ln(4) - 1 + \frac{(4-e)^2}{2e^2} + 6[L^{-1}e^{-4u}]_{x=4-e}$$

$$u'(0) = \frac{(4-e)(32+e^2)}{96e^2} + \frac{\ln(4)-1}{4-e} - (4-e)\int_0^{4-e}\int_0^x e^{-4u}dxdx + \frac{6}{4-e}[L^{-1}e^{-4u}]_{x=4-e} \qquad (21)$$

$$(4-e)u'''(0) = \frac{1}{e^2} - \frac{1}{16} + 6\int_0^{4-e}\int_0^x e^{-4u}\,dxdx$$

$$u'''(0) = \frac{(4+e)}{16e^2} + \frac{6}{4-e}\int_0^{4-e}\int_0^x e^{-4u}dxdx \qquad (22)$$

Substituting eqns. (21) and (22) into (18)

$$u(x) = 1 + x\left[\frac{(4-e)(32+e^2)}{96e^2} + \frac{\ln(4)-1}{(4-e)}\right] - \frac{x^2}{2e^2} + \frac{(4+e)x^3}{96e^2} - (4-e)x\int_0^{4-e}\int_0^x e^{-4u}dxdx + \frac{6x}{4-e}[L^{-1}e^{-4u}]_{x=4-e} + \frac{x^3}{4-e}\int_0^{4-e}\int_0^x e^{-4u}dxdx - 6L^{-1}e^{-4u}$$

Therefore,

$$u_0 = 1$$

$$u_1 = x\left[\frac{(4-e)(32+e^2)}{96e^2} + \frac{\ln(4)-1}{(4-e)}\right] - \frac{x^2}{2e^2} + \frac{(4+e)x^3}{96e^2} - (4-e)x\int_0^{4-e}\int_0^x A_0dxdx + \frac{6x}{4-e}[L^{-1}A_0]_{x=4-e} + \frac{x^3}{4-e}\int_0^{4-e}\int_0^x A_0dxdx - 6L^{-1}A_0$$

$$u_n = -(4-e)x\int_0^{4-e}\int_0^x A_{n-1}dxdx + \frac{6x}{4-e}[L^{-1}A_{n-1}]_{x=4-e} + \frac{x^3}{4-e}\int_0^{4-e}\int_0^x A_{n-1}dxdx - 6L^{-1}A_{n-1}, \quad n \ge 2$$

Where the Adomian polynomials are

$$A_0 = e^{-4u_0}$$

$$A_1 = -4e^{-4u_0}u_1$$

$$A_2 = 4e^{-4u_0}(2u_1^2 - u_2)$$

$$A_3 = -4e^{-4u_0}\left(\frac{8}{3}u_1^3 - 4u_1u_2 + u_3\right)\cdots$$

Thus we can determine as

$$u_1 = x\left[\frac{(4-e)(32+e^2)}{96e^2} + \frac{\ln(4)-1}{4-e}\right] - \frac{x^2}{2e^2} + \frac{(4+e)x^3}{96e^2} - (4-e)x\int_0^{4-e}\int_0^x e^{-4}dxdx + \frac{6x}{4-e}\left[\int_0^x\int_0^x\int_0^x\int_0^x e^{-4}dxdxdxdx\right]_{x=4-e} + \frac{x^3}{4-e}\int_0^{4-e}\int_0^x e^{-4}dxdx - 6\int_0^x\int_0^x\int_0^x\int_0^x e^{-4}\,dxdxdxdx$$

$$u_1 = 0.3629183x - 0.0676676x^2 + 0.0212088x^3 - 0.00457891x^4$$

$$u_2 = 0.00746686x - 0.00658100x^3 + 0.00132942x^5 - 0.000082625x^6 + 0.000011099x^7 - 1.1980808 \times 10^{-6}x^8$$

$$u_3 = -0.003358787x + 0.002575707x^3 + 0.000027352x^5 - 0.000321646x^6 + 0.00004796097x^7 - 0.00001045196x^8 + 1.9939469 \times 10^{-6}x^9 - 1.9376435 \times 10^{-7}x^{10} + 2.2175865 \times 10^{-8}x^{11} - 1.5958993 \times 10^{-9}x^{12}$$

we plot the error functions

$$E_n(x) = \varphi_n(x) - u^*(x)$$

$for\ n = 2,3,4\ and\ n = 5,6,7,$ respectively. we show the logarithmic scheme of the maximal errors $ME_n = \max_{0\le x\le 4-e}|E_n(x)|\ versus\ n\ for\ n = 2,3,4,5,6\ and\ 7.$

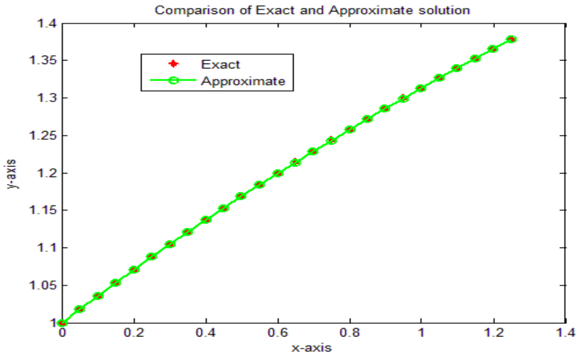

Figure 2: Graph of Example 2

Example 3

Solve the following BVP with quadratic non-linearity

$$u'' = 6u^2, \qquad 0 \le x \le 1$$

$$u(0) = 1, u(1) = \frac{1}{4}$$

Solution:

The exact solution is$u^*(x) = (1+x)^{-2}$, we get

$u = 1 - \frac{3}{4}x + 6L^{-1}u^2 - 6x[L^{-1}u^2]_{x=1}$

Where $L^{-1}(.) = \int_0^x\int_0^x(.)dxdx$

The Adomian polynomials of the nonlinear f(u)=u^2 with the decomposition $u = \sum_{k=0}^{\infty} u_n$ are

$$A_0 = u_0^2, \qquad A_1 = 2u_0u_1, \qquad A_2 = 2u_0u_2 + u_1^2, ...,$$

$$A_n = \sum_{k=0}^{\infty} u_k u_{n-k}$$

The following un-parameterized recursion system

$$u_0 = 1 - \frac{3}{4}x$$

$$u_{n+1} = 6L^{-1}A_n - 6x[L^{-1}A_n]_{x=1}, \quad n \ge 0$$

We obtain

$$u_1 = 6L^{-1}A_0 - 6x[L^{-1}A_0]_{x=1}, \quad n \ge 0$$

$6L^{-1}u_0^2 - 6x[L^{-1}u_0^2]_{x=1} = 6L^{-1}\left(1-\frac{3}{4}x\right)^2 - 6x[L^{-1}\left(1-\frac{3}{4}x\right)^2]_{x=1}$

$= \frac{6x^2}{2} + \frac{54x^4}{192} - \frac{18x^3}{12} - 6x\left[\frac{1}{2} + \frac{9}{192} - \frac{1}{4}\right]$

$u_1 = 3x^2 + \frac{9}{32}x^4 - \frac{3}{2}x^3 - \frac{57}{32}x$

$u_2 = \frac{873x}{896} - \frac{57x^3}{16} + \frac{555x^4}{128} - \frac{9x^5}{4} + \frac{9x^6}{16} - \frac{27x^7}{448}...$

This series of partial sums $\varphi_{n(x)} = \sum_{k=0}^{n-1} u_k(x)$does not converge to the accurate solution $u^*(x)$ on the period$\le x \le 1$,but as an alternativeshow divergence. That can be display by the plot of the absolute errors $|E_n(x)| = |\varphi_n(x) - u^*(x)|$.we show the $|E_n(x)|\ for\ n = 3,4,5\ and\ 6.$

The nonlinear BVPs with the recursion method

$$u_0 = c + 1 - \frac{3}{4}x$$

$$u_1 = -c + 6L^{-1}A_0 - 6x[L^{-1}A_0]_{x=1}$$

$$u_{n+1} = 6L^{-1}A_n - 6x[L^{-1}A_n]_{x=1}, \qquad n \ge 1$$

We have checked that, for $c = -0.2, -0.3, and - 0.4,$ respectively, the maximal errors $ME_n = \max_{0\le x\le 1}|\varphi_n(x) - u^*(x)|$ develop into steadily lesser and come near zero for n from 1 to 30. For example, as $c = -0.2$

$u_0 = \frac{4}{5} - \frac{3}{4}x$

$u_1 = -c + 6L^{-1}A_0 - 6x[L^{-1}A_0]_{x=1}$

$u_1 = \frac{1}{5} - \frac{801}{800}x + \frac{48}{25}x^2 - \frac{6}{5}x^3 + \frac{9}{32}x^4$

$u_2 = -\frac{33003x}{112000} + \frac{24x^2}{25} - \frac{951x^3}{500} + \frac{36591x^4}{16000} - \frac{36x^5}{25} + \frac{9x^6}{20} - \frac{27x^7}{448}$

The errors functions of the absolute values are $|E_n(x)|, for\ n = 2,3,4\ and\ 5,$ are plotted. The maximal errors $ME_n, for\ n = 1 through\ 16$

Otherwise, we can also use the parameterized recursion method

$u_0 = c$

$u_1 = -c + 1 - \frac{3}{4}x + 6L^{-1}A_0 - 6x[L^{-1}A_0]_{x=1}$

$u_{n+1} = 6L^{-1}A_n - 6x[L^{-1}A_n]_{x=1}, \quad n \ge 1$

For c = 0.1, 0.2, 0.3, 0.4 and 0.5, correspondingly, the maximal errors ME_nbecome gradually lesser and come near zero. For example, for c = 0.4

$u_0 = \frac{2}{5}$

$u_1 = \frac{3}{5} - \frac{123x}{100} + \frac{12x^2}{25}$

$u_2 = -\frac{81x}{125} + \frac{36x^2}{25} - \frac{123x^3}{125} + \frac{24x^4}{125}\cdots$

And the maximal errors

$ME_n = \max_{0\le x\le 1}|\varphi_n(x) - u^*)x)|, for\ n = 1,2,\cdots,16$

$u_0 = c + 1 - \frac{3}{4}x$

$u_{n+1} = -\frac{c}{2^{n+1}} + 6L^{-1}A_n - 6x[L^{-1}A_n]_{x=1}, \quad n \ge 0$

For c= - 0.4 we have

$u_0 = \frac{3}{5} - \frac{3x}{4}$

$u_1 = \frac{1}{5} - \frac{369x}{800} + \frac{27x^2}{25} - \frac{9x^3}{10} + \frac{9x^4}{32}$

$u_2 = \frac{1}{10} - \frac{36699x}{112000} + \frac{18x^2}{25} - \frac{1707x^3}{2000} + \frac{15903x^4}{16000} - \frac{81x^5}{100} + \frac{27x^6}{80} - \frac{27x^7}{448}$

Table 1: For c=-0.2, maximal errors ME_n for $n = 1,2,\cdots,16$

N	1	2	3	4
ME_n	0.2	0.027527	0.013583	0.005345
N	5	6	7	8
ME_n	0.001447	0.000180	0.000464	0.000375
N	9	10	11	12
ME_n	0.000199	0.000065	0.000009	0.000026
N	13	14	15	16
ME_n	0.000022	0.000013	0.000004	0.0000007

Table 2: For c=0.4, maximal errors $ME_n\ for\ n = 1,2,\cdots,16$

n	1	2	3	4
ME_n	0.6	0.082902	0.014544	0.01042
n	5	6	7	8
ME_n	0.007595	0.002760	0.000979	0.001312
n	9	10	11	12
ME_n	0.000660	0.000115	0.000285	0.000185
n	13	14	15	16
ME_n	0.000033	0.000067	0.000055	0.000015

Table 3: For c=-0.4, maximal errors $ME_n\ for\ n = 1,2,\cdots,16.$

n	1	2	3	4
ME_n	0.4	0.2	0.1	0.05
n	5	6	7	8
ME_n	0.025	0.0125	0.00625	0.003125
n	9	10	11	12
ME_n	0.001563	0.000781	0.000391	0.000195
n	13	14	15	16
ME_n	0.000098	0.000049	0.000024	0.000012

The maximal errors $ME_n = \max_{0\le x\le 1}|\varphi_n(x) - u^*(x)|$, for n = 1

Graphical

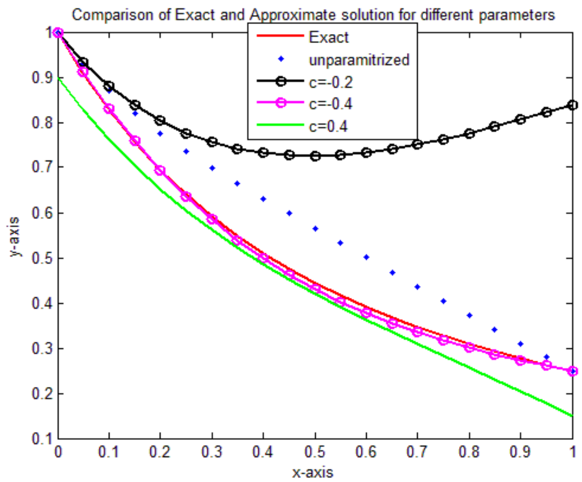

Figure 3: Graph of example 3 with iterations

4. CONCLUSION

The solution of higher order non-linear ordinary differential equations with different boundary conditions has been analyzed in this research. The advance Adomian decomposition method is used for the solution of the two-point BVPs with Different conditions, based on inverse linear operator.

The approximate and exact solution becomes nearly identical for a sufficiently small number of components; a unique solution is obtained here. Furthermore, to solve linear and non-linear BVPs with the help of some conditions named Robin boundary condition through ADM. A new approach is described here which extend the different ways of ADM used for BVPs. When the unique calculated sequence does not converge over the particular field or the nonlinear differential problem is focused to a position of Neumann boundary form.

REFERENCES

[1] G. Adomian, "Solving Frontier Problems of Physics: The Decomposition Method," *KluwerAcad.*, Dordrecht (1994).

[2] G. Adomian, "Explicit solutions of nonlinear partial differential equations," *Appl. Math. Comput*. **88**: 117–126 (1997).

[3] G. Adomian, "Solutions of nonlinear P.D.E," *Appl. Math. Lett*. 11: 121–123 (1998).

[4] G. Adomian, Y. Cherruault, K. Abbaoui, "A nonperturbative analytical solution of immune response with time-delays and possible generalization," *Math.Comput. Modeling* **24**: 89–96 (1996).

[5] Y. Cherruault, M. Inc, K. Abbaoui, "On the solution of the nonlinear Korteweg–de Vries equation by the decomposition method," *Kybernetes* **31**: 766–772 (2002).

[6] D. Kaya, S.M. El-Sayed, "A numerical method for solving Jaulent–Miodek equation," *Phys. Lett. A* **318**: 345–353 (2003).

[7] J. Biazar, M. Ilie, A. Khoshkenar, "A new approach to the solution of the prey and predator problem and comparison of the results with the Adomian method," *Appl. Math. Comput.* **171**: 486–491 (2005).

[8] I. Hashim, M.S.M. Noorani, R. Ahmad, S.A. Bakar, E.S. Ismail, A.M. Zakaria, "Accuracy of the Adomian decomposition method applied to the Lorenzsystem," *Chaos Solitons Fractals* **28**: 1149–1158 (2006).

[9] D. Lesnic, "Blow-up solutions obtained using the decomposition method," Chaos Solitons Fractals **28:** 776–787 (2006).

[10] Sana. A., N. A. Chaudhry, M.Saeed and M.F.Tabassum "Adomian Decomposition Method with Neumann Boundary Conditions for Solution of Nonlinear Boundary Value Problem" *Sci.Int.* **27**(1): 338-388 (2015).

[11] Y. Cherruault, "Convergence of Adomian's method," *Kybernetes* **18**: 31–38 (1989).

[12] G. Adomian, R. Rach, "Inversion of nonlinear stochastic operators," *J. Math. Anal. Appl*. **91**: 39–46 (1983).

[13] G. Adomian, "Stochastic Systems", *Academic*, New York, (1983).