NUMERICAL SOLUTION FOR NONLINEAR COUPLED PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT: *In this paper, numerical methods have been used to solve non-linear Partial Differential Equations (PDE's). Solution of non-linear PDE's having coupled partial differential equations have been discussed using Laplace Decomposition Method (LDM), Adomian Decomposition Method (ADM) and Homotopy Perturbation Method (HPM) respectively. Numerical examples of non-linear coupled PDE's are solved using above mentioned methods to explain the methods and to check the similarity of the results.*

Keywords: non-linear coupled partial differential equations, laplace decomposition method, adomian decomposition method, homotopy perturbation method

1. INTRODUCTION

The Laplace Decomposition method (LDM) is a numerical algorithm to solve nonlinear ordinary, Partial differential equations. Khuri [1, 2] used this method for the approximate solution of a class of nonlinear ordinary differential equations. Adadjanov [3] applied this method for the solution of Duffing equation. Elgazery [4] exploit this method to solve Falkuer-skan equation. Majid khan also used this method for the solution to the existing ones [5] concerning to the non-linear coupled Partial differential equations.

The Adomian Decomposition method was developed from the 1970s to the 1990s George Adomian (1923-1996). Adomian Decomposition method [7] (Wazwaz 2000; EL-Sayad etal. 2010) were used for solving non-linear problems. Adomian Decomposition method was successfully applied to nonlinear differential delay equations [6],a non-linear dynamic systems, the heat equation [8,9], the wave equation [10], coupled non-linear Partial differential equations [11,12], linear and non-linear integro-differential equations [13].

The HPM, proposed first by Ji-Huan. He [14,15] for solving differential and integral equations, linear and nonlinear, has been the subject of extensive analytical and numerical studies. The HPM is applied to volterra's integro-differential equation [16], nonlinear oscillators [17], bifurcation of nonlinear problems [18], bifurcation of delay-differential equation [19], non-linear wave equations [20]. Several techniques including the method of characteristic, Riemann invariants, combination of wave form relaxation and multi-grid, periodic multi-grid wave form, variational iteration homotopy perturbation method have been used for the solutions of such problems.

2. MATERIAL AND METHODS

Many problems in natural and engineering sciences are modeled by Partial differential equations (PDE's). After studying the Laplace decomposition method that was applied to solve some examples which were nonlinear coupled partial differential equations. Now the basic motivation of the present paper is the implementation of two methods on the same examples, which are Adomian decomposition method and Homotopy perturbation method. In these methods we obtain the same exact or approximate solutions. Results of all three methods are shown by graphically and analytically.

2.1 Laplace Decomposition Method

We consider the general form of in-homogenous nonlinear partial differential equations with initial conditions as given below

$$Lu + Ru + Nu = h(x,t) \quad (1)$$

$$u(x,0) = f(x)\,,\ u_t(x,0) = g(x) \quad (2)$$

wherever $L=\frac{\partial^2}{\partial x^2}$, R is the residual linear operator, Nu represents a general non-linear differential operator and h(x,t) is resource term. The method consists of applying Laplace transform first on both sides of equation (1), we get the result

$$L\{u(x,t)\} = \frac{f(x)}{s} + \frac{g(x)}{s^2} + \frac{1}{s^2}L\{h(x,t)\} - \frac{1}{s^2}L\{Ru(x,t)\} - \frac{1}{s^2}L\{Nu(x,t)\} \quad (3)$$

The second step in Laplace decomposition method is that we symbolize solution as an infinite series given below

$$u = \sum_{m=0}^{\infty} u_m(x,t) \quad (4)$$

The nonlinear operator is decompose as

$$Nu(x,t) = \sum_{m=0}^{\infty} A_m \quad (5)$$

Where A_m are Adomian polynomials [9] of $u_0, u_1, u_2, \ldots, u_n$ and it can be calculated by formula given below

$$A_m = \frac{1}{m!}\frac{d^m}{d\lambda^m}\left[N\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right]\ ,\ \ m = 0,1,2,\ldots, \quad (6)$$

Putting Equations (4), (5) and (6) in Equation (3), we get

$$\sum_{m=0}^{\infty} L\{u_m(x,t)\} = \frac{f(x)}{s} + \frac{g(x)}{s^2} + \frac{1}{s^2}L\{h(x,t)\} - \frac{1}{s^2}L\{Ru(x,t)\} - \frac{1}{s^2}L\left\{\sum_{m=0}^{\infty} A_m\right\} \quad (7)$$

On comparing both sides of the Eq. (7) we obtain

$$L\{u_0(x,t)\} = \frac{f(x)}{s} + \frac{g(x)}{s^2} + \frac{1}{s^2}L\{h(x,t)\} = K(x,s) \quad (8)$$

In general, the recursive relation is specified by

$$L\{u_{n+1}(x,t)\} = -\frac{1}{s^2}L\{Ru_n(x,t)\} - \frac{1}{s^2}L\{A_n\} \quad (9)$$

Applying inverse Laplace transform to Eqs. (8)-(9), our requisite recursive relation is given as follows:

$$u_0(x,t) = K(x,t) \quad (10)$$

$$u_{n+1}(x,t) = -L^{-1}\left\{\frac{1}{s^2}L\{Ru_n(x,t)\} + \frac{1}{s^2}L\{A_n\}\right\} \quad (11)$$

where $\mathrm{K}(x,t)$ represents the expression arising from resource term and prescribes initial conditions. Now initially we applying Laplace transform of the expressions on the right hand side of Eq. (11).then applying inverse Laplace transform we get the values of $u_1, u_2, \ldots, u_n$correspondingly.

2.2 Adomian Decomposition Method

Suppose the following coupled PDE's

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial t^2}+\alpha uv=g_1(x,t)$$
$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial t^2}+\beta uv=g_2(x,t)$$

Given (uncoupled) boundary conditions are
$u(x_1,t)=\xi_1(t), u(x,0)=\tau_1(x)u(x_2,t)=\xi_2(t), u_t(x,0)=\tau_2(x), v(x_1,t)=\eta_1(t), v(x,0)=\sigma_1(x), v(x_2,t)=\eta_2(t), v_t(x,0)=\sigma_2(x)$,

Solving for $\frac{\partial^2 u}{\partial t^2}$ with the initial conditions yield the same solution [10]. In operator form using the given conditions on x,

Where $L=\frac{\partial^2}{\partial x^2}$ and L^{-1} is a two-fold indefinite integration. Operating with

$$L^{-1}u=u_0-L^{-1}\left(\frac{\partial^2}{\partial t^2}\right)u-L^{-1}\alpha uv, v=v_0-L^{-1}\left(\frac{\partial^2}{\partial t^2}\right)v-L^{-1}\beta uv,$$

and

$$u_0=A_0(t)+xB_0(t)+L^{-1}g_1(x,t), v_0=C_0(t)+xD_0(t)+L^{-1}g_2(x,t),$$

We write $u=\sum_{n=0}^{\infty}u_n$ and $v=\sum_{n=0}^{\infty}v_n$. The uv term can be written

$$uv=\sum_{n=0}^{\infty}[\sum_{\lambda=0}^{\infty}u_{n-\lambda}\,v_\lambda],$$

Magnitude in the bracket is the Adomian Polynomial of order n. The preliminary expressions u_0, v_0 have been given , and terms after n=0 are given by

$$u_n=A_n(t)+B_n(t)x-L^{-1}\left(\frac{\partial^2}{\partial t^2}\right)u_{n-1}-L^{-1}\alpha\sum_{\lambda=0}^{n-1}u_{n-1-\lambda}v_\lambda, v_n=C_n(t)+D_n(t)x-L^{-1}\left(\frac{\partial^2}{\partial t^2}\right)v_{n-1}-L^{-1}\beta\sum_{\lambda=0}^{n-1}u_{n-1-\lambda}v_\lambda.$$

The approximant
$\varphi_1[u]=u_0$ and$\varphi_2[v]=v_0.\varphi_{\lambda+1}[u]=\sum_{n=0}^{\lambda}u_n$ and $\varphi_{\lambda+1}[v]=\sum_{n=0}^{\lambda}v_n$ converge to $u=\sum_{n=0}^{\infty}u_n$ and $v=\sum_{n=0}^{\infty}v_n$.

The succeeding approximants must satisfy the applicable boundary conditions, and hence for φ_1 or u_0,

$$u_0(x_1,t)=\varphi_1[u(x_1,t)]=\xi_1(t), u_0(x_2,t)=\varphi_1[u(x_2,t)]=\xi_2(t),$$
$$v_0(x_1,t)=\varphi_1[v(x_1,t)]=\eta_1(t), v_0(x_2,t)=\varphi_1[v(x_2,t)]=\eta_2(t).$$

Since each successive approximant $\varphi_1,\varphi_2,\ldots$ must satisfy the given conditions, we have for all n > 0: $\varphi_n[u(x_1,t)]=\xi_1(t), \varphi_n[u(x_2,t)]=\xi_2(t), \varphi_n[v(x_1,t)]=\eta_1(t), \varphi_n[v(x_2,t)]=\eta_2(t)$.

We observe that $\varphi_{n+1}[u]=\varphi_n[u]+u_n$ and $\varphi_{n+1}[v]=\varphi_n[v]+v_n$, and consequently for $n\geq 1$,
$u_n(x_1,t)=u_n(x_2,t)=0, v_n(x_1,t)=v_n(x_2,t)=0$,
Consequently, for

$$\varphi_1=u_0, A_0(t)+x_1B_0(t)+L^{-1}g_1(x_1,t)=\xi_1(t), A_0(t)+x_2B_0(t)+L^{-1}g_1(x_2,t)=\xi_2(t), C_0(t)+x_1D_0(t)+L^{-1}g_1(x_1,t)=\eta_1(t), C_0(t)+x_2D_0(t)+g_1(x_2,t)=\eta_2(t).$$

Let us rewrite this as

$$A_0(t)+x_1B_0(t)=\xi_1^{(0)}(t)=\xi_1(t)-L^{-1}g_1(x_1,t), A_0(t)+x_2B_0(t)=\xi_2^{(0)}(t)=\xi_2(t)-L^{-1}g_2(x_2,t), \quad C_0(t)+x_1D_0(t)=\eta_1^{(0)}(t)=\eta_1(t)-L^{-1}g_1(x_1,t), C_0(t)+x_2D_0(t)=\eta_1^{(0)}(t)=\eta_2(t)-L^{-1}g_1(x_2,t)$$
(12)

We can also write as a result of the decomposition

$$\xi_1^{(n)}(t)=L^{-1}\left(\frac{\partial^2}{\partial t^2}\right)u_{n-1}(x_1,t)+L^{-1}\alpha\sum_{\lambda=0}^{n}u_{n-\lambda}(x_1,t)v_\lambda(x_1,t),$$
$$\xi_2^{(n)}(t)=\left(\frac{\partial^2}{\partial t^2}\right)u_{n-1}(x_2,t)+L^{-1}\alpha\sum_{\lambda=0}^{n}u_{n-\lambda}(x_2,t)v_\lambda(x_2,t),$$
$$\eta_1^{(n)}(t)=L^{-1}\left(\frac{\partial^2}{\partial t^2}\right)v_{n-1}(x_1,t)+L^{-1}\beta\sum_{\lambda=0}^{n}u_{n-\lambda}(x_1,t)v_\lambda(x_1,t),$$
$$\eta_2^{(n)}(t)=\left(\frac{\partial^2}{\partial t^2}\right)v_{n-1}(x_2,t)+L^{-1}\beta\sum_{\lambda=0}^{n}u_{n-\lambda}(x_1,t)v_\lambda(x_1,t).$$

Also corresponding to (12), we write for determination of the constants of integration for $n\geq 1$, $A_n(t)+x_1B_n(t)=\xi_1^{(n)}(t), A_n(t)+x_2B_0(t)=\xi_2^{(n)}(t), C_n(t)+x_1D_n(t)=\eta_1^{(n)}(t), C_n(t)+x_2D_n(t)=\eta_2^{(n)}(t)$.

We can now write

$$\begin{bmatrix}1 & x_1\\1 & x_2\end{bmatrix}.\begin{bmatrix}A_0(t)\\B_0(t)\end{bmatrix}=\begin{bmatrix}\xi_1^{(0)}(t)\\\xi_2^{(0)}(t)\end{bmatrix},$$
$$\begin{bmatrix}1 & x_1\\1 & x_2\end{bmatrix}.\begin{bmatrix}C_0(t)\\D_0(t)\end{bmatrix}=\begin{bmatrix}\eta_1^{(0)}(t)\\\eta_2^{(0)}(t)\end{bmatrix},$$
$$\begin{bmatrix}1 & x_1\\1 & x_2\end{bmatrix}.\begin{bmatrix}A_n(t)\\B_n(t)\end{bmatrix}=\begin{bmatrix}\xi_1^{(n)}(t)\\\xi_2^{(n)}(t)\end{bmatrix},$$
$$\begin{bmatrix}1 & x_1\\1 & x_2\end{bmatrix}.\begin{bmatrix}C_n(t)\\D_n(t)\end{bmatrix}=\begin{bmatrix}\eta_1^{(n)}(t)\\\eta_2^{(n)}(t)\end{bmatrix}. \tag{13}$$

Inversion of the first matrix to $\left(1/(x_2-x_1)\right)\begin{bmatrix}x_2 & -x_1\\-1 & 1\end{bmatrix}$ allows determination of the integration constants, e.g.,

$$A_0(t)=\frac{x_2\xi_1^{(0)}(t)-x_1\xi_2^{(0)}(t)}{x_2-x_1}, B_0(t)=\frac{\xi_2^{(0)}(t)-\xi_1^{(0)}(t)}{x_2-x_1},$$
$$C_0(t)=\frac{x_2\eta_1^{(0)}(t)-x_1\eta_2^{(0)}(t)}{x_2-x_1}, D_0(t)=\frac{\eta_2^{(0)}(t)-\eta_1^{(0)}(t)}{x_2-x_1},$$

So that u_0 and v_0 are determined.

To calculate the u_1 and v_1, or the $\varphi_2[u]$ and $\varphi_2[v]$ approximants, we maintain it with the matrix equations in (13)

$$A_n(t)=\frac{x_2\xi_1^{(n)}(t)-x_1\xi_2^{(n)}(t)}{x_2-x_1}, B_n(t)=\frac{\xi_2^{(n)}(t)-\xi_1^{(n)}(t)}{x_2-x_1},$$
$$C_n(t)=\frac{x_2\eta_1^{(n)}(t)-x_1\eta_2^{(n)}(t)}{x_2-x_1}, D_n(t)=\frac{\eta_2^{(n)}(t)-\eta_1^{(n)}(t)}{x_2-x_1}.$$

We have

$$u_1=A_1(t)+xB_1(t)-L^{-1}\left(\frac{\partial^2}{\partial t^2}\right)u_0-L^{-1}\alpha u_0v_0, v_1=C_1(t)+xD_1(t)-L^{-1}\left(\frac{\partial^2}{\partial t^2}\right)v_0-L^{-1}\beta u_0v_0,$$

And $\varphi_2[u]=\varphi_1[u]+u_1, \quad \varphi_2[v]=\varphi_1[v]+v_1$.

We can maintain it for finding components u_n and v_n and approximants are

$u_n =$

$A_n(t) + xB_n(t) - L^{-1}\left(\frac{\partial^2}{\partial t^2}\right)u_{n-1} -$

$L^{-1}\alpha\sum_{\lambda=0}^{n-1} u_{n-1-\lambda}v_\lambda,$

$v_n =$

$C_n(t) + xD_n(t) - L^{-1}\left(\frac{\partial^2}{\partial t^2}\right)v_{n-1} -$

$L^{-1}\beta\sum_{\lambda=0}^{n-1} u_{n-1-\lambda}v_\lambda,$

so that the approximants $\varphi_{n+1}[u] = \varphi_n[u] + u_n$, and $\varphi_{n+1}[v] = \varphi_n[v] + v_n$ can be formed.

2.3 Homotopy Perturbation Method

Suppose a general equation

$$L(u) = 0\,, \qquad (14)$$

where L is any integral or differential operator. We define a convex homotopyH(u,p) by

$$H(u,p) = (1-p)F(u) + pL(u)\,, \qquad (15)$$

where F(u) is a functional operator with known solutions v_0,which can be obtained easily. It is clear that for

$$H(u,0) = 0 \qquad (16)$$

$H(u,0) = F(u)\,,\ \ H(u,1) = L(u)\,.$

This shows that H(u,p) continuously traces an implicitly defined curve from a starting point $H(v_0,0)$ to a solution function $H(f,1)$.The embedding parameter momotonically increases from zero to unity as the trivial problem $F(u) = 0$ continuously deforms the original problem $L(u) = 0$.The embedding parameter $p \in (0,1]$ can be considered as an expanding parameter [11]. The HPM uses the Homotopy parameter p as an expanding parameter to obtain

$$u = \sum_{i=0}^{\infty} p^i u_i = u_0 + pu_1 + p^2u_2 + p^3u_3 + \cdots. \ (17)$$

If $p \to 1$,then (17) corresponds to (15) and becomes the approximate solution of the form

$$f = \lim_{p\to 1} u_i = \sum_{i=0}^{\infty} u_i = u_0 + u_1 + u_2 + \cdots. \ (18)$$

It is well known that series (18) is convergent foe most of the cases and also the rate of convergence of the HPM [11].The comparisons of equal powers of p give solutions of various orders. In sum, according to [11,12] He's HPM considers the solution $u(x)$ of the homotopy equation in a series of p as

$u(x) = \sum_{i=0}^{\infty} p^i u_i = u_0 + pu_1 + p^2u_2 + \cdots,$

And the method considers the nonlinear term $N(u)$ as

$N(u) = \sum_{i=0}^{\infty} p^i H_i = H_0 + pH_1 + p^2H_2 + \cdots,$

Where H_n are the so-called He's polynomials [11,12] which can be calculated by using the formula

$H_n(u_0, \ldots, u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left(N(\sum_{i=0}^{n} p^i u_i)\right)_{p=0}$

$n = 0,1,2, \ldots.$

3. NUMERICAL RESULTS

Here we discuss solutions of non-linear coupled PDE's using LDM, ADM and HPM respectively.

Sample Example 1

Suppose system of nonlinear coupled partial differential equations [30]

$\frac{\partial u(x,y,t)}{\partial t} - v_x w_y = 1$

$\frac{\partial v(x,,y,t)}{\partial t} - w_x u_y = 5$

$\frac{\partial w(x,y,t)}{\partial t} - u_x v_y = 5$

with initial conditions

$u(x,y,0) = x + 2y$

$v(x,y,0) = x - 2y$

$w(x,y,0) = -x + 2y$

Solution using LDM

Applying the Laplace decomposition method with initial conditions and then taking inverse Laplace transform on the above equations respectively, we obtain the recursive relations are

$u_0(x,y,t) = t + (x + 2y)$

$u_{n+1}(x,y,t) = L^{-1}\left[\frac{1}{s}L[\sum_{n=0}^{\infty} C_n(v,w)]\right]\,,\ n \geq 0$

$v_0(x,y,t) = 5t + (x - 2y)$

$v_{n+1}(x,y,t) = L^{-1}\left[\frac{1}{s}L[\sum_{n=0}^{\infty} D_n(u,w)]\right]\,,\ n \geq 0$

$w_0(x,y,t) = 5t + (-x + 2y)$

$w_{n+1}(x,y,t) = L^{-1}\left[\frac{1}{s}L[\sum_{n=0}^{\infty} E_n(u,v)]\right]\,,\ n \geq 0$

where $C_n(v,w), D_n(u,w)$ and $E_n(u,v)$ are Adomian polynomials representing the nonlinear expressions [9]. Adomian polynomials are given as follows

$C_0(v.w) = v_{0x}w_{0y},,$

$C_n(v,w) = \sum_{i=0}^{n} v_{ix}w_{n-iy}$

$D_0(u,w) = u_{0y}w_{0x}\,,$

$D_n(v,w) = \sum_{i=0}^{n} w_{ix}u_{n-iy}$

$E_0(u,v) = u_{0x}v_{oy},$

$E_n(u,v) = \sum_{i=0}^{n} u_{ix}v_{n-iy}$

Gives the final result as

$\sum_{m=0}^{\infty} u_m(x,t) = x + 2y + 3t$

$\sum_{m=0}^{\infty} v_m(x,t) = x - 2y + 3t$

$\sum_{m=0}^{\infty} w_m(x,t) = -x + 2y + 3t$

Solution using ADM

Define $L(.) = \frac{\partial}{\partial t}$ then apply the above condition on the equations respectively, we get

$Lu(x,y,t) = 1 + v_x w_y,$

$Lv(x,y,t) = 5 + w_x u_y,$

$Lw(x,y,t) = 5 + u_x v_y$

Define $L^{-1}(.) = \int_0^t (.)dt$ then apply Laplace operator on the above Equations. Respectively and using initial conditions we get

$u(x,y,t) = x + 2y + t + L^{-1}(v_x w_y),$

$v(x,y,t) = x - 2y + 5t + L^{-1}(w_x u_y)$

$w(x,y,t) = -x + 2y + 5t + L^{-1}(u_x v_y)$

The recursive relations are

$u_0(x,y,t) = x + 2y + t,$

$u_{n+1}(x,y,t) = L^{-1}(\sum_{n=0}^{\infty} C_n(v,w))\,,\ \ n \geq 0,$

$v_0(x,y,t) = 5t + x - 2y,$

$v_{n+1}(x,y,t) = L^{-1}(\sum_{n=0}^{\infty} D_n(u,w))\,,\ n \geq 0$

$w_0(x,y,t) = 5t - x + 2y,$

$w_{n+1}(x,y,t) = L^{-1}(\sum_{n=0}^{\infty} E_n(u,v))\ ,n \geq 0$

$C_0(v,w) = v_{0x}w_{0y},$

$C_n(v,w) = \sum_{i=0}^{n} v_{ix}w_{n-iy}$

$D_0(u,w) = u_{0y}w_{0x},$

$D_n(u,w) = \sum_{i=0}^{n} w_{ix}u_{n-iy}$

$E_0(u,v) = u_{0x}v_{0y},$

$E_n(u,v) = \sum_{i=0}^{n} u_{ix}v_{n-iy}$

So the solution of above system of nonlinear partial differential equations are given below

$\sum_{m=0}^{\infty} u_m(x,t) = x + 2y + 3t,$
$\sum_{m=0}^{\infty} v_m(x,t) = x - 2y + 3t,$
$\sum_{m=0}^{\infty} w_m(x,t) = -x + 2y + 3t$

Solution using HPM

Applying the convex homotopy method on the above equations

$$\frac{\partial u_0}{\partial t} + p\frac{\partial u_1}{\partial t} + p^2\frac{\partial u_2}{\partial t} + .. = 1 + p\left(\frac{\partial v_0}{\partial x} + p\frac{\partial v_1}{\partial x} + p^2\frac{\partial v_2}{\partial x} + \cdots\right)\left(\frac{\partial w_0}{\partial y} + p\frac{\partial w_1}{\partial y} + p^2\frac{\partial w_2}{\partial y} + \cdots\right)$$

$$\frac{\partial v_0}{\partial t} + p\frac{\partial v_1}{\partial t} + p^2\frac{\partial v_2}{\partial t} + \cdots = 5 + p\left(\frac{\partial w_0}{\partial x} + p\frac{\partial w_1}{\partial x} + p^2\frac{\partial w_2}{\partial x} + \cdots\right)\left(\frac{\partial u_0}{\partial y} + p\frac{\partial u_1}{\partial y} + p^2\frac{\partial u_2}{\partial y} + \cdots\right)$$

$$\frac{\partial w_0}{\partial t} + p\frac{\partial w_1}{\partial t} + p^2\frac{\partial w_2}{\partial t} + \cdots = 5 + p\left(\frac{\partial u_0}{\partial x} + p\frac{\partial u_1}{\partial x} + p^2\frac{\partial u_2}{\partial x} + \cdots\right)\left(\frac{\partial v_0}{\partial y} + p\frac{\partial v_1}{\partial y} + p^2\frac{\partial v_2}{\partial y} + \cdots\right)$$

And comparing the coefficients of like powers of p and applying initial conditions gives the solution as

$u(x,y,t) = x + 2y + 3t,$
$v(x,y,t) = x - 2y + 3t$
$w(x,y,t) = -x + 2y + 3t$

Table 1: Summary of results obtained in example 1

x	**y**	**t**	$\boldsymbol{u = x + 2y + 3t}$	$\boldsymbol{v = x - 2y + 3t}$	$\boldsymbol{w = -x + 2y + 3t}$
0	0	0	0	0	0
0.1	0.1	0.1	0.6	0.2	0.4
0.2	0.2	0.2	1.2	0.4	0.8
0.3	0.3	0.3	1.8	0.6	1.2
0.4	0.4	0.4	2.4	0.8	1.6
0.5	0.5	0.5	3	1.0	2.0
0.6	0.6	0.6	3.6	1.2	2.4
0.7	0.7	0.7	4.2	1.4	2.8
0.8	0.8	0.8	4.8	1.6	3.2
0.9	0.9	0.9	5.4	1.8	3.6
1	1	1	6	2	4

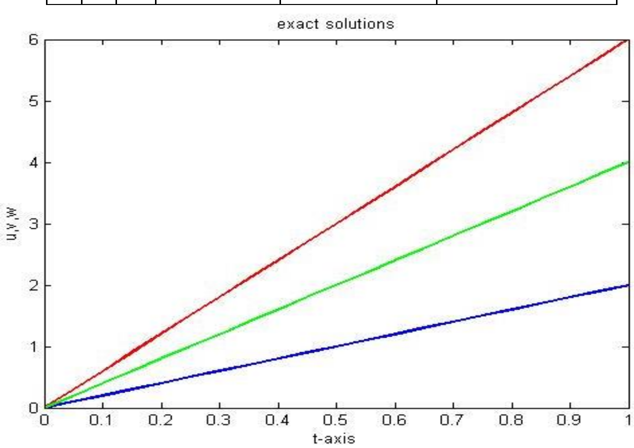

Figure: 1 Graph of Example :1

Sample Example 2

Suppose system of nonlinear coupled partial differential equations [30]

$\frac{\partial u(x,y,t)}{\partial t} + v_x w_y - v_y w_x = -u,$
$\frac{\partial v(x,y,t)}{\partial t} + w_x u_y + u_x w_y = v,$
$\frac{\partial w(x,y,t)}{\partial t} + u_x v_y + u_y v_x = w$

With initial conditions

$u(x,y,0) = e^{x+y}$
$v(x,y,0) = e^{x-y}$
$w(x,y,0) = e^{-x+y}$

Solution using LDM

Applying the Laplace decomposition method and then applying initial conditions we get,

$u(x,y,s) = \frac{1}{s}e^{x+y} + \frac{1}{s}L[v_y w_x - v_x w_y - u]$,
$v(x,y,s) = \frac{1}{s}e^{x-y} + \frac{1}{s}L[-w_x u_y - u_x w_y + v]$
$w(x,y,s) = \frac{1}{s}e^{-x+y} + \frac{1}{s}L[-u_x v_y - u_y v_x + w]$

Applying inverse Laplace transform we get,

$u(x,y,t) = e^{x+y} + L^{-1}\left[\frac{1}{s}L[v_y w_x - v_x w_y - u]\right]$
$v(x,y,t) = e^{x-y} + L^{-1}\left[\frac{1}{s}L[-w_x u_y - u_x w_y + v]\right]$
$w(x,y,t) = e^{-x+y}L^{-1}\left[\frac{1}{s}L[-u_x v_y - u_y v_x + w]\right]$

The recursive relations are

$u_0(x,y,t) = e^{x+y}$
$u_{n+1}(x,y,t) = L^{-1}\left[\frac{1}{s}L[\sum_{n=0}^{\infty} F_n(v,w) - \sum_{n=0}^{\infty} G_n(v,w) - u_n]\right], n \geq 0$
$v_0(x,y,t) = e^{x-y},$
$v_{n+1}(x,y,t) = L^{-1}\left[\frac{1}{s}L[v_n - \sum_{n=0}^{\infty} H_n(u,w) - \sum_{n=0}^{\infty} I_n(u,w)]\right], n \geq 0$
$w_0(x,y,t) = e^{-x+y}$
$w_{n+1}(x,y,t) = L^{-1}\left[\frac{1}{s}L[w_n - \sum_{n=0}^{\infty} J_n(u,v) - \sum_{n=0}^{\infty} K_n(u,v)]\right], n \geq 0$

Where $F_n(v,w)$, $G_n(v,w)$, $H_n(u,w)$, $I_n(u,w)$, $J_n(u,v)$, and $K_n(u,v)$ are Adomian polynomials [9]

Representing nonlinearities arising in above system of non-linear coupled partial differential equations. A small number of components of above Adomian polynomials are given below:

$F_0(v,w) = v_{0y}w_{0x},$ $F_n(v,w) = \sum_{i=0}^{n} v_{iy}w_{n-ix}$
$G_0(v,w) = v_{0x}w_{oy},$ $G_n(v,w) = \sum_{i=0}^{n} v_{ix}w_{n-iy}$
$H_0(u,w) = w_{0x}u_{0y},$ $H_n(u,w) = \sum_{i=0}^{n} w_{ix}u_{n-iy}$
$I_0(u,w) = w_{0y}u_{0x},$ $I_n(u,w) = \sum_{i=0}^{n} w_{iy}u_{n-ix}$
$J_0(u,v) = u_{0x}v_{0y},$ $J_n(u,v) = \sum_{i=0}^{n} u_{ix}v_{n-iy}$
$K_0(u,v) = u_{0y}v_{0x}$ $K_n(u,v) = \sum_{i=0}^{n} u_{iy}v_{n-ix}$

So our required solutions are given below:

$\sum_{n=0}^{\infty} u_n(x,y,t) = e^{x+y+t},$
$\sum_{n=0}^{\infty} v_n(x,y,t) = e^{x-y+t},$
$\sum_{n=0}^{\infty} w_n(x,y,t) = e^{-x+y+t}$

Solution using ADM

Define $L(.) = \frac{\partial}{\partial t}$ then apply the above condition on the equations respectively we get,

$Lu(x,y,t) = -u + v_y w_x - v_x w_y$
$Lv(x,y,t) = v - w_x u_y - u_x w_y$

$Lw(x,y,t) = w - u_x v_y - u_y v_x$

Define $L^{-1}(.) = \int_0^t (.)dt$ then apply Laplace operator and initial conditions we get,

$u(x,y,t) = e^{x+y} + L^{-1}(-u + v_y w_x - v_x w_y)$,

$v(x,y,t) = e^{x-y} + L^{-1}(v - w_x u_y - u_x w_y)$

$w(x,y,t) = e^{-x+y} + L^{-1}(w - u_x v_y - u_y v_x)$

The recursive relations are

$u_0(x,y,t) = e^{x+y}$

$u_{n+1}(x,y,t) = L^{-1}(-u_n + \sum_{n=0}^{\infty} F_n(v,w) - \sum_{n=0}^{\infty} G_n(v,w)), n \geq 0$

$v_0(x,y,t) = e^{x-y}$

$v_{n+1}(x,y,t) = L^{-1}(v_n - \sum_{n=0}^{\infty} H_n(u,w) - \sum_{n=0}^{\infty} I_n(u,w)), n \geq 0$

$w_0(x,y,t) = e^{-x+y}$

$w_{n+1}(x,y,t) = L^{-1}(w_n - \sum_{n=0}^{\infty} J_n(u,v) - \sum_{n=0}^{\infty} K_n(u,v)), n \geq 0$

So our required solutions are given below:

$\sum_{n=0}^{\infty} u_n(x,y,t) = e^{x+y+t}$

$\sum_{n=0}^{\infty} v_n(x,y,t) = e^{x-y+t}$

$\sum_{n=0}^{\infty} w_n(x,y,t) = e^{-x+y+t}$

Solution using HPM

Applying the convex homotopy method

$$\frac{\partial u_0}{\partial t} + p\frac{\partial u_1}{\partial t} + p^2\frac{\partial u_2}{\partial t} + \cdots = -p(u_0 + pu_1 + p^2u_2 + \cdots) + p\left(\frac{\partial v_0}{\partial y} + p\frac{\partial v_1}{\partial y} + p^2\frac{\partial v_2}{\partial y} + \cdots\right)\left(\frac{\partial w_0}{\partial x} + p\frac{\partial w_1}{\partial x} + p^2\frac{\partial w_2}{\partial x} + \cdots\right) - p\left(\frac{\partial v_0}{\partial x} + p\frac{\partial v_1}{\partial x} + p^2\frac{\partial v_2}{\partial x} + \cdots\right)\left(\frac{\partial w_0}{\partial y} + p\frac{\partial w_1}{\partial y} + p^2\frac{\partial w_2}{\partial y} + \cdots\right)\frac{\partial v_0}{\partial t} + p\frac{\partial v_1}{\partial t} + p^2\frac{\partial v_2}{\partial t} + \cdots$$

$$= p(v_0 + pv_1 + p^2v_2 + \cdots) - p\left(\frac{\partial u_0}{\partial y} + p\frac{\partial u_1}{\partial y} + p^2\frac{\partial u_2}{\partial y} + \cdots\right)\left(\frac{\partial w_0}{\partial x} + p\frac{\partial w_1}{\partial x} + p^2\frac{\partial w_2}{\partial x} + \cdots\right) - p\left(\frac{\partial u_0}{\partial x} + p\frac{\partial u_1}{\partial x} + p^2\frac{\partial u_2}{\partial x} + \cdots\right)\left(\frac{\partial w_0}{\partial y} + p\frac{\partial w_1}{\partial y} + p^2\frac{\partial w_2}{\partial y} + \cdots\right)$$

$$\frac{\partial w_0}{\partial t} + p\frac{\partial w_1}{\partial t} + p^2\frac{\partial w_2}{\partial t} + \cdots = p(w_0 + pw_1 + p^2w_2 + \cdots) - p\left(\frac{\partial u_0}{\partial x} + p\frac{\partial u_1}{\partial x} + p^2\frac{\partial u_2}{\partial x} + \cdots\right)\left(\frac{\partial v_0}{\partial y} + p\frac{\partial v_1}{\partial y} + p^2\frac{\partial v_2}{\partial y} + \cdots\right) - p\left(\frac{\partial u_0}{\partial y} + p\frac{\partial u_1}{\partial y} + p^2\frac{\partial u_2}{\partial y} + \cdots\right)\left(\frac{\partial v_0}{\partial x} + p\frac{\partial v_1}{\partial x} + p^2\frac{\partial v_2}{\partial x} + \cdots\right)$$

And comparing the coefficients of like powers of p and initial conditions we get the solution as

$u(x,y,t) = e^{x+y+t}$,

$v(x,y,t) = e^{x-y+t}$,

$w(x,y,t) = e^{-x+y+t}$

Table 2: Summary of results obtained in example 2

X	Y	T	$u = e^{x+y+t}$	$v = e^{x-y+t}$	$w = e^{-x+y+t}$
0	0	0	1	1	1
0.1	0.1	0.1	1.349859	1.105171	1.105171
0.2	0.2	0.2	1.822119	1.221403	1.221403
0.3	0.3	0.3	2.459603	1.349859	1.349859
0.4	0.4	0.4	3.320117	1.491825	1.491825
0.5	0.5	0.5	4.481689	1.648721	1.648721
0.6	0.6	0.6	6.049647	1.822119	1.822119
0.7	0.7	0.7	8.166170	2.013753	2.013753
0.8	0.8	0.8	11.023176	2.225541	2.225541
0.9	0.9	0.9	14.879732	2.459603	2.459603
1	1	1	20.085537	2.718282	2.718282

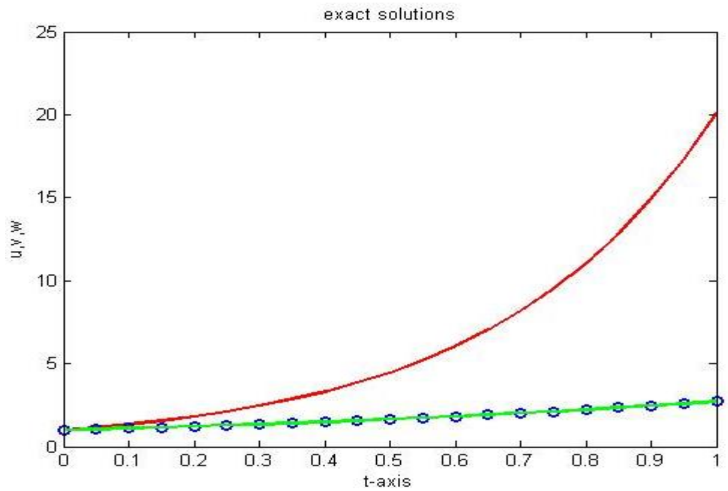

Figure: 2 Graph of Example 2

4. CONCLUSION

In this paper, Laplace decomposition method is applied to solve non-linear coupled partial differential equations bu using the initial values. Two examples are solved to explain the method. These examples are also solved by Adomian decomposition method and Homotopy perturbation method. In all these methods we obtain the same exact or approximate solutions. The results of these examples tell us that all these methods can be used alternatively for the solution of Higher-order initial value applications.

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