# MODIFIED ADOMIAN DECOMPOSITION METHOD AND HOMOTOPY PERTURBATION METHOD FOR HIGHER-ORDER SINGULAR BOUNDARY VALUE PROBLEMS

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***ABSTRACT:*** *In this paper, we presented effective numerical methods for the approximate solution of nonlinear higher order Boundary Value Problems (BVPs). We proposed a reliable modification of the Adomian Decomposition Method (MADM) that will accelerate the rapid convergence of the series solution. He,s polynomials are also used to overcome the difficult calculation. We also proposed the Homotopy Perturbation Method (HPM) which is highly accurate in witch only a few terms are required to obtain accurate computable solution. The validity of the methods is verified through illustrative examples. The results obtained demonstrate the accuracy and efficiency of the proposed methods.*

**Keywords:** Boundary Value Problems, Adomian Decomposition Method, Homotopy Perturbation Method. He,s polynomials

## 1. INTRODUCTION

Adomian Decomposition Method (ADM) is one of the powerful and reliable methods for solving BVP's, linear and nonlinear. This method accurately computes the series solution [1] with rapid convergence, if the solution is unique, and it is successfully applied to a wide range of problems arising in applied sciences [2-3]. The convergence of the decomposition series is studied by various authors$[4-5]$ . In recent years, the Adomian method has been modified so as to solve boundary value problems with singular nature by several authors [6-8]. The difficulty of those singular problems is due to the singularity behaviour that occurs at the point $x = 0$.In this paper we introduce reliable modification of the ADM to overcome the singularity difficulty for higher-order boundary value problems. For a comprehensive survey on the method, new interpretation, and its applications are also given in $[9-10]$. Some numerical examples [12] will be considered to illustrate that the proposed framework is suitable to get an accurate solution for higher-order singular boundary value problems, revealing its reliability and applicability.

He,s Homotopy Perturbation Method which is proved to be very effective, simple, and convenient to solve nonlinear boundary value problems. For relatively comprehensive survey on the method and its applications, the reader is referred to He,s review article and monographs [11]. The papers$[13-15,16-20]$ constitute a guided tour through the mathematics needed for a proper understanding of Homotopy Perturbation method as applied to various nonlinear problems.

## 2. MATERIAL AND METHODS

### 2.1 Modified Adomian Decomposition Method

Consider the singular BVP of (n +1 ) order ordinary differential equation in the following form

$$x^{(n+1)} + \frac{m}{t}x^n + Nx = f(t) \quad (1)$$

$$x(0) = a_0, x'(0) = a_1, \dots x^{(r-1)}(0) = a_{r-1}$$

$$x(b) = C_0, x'(b) = C_1, \dots x^{(n-r)}(b) = C_{n-r} \quad (2)$$

where N is nonlinear differential operator of order less than n, f(t) is given function,

$$a_0 a_1, \dots a_{r-1}, C_0, C_1, \dots C_{n-r}$$

are given constants, where $m \le r \le n, r \ge 1$ .

We write equation (1) in the form

$$t^{-2}x^{(n-1)}\left[t^{2n-m}x'\left[t^{(m-2n+2)}x'\right] + (n-m)(n-1)t^{-2}x^{n-1}\right] + Nx = f(t) \quad (3)$$

We write equation (3) in the operator form as

$$L_2L_1x = f(t) + (m-n)(n-1)L_2x - Nx \quad (4)$$

where

$$L_1 = t^{2n-m}\frac{d}{dt}\left(t^{m-2n+2}\frac{d}{dt}\right) \quad (5)$$

$$L_2 = t^{-2}\frac{d}{dt}^{(n-1)} \quad (6)$$

To overcome the singularity behavior at t=0, the integral operator is defined as

$$L_1^{-1}(.) = \int_b^t t^{2n-m-2}\int_0^t t^{m-2n}(.)dtdt \quad (7)$$

$$L_2^{-1}(.) = \int_0^t \dots \int_0^t t^2(.)dt \dots dt \quad (8)$$

Taking $L_2^{-1}$ on equation (4), we get

$$L_1x = L_2^{-1}f(t) + \emptyset_1(t) - L_2^{-1}Nx \quad (9)$$

As $L_2\emptyset_1(t) = 0$

Taking $L_1^{-1}$ on equation (9) we get,

$$x(t) = L_1^{-1}L_2^{-1}f(t) + L_1^{-1}\emptyset_1(t) - L_1^{-1}L_2^{-1}Nx + \emptyset_2(t) \quad (10)$$

such that $L_1\emptyset_2(t) = 0$

The ADM defines the solution of x(t) through infinite series,

$$x(t) = \sum_{n=0}^{\infty} x_n(t) \quad (11)$$

The nonlinear term approximated by a series of Adomian polynomials

$$Fx = \sum_{n=0}^{\infty} A_n(x_0, x_1, \dots x_n) \quad (12)$$

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{n=0}^{\infty}\lambda^i x_i\right)\right]\lambda = 0 \quad (13)$$

The components $x_n(t)$ of the solution $x(t)$ will be detrermined recurrently, and Adomian Polynomial can be constructed for several classes of nonlinearity according to specific algorithm presented in [3].

Substituting equation (11) and equation (12) into equation (10), we get

$$\sum_{n=0}^{\infty} x_n = L_1^{-1}L_2^{-1}f(t) + L_1^{-1}\emptyset_1(t) - L_1^{-1}L_2^{-1}\sum_{n=0}^{\infty} A_n + \emptyset_2(t)$$

Identifying

$$x_0 = L_1^{-1}L_2^{-1}f(t) + L_1^{-1}\emptyset_1(t) + \emptyset_2(t)$$

The Adomian method admits the use of recursive relation
$x_0 = L_1^{-1}L_2^{-1}f(t) + L_1^{-1}\emptyset_1(t) + \emptyset_2(t)$
$x_{n+1} = -L^{-1}A_n, n \geq 0$
which gives

$$x_0 = \emptyset(t) + L^{-1}f(t)$$
$$x_1 = -L^{-1}A_0$$
$$x_2 = -L^{-1}A_1.$$
$$x_3 = -L^{-1}A_2 \tag{14}$$

This helps the complete resolution of $x_n(t)$ of $x(t)$.The solution of $x(t)$ defines by (11) follows for numerical purposes

$$\emptyset_n = \sum_{j=0}^{n-1} x_j$$

used for approximated solution.

**2.2 Homotopy Perturbation Method**

We consider the nonlinear equation $f(x) = 0$ and use the basic idea of HPM [17]. We construct a homotopy $H(u,p): R \times [0,1] \to R$ which satisfies

$$H(u,P) = Pf(u) + (1-p)[f(u) - f(x_0)] = 0, \ u \in R, \tag{15}$$

where $p \in [0, 1]$ is an embedding parameter, and $x_0$ is initial approximation of (1) . From (15), we have

$$H(u,0) = f(u) - f(x_0) = 0$$

and $H(u,1) = f(u) = 0$

The parameter p increases from 0 to 1 as the trivial problem $H(u,0) = f(u) - f(x_0) = 0$ continuously transformed to$(u,1) = f(u) = 0$. The HPM method uses the parameter $p$ as a small parameter and writes the solution of (15) in power series of $p$ as

$$u = x_0 + \tilde{x}_1 P + \tilde{x}_2 P^2 + \ldots, \tag{16}$$

For $P = 1$, we have
$X = \lim_{p\to 1} u = x_0 + \tilde{x}_1 + \tilde{x}_2 + \ldots,$
Note that

$$x_0 = u(0), \tilde{x}_1 = u'(0), \tilde{x}_2 = \frac{1}{2!}u''(0), \ldots, \tilde{x}_n = \frac{1}{n!}u^n(0), \ldots,$$

and $u^{(n)}(0)$ can be determined from

$$f(x_0), f'(x_0), \ldots, f^{(n)}(x_0)$$

by the Eq. (15) or by the equation
$f\big(u(P)\big) - f(x_0) + Pf(x_0) = 0$ recursively.
We define another convex homotopy $H(u,P)$ by $H(u,P) = Pf(u) + (1-P)F(u)$,
where $F(u)$is a proper function with known solution $v_0$ which is obtained easily. We take $F(u)$ as the deformation of $f(u)$. As in (15), $F(u) = f(u) - f(x_0)$ and $x_0$ is an initial guess. From $H(u,P) = 0$, we get $H(u,0) = F(u) = 0$ and $H(u,1) = f(u) = 0$ . The parameter $p$ increases from 0 to 1 as $F(v) = 0$ is continuously transformed $f(u) = 0$.

## 3. NUMERICAL RESULTS

**Example 1:**

Consider non-linear BV problem

$$x'' - \frac{1}{t}x' = \frac{t^2}{3}x^5 \tag{17}$$

$$x(0) = 1, \quad x'(1) = \frac{\sqrt{3}}{8} \tag{18}$$

**3.1 Solution using MADM**

We define

$$L_1 = t^3 \frac{d}{dt}\left(t^{-1}\frac{d}{dt}\right)$$

$$L_1^{-1}(.) = \int_0^t t \int_1^t t^{-3}\,(.)dt\,dt$$

Equation (17) becomes

$$L_1 x = \frac{t^4}{3}x^5$$

Applying $L_1^{-1}$ on both sides we get

$$x(t) = x(0) + \frac{1}{2}x'(1)t^2 + L_1^{-1}\frac{t^4}{3}x^5$$

Using (12) and (13), we get

$$\sum_{n=0}^{\infty} x_n = x(0) + \frac{1}{2}x'(1)t^2 + L_1^{-1}\left(\frac{t^4}{3}\sum_{n=0}^{\infty} A_n\right)$$

This gives

$$x_0(t) = x(0) + \frac{1}{2}x'(1)t^2 \tag{18}$$

$$x_0(t) = 1 - \frac{\sqrt{3}}{16}t^2 = 1 - 0.108253t^2$$

$$x_{n+1} = L_1^{-1}\left(\frac{t^4}{3}A_n\right) \qquad n \geq 0 \tag{19}$$

Find out the Adomian Polynomials for the nonlinear term $x^5$using (13).then we get following results

$$x_0 = 1 - \frac{\sqrt{3}}{16}t^2 = 1 - 0.108253t^2$$

$x_1 = -0.0637827t^2 + 0.041666t^4 - 0.0075176t^6 + 0.00081380t^8 - 0.00005286t^{10} + 1.9073486 \times 10^{-6}t^{12} - 2.9496649 \times 10^{-8}t^{14}$

Successive iterations give us the following results

$x_2 = .0062577t^2 - .0044294t^6 + .0024057t^8 - .0006259t^{10} + .0001017t^{12} - .0000114t^{14} + \ldots,$

$x_3 = -.0010668t^2 + .000436t^6 + .0001884t^8 - .0003596t^{10} + .0001864t^{12} - .0000553t^{14} + \ldots,$

So the Approximated Solution is

$x = x_0 + x_1 + x_2 + x_3$

$x = 1 - .1668450t^2 + .0416667t^4 - .0115124t^6 + .0034079t^8 - .0010384t^{10} + .0002900t^{12} - .0000668t^{14} + \ldots,$

**3.2 Solution using HPM**

We define

$$L_1 = t^3 \frac{d}{dt}\left(t^{-1}\frac{d}{dt}\right)$$

$$L_1^{-1}(.) = \int_0^t t \int_1^t t^{-3}\,(.)dt\,dt \tag{20}$$

Equation (17) becomes

$$L_1 x = \frac{t^4}{3}x^5$$

Applying $L_1^{-1}$ on both sides

$$x(t) - x(0) - \frac{1}{2}x'(1)t^2 - L_1^{-1}\frac{t^4}{3}x^5 = 0$$

We make the following homotopy:

$$x(t) - x(0) - \frac{1}{2}x'(1)t^2 - PL_1^{-1}\frac{t^4}{3}F(x) = 0 \tag{21}$$

where $F(x) = x^5$, $P \in [0,1]$ is the embedding Parameter, $L_1^{-1}(.)$ is defined by (17)
According to He,s HPM , we assume

$$x = x_0 + Px_1 + P^2x_2 + \ldots.. \tag{22}$$

The nonlinear term $F(y)$ is such that

$$F(x) = H(x_0) + PH(x_0,x_1) + P^2H(x_0,x_1,x_2) + . \tag{23}$$

where

$$H(x_0,x_1,\ldots x_n) = \frac{1}{n!}\frac{d^n}{dP^n}F\left(\sum_{k=0}^{n} P^k x_k\right)_{P=0}$$

Substituting (20) and (21) into (19), and collecting the coefficients of like powers of p, we have

$$\left[x_0 - x(0) - \frac{1}{2}x'(1)t^2\right]P^0 + \left[x_1 - L_1^{-1}\frac{t^4}{3}H(x_0)\right]P^1 + \left[x_2 - L_1^{-1}\frac{t^4}{3}\{H(x_0,x_1)\}\right]P^2 + \left[x_3 - L_1^{-1}\frac{t^4}{3}\{H(x_0,x_1,x_2)\}\right]P^3 + \left[x_4 - L_1^{-1}\frac{t^4}{3}\{H(x_0,x_1,x_2,x_3)\}\right]\ldots = 0 \quad (24)$$

Using He,s polynomials in (24) we get the following results

$x_0(t) = 1 - 0.108253t^2$

$x_1 = -.0637827t^2 + .0416666t^4 - .0075176t^6 + .00081380t^8 - .00005286t^{10} + 1.9073486 \times 10^{-6}t^{12} - 2.9496649 \times 10^{-8}t^{14}$

$x_2 = .0062577t^2 - .0044294t^6 + .0024057t^8 - .0006259t^{10} + .0001017t^{12} - .0000114t^{14} + \ldots,$

$x_3 = -.0010668t^2 + .000436t^6 + .0001884t^8 - .0003596t^{10} + .0001864t^{12} - .0000553t^{14} + \ldots,$

Approximated Solution is $x = x_0 + x_1 + x_2 + x_3$

$x = 1 - 0.1668450t^2 + 0.04167t^4 - 0.0115124t^6 + 0.0034079t^8 - 0.0010384t^{10} + 0.0002900t^{12} - 0.0000668t^{14} + \ldots,$

**Table-1:** Summary of the results obtained in example 1

| T | Exact | Approximate | Error |
|---|---|---|---|
| 0.1 | 0.998337488459583 | 0.998335705191575 | 1.7832680080E-06 |
| 0.15 | 0.996270962773436 | 0.996266951000901 | 4.0117725350E-06 |
| 0.2 | 0.993399267798783 | 0.993392138545469 | 7.1292533139E-06 |
| 0.25 | 0.989743318610787 | 0.989732188429512 | 1.1130181275E-05 |
| 0.3 | 0.985329278164293 | 0.985313275341994 | 1.6002822299E-05 |
| 0.35 | 0.980188050780009 | 0.980166325291874 | 2.1725488135E-05 |
| 0.4 | 0.974354703692446 | 0.974326441932890 | 2.8261759556E-05 |
| 0.45 | 0.967867836991654 | 0.967832282253856 | 3.5554737798E-05 |
| 0.5 | 0.960768922830523 | 0.960725402270508 | 4.3520560015E-05 |
| 0.55 | 0.953101634249032 | 0.953049592551691 | 5.2041697341E-05 |
| 0.6 | 0.944911182523068 | 0.944850221531164 | 6.0960991904E-05 |
| 0.65 | 0.936243679766961 | 0.936173601673488 | 7.0078093473E-05 |
| 0.7 | 0.927145540823120 | 0.927066389666099 | 7.9151157021E-05 |
| 0.75 | 0.917662935482247 | 0.917575026686241 | 8.7908796006E-05 |
| 0.8 | 0.907841299003204 | 0.907745217880644 | 9.6081122560E-05 |
| 0.85 | 0.897724905923812 | 0.897621440407652 | 1.0346551616E-04 |
| 0.9 | 0.887356509416114 | 0.887246454882579 | 1.1005453353E-04 |
| 0.95 | 0.876777046043594 | 0.876660772972146 | 1.1627307145E-04 |
| 1 | 0.866025403784439 | 0.865902000000000 | 1.2340378444E-04 |

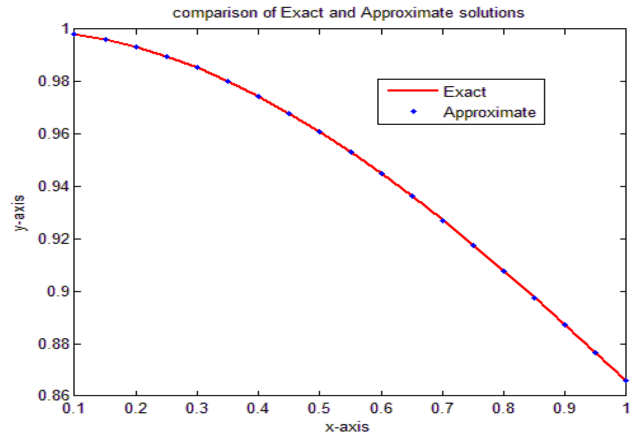


**Figure 1:** Graphical comparison of exact and approximate solutions of example 1.

**Example 2:**

Consider the nonlinear BV Problem

$$x''' + \frac{3}{t}x'' - x^3 = g(t) \quad (25)$$

$$x(0) = 0,\ x'(0) = 0,\ x(1) = e, \quad (26)$$

where $g(t) = 24e^t + 36te^t + 12t^2e^t + t^3e^t - t^9e^{3t}$.

Using the Taylor series of g(t) with order 10,

$$g(t) \approx g(T) = 24 + 60t + 60t^2 + 35t^3 + 14t^4 + \frac{21}{5}t^5 + t^6 + \frac{11}{56}t^7 + \frac{11}{336}t^8 - \frac{30097}{30240}t^9 - \frac{64787}{21600}t^{10}$$

### 3.3 Solution using MADM

We define

$$L_1 = t\frac{d}{dt}\left(t\frac{d}{dt}\right),$$

$$L_2 = t^{-2}\frac{d}{dt},$$

$$L_1^{-1} = \int_0^t t^{-1}\int_0^t t^{-1}(.)dtdt.$$

$$L_2^{-1} = \int_0^t t^2(.)dt.$$

Equation (25) becomes

$$L_2L_1x = gT(t) + L_2x + x^3 \quad (27)$$

Applying $L_2^{-1}, L_1^{-1}$ on both sides of (27) and then incorporating the given boundary conditions, we get

$$x(t) = L_1^{-1}L_2^{-1}gT(t) + L_1^{-1}x + L_1^{-1}L_2^{-1}x^3. \quad (28)$$

Using decomposition series for $x(t)$ and using polynomial series for $x^3$, we obtain

$$\sum_{n=0}^{\infty} x_n = L_1^{-1}L_2^{-1}gT(t) + L_1^{-1}\left(\sum_{n=0}^{\infty} x_n\right) + L_1^{-1}L_2^{-1}\left(\sum_{n=0}^{\infty} A_n\right)$$

This provides recursive relation

$$x_0(t) = L_1^{-1}L_2^{-1}gT(t) \quad (29)$$

$$x_{k+1}(t) = L_1^{-1}x_k + L_1^{-1}L_2^{-1}A_k, \quad k \geq 0$$

The adomian polynomials for $x^3$ are computed by (13).Substituting $L_1^{-1}$ ,$L_2^{-1}$into (29) The ADM leads to the following scheme:

$x_0 = 0.8888889t^3 + 0.9375000t^4 + 0.4800000t^5 + 0.1620370t^6 + 0.0408163t^7 + 0.0082031t^8 + 0.0013717t^9 + 0.0001964t^{10} + 0.0000246t^{11} - 0.0005759t^{12} - 0.0013652t^{13},$

$x_1 = 0.0987654t^3 + 0.0585938t^4 + 0.0192000t^5 + 0.0045010t^6 + 0.0008329t^7 + 0.0001282t^8 + 0.0000169t^9 + 0.0000019t^{10} + 0.0000002t^{11} + \ldots,$

Successive iterations give us the following results

$x_2 = 0.0109739t^3 + 0.0036621t^4 + 0.0007680t^5 + 0.0001250t^6 + 0.0000169t^7 + 0.0000020t^8 +$

$2.0907516 \times 10^{-7}t^9 + 1.9642857 \times 10^{-8}t^{10}.6 + 1799819 \times 10^{-9}t^{11} + \ldots,$

$x_3 = 0.0012193t^3 + 0.0002289t^4 + 0.0000307t^5 + 0.00000347t^6 + 3.4693305 \times 10^{-7}t^7 + 3.1292439 \times 10^{-8}t^8 + 2.5811748 \times 10^{-9}t^9 + 1.9642857 \times 10^{-10}t^{10} + 1.3884148 \times 10^{-11}t^{11} + \ldots,$

. Approximated solution is

$x = x_0 + x_1 + x_2 + x_3$

$x = .9998476t^3 + .9999847t^4 + .4999987t^5 + .1666666t^6 + .0416667t^7 + .0083333t^8 + .0013888t^9 + 0.0001984t^{10} + 0.0000248t^{11} + \ldots,$

### 3.4 Solution using HPM

With equation (28), we construct the following homotopy:

$x(t) - {L_1}^{-1}{L_2}^{-1}gT(t) - {L_1}^{-1}x - P{L_1}^{-1}{L_2}^{-1}F(x) = 0$ (30)

where $F(x) = x^3$ ,P $\in[0,1]$ is the embedding Parameters ${L_1}^{-1}(.)$ and ${L_2}^{-1}(.)$ are defined respectively.

According to He,s HPM ,We assume that the solution of (30) is

$$x = x_0 + Px_1 + P^2x_2 + \ldots. \quad (31)$$

The nonlinear term $F(x)$ can be expressed in the form

$F(x) = H(x_0) + PH(x_0, x_1) + P^2H(x_0, x_1, x_2) + ..$ (32)

where $H(x_0, x_1, \ldots, x_n)$ is called He, polynomial defind by

$H(x_0, x_1, \ldots x_n) = \frac{1}{n!}\frac{d^n}{dP^n}F(\sum_{k=0}^{n} P^k x_k)_{P=0}$

Substituting (31) and (32) into (30) ,and collecting the coefficients of like powers of p, we have

$[x_0 - {L_1}^{-1}{L_2}^{-1}gT]P^0 + [x_1 - {L_1}^{-1}x_0 - {L_1}^{-1}{L_2}^{-1}H(x_0)]P^1 + [x_2 - {L_1}^{-1}x_1 - {L_1}^{-1}{L_2}^{-1}\{H(x_0, x_1)\}]P^2 + [x_3 - {L_1}^{-1}x_2 - {L_1}^{-1}{L_2}^{-1}\{H(x_0, x_1, x_2)\}]P^3 + [x_4 - {L_1}^{-1}x_3 - {L_1}^{-1}{L_2}^{-1}\{H(x_0, x_1, x_2, x_3)\}]P^4 \ldots = 0$

Using ${L_1}^{-1}$ ,${L_2}^{-1}$and He,s polynomials in above equation we have

$x_0 = .8888889t^3 + .9375000t^4 + .4800000t^5 + .1620370t^6 + .0408163t^7 + .0082031t^8 + .0013717t^9 + .0001964t^{10} + .0000246t^{11} - .0005759t^{12} - .0013652t^{13},$

$x_1 = .0987654t^3 + 0.0585938t^4 + 0.0192000t^5 + 0.0045010t^6 + 0.0008329t^7 + 0.0001282t^8 + 0.0000169t^9 + 0.0000019t^{10} + 0.0000002t^{11} + \ldots,$

$x_2 = 0.0109739t^3 + 0.0036621t^4 + 0.0007680t^5 + 0.0001250t^6 + 0.0000169t^7 + 0.0000020t^8 + 2.0907516 \times 10^{-7}t^9 + 1.9642857 \times 10^{-8}t^{10} + 1.6799819 \times 10^{-9}t^{11} + \ldots,$

$x_3 = 0.0012193t^3 + 0.0002289t^4 + 0.0000307t^5 + 0.00000347t^6 + 3.4693305 \times 10^{-7}t^7 + 3.1292439 \times 10^{-8}t^8 + 2.5811748 \times 10^{-9}t^9 + 1.9642857 \times 10^{-10}t^{10} + 1.3884148 \times 10^{-11}t^{11} + \ldots,$

Approximated solution is

$x = x_0 + x_1 + x_2 + x_3$

$x = .9998476t^3 + .9999847t^4 + .4999987t^5 + .1666666t^6 + .0416667t^7 + .0083333t^8 + .0013888t^9 + 0.0001984t^{10} + 0.0000248t^{11} + \ldots,$

**Table-2: Summary of the Results in Example** 2.

| T | Exact | Approximate | Error |
|---|---|---|---|
| 0.1 | 0.001105170918076 | 0.001105016975012 | 1.5394306400E-07 |
| 0.15 | 0.003921190569208 | 0.003920668374149 | 5.2219505900E-07 |
| 0.2 | 0.009771222065281 | 0.009769977965298 | 1.2440999830E-06 |
| 0.25 | 0.020062897135746 | 0.020060454835320 | 2.4423004260E-06 |
| 0.3 | 0.036446187804552 | 0.036441945868718 | 4.2419358340E-06 |
| 0.35 | 0.060842521145936 | 0.060835750446836 | 6.7706991000E-06 |
| 0.4 | 0.095476780649041 | 0.095466621743890 | 1.0158905151E-05 |
| 0.45 | 0.142912447902792 | 0.142897908312676 | 1.4539590116E-05 |
| 0.5 | 0.206090158837516 | 0.206070110156250 | 2.0048681266E-05 |
| 0.55 | 0.288369970847688 | 0.288343145530634 | 2.6825317054E-05 |
| 0.6 | 0.393577660884350 | 0.393542648421655 | 3.5012462695E-05 |
| 0.65 | 0.526055400167941 | 0.526010642095515 | 4.4758072426E-05 |
| 0.7 | 0.690717178662373 | 0.690660961451317 | 5.6217211056E-05 |
| 0.75 | 0.893109382008472 | 0.893039826223183 | 6.9555785289E-05 |
| 0.8 | 1.139476955388140 | 1.139391998513380 | 8.4956874760E-05 |
| 0.85 | 1.436835622939040 | 1.436732991814190 | 1.0263112485E-04 |
| 0.9 | 1.793050668033410 | 1.792927834728340 | 1.2283330507E-04 |
| 0.95 | 2.216922819155920 | 2.216776931163660 | 1.4588799226E-04 |
| 1 | 2.718281828459040 | 2.718109600000000 | 1.7222845904E-04 |

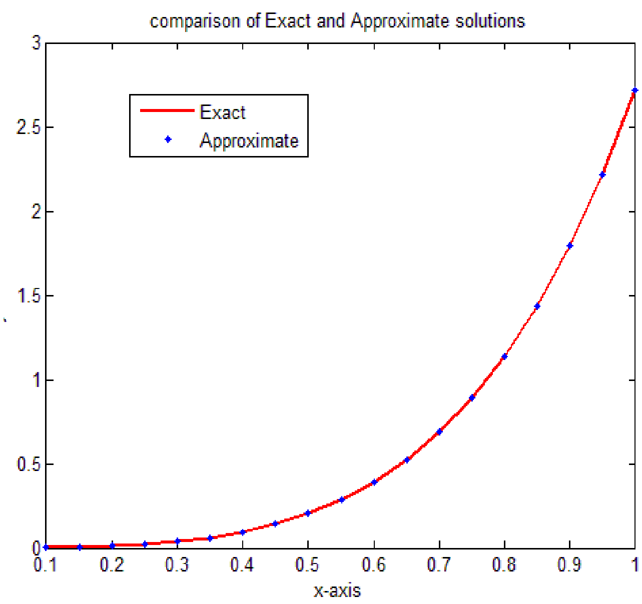


**Figure 2: Graphical comparison of Exact and Approximate solutions of example 2.**

**Example3:**

Consider the nonlinear BV problem

$x^4 + \frac{3}{t}x^3 + x^2 - x^3 = g(t)$ (33)

$x(0) = 0, x'(0) = 0, x(1) = e, x'(1) = 3e,$ (34)

where $g(t) = 18t^{-1}e^t + 30e^t + 11te^t + t^2e^t + t^4e^{2t} - t^6e^{3t}.$

We use the 11 terms of the Taylor series of g($t$),

$$g(t) \approx \frac{18}{t} + 48 + 50t + 30t^2 + \frac{49}{4}t^3 + \frac{71}{15}t^4 + \frac{29}{10}t^5 + \frac{33}{28}t^6 - \frac{6599}{4032}t^7 - \frac{9649}{2520}t^8 - \frac{1279991}{302400}t^9.$$

After solving this problem by both methods we obtain the following approximated solution

$$x = x_0 + x_1 + x_2 + x_3$$

$$x = .9986603t^2 + t^3 + .5t^4 + .1666667t^5 + .0416667t^6 + .0083333t^7 + .0013899t^8 + .0001990t^9 + 0.0000244t^{10} + 2.0591302 \times 10^{-6}t^{11} + \ldots$$

**Table 3:** Summary of the results in example 3.

| T | Exact | Approximate | Error |
|---|---|---|---|
| 0.1 | 0.011051709180756 | 0.011038312181130 | 1.3396999626E-05 |
| 0.15 | 0.026141270461386 | 0.026111127214519 | 3.0143246867E-05 |
| 0.2 | 0.048856110326407 | 0.048802522341613 | 5.3587984794E-05 |
| 0.25 | 0.080251588542984 | 0.080167857348742 | 8.3731194242E-05 |
| 0.3 | 0.121487292681840 | 0.121366719853993 | 1.2057282785E-04 |
| 0.35 | 0.173835774702674 | 0.173671661922789 | 1.6411277989E-04 |
| 0.4 | 0.238691951622603 | 0.238477600786332 | 2.1435083627E-04 |
| 0.45 | 0.317583217561759 | 0.317311930960394 | 2.7128660136E-04 |
| 0.5 | 0.412180317675032 | 0.411845398271060 | 3.3491940397E-04 |
| 0.55 | 0.524309037904887 | 0.523903789709017 | 4.0524819587E-04 |
| 0.6 | 0.655962768140583 | 0.655480496666301 | 4.8227147428E-04 |
| 0.65 | 0.809316000258371 | 0.808750012970921 | 5.6598728745E-04 |
| 0.7 | 0.986738826660533 | 0.986082433237721 | 6.5639342281E-04 |
| 0.75 | 1.190812509344630 | 1.190059021410920 | 7.5348793371E-04 |
| 0.8 | 1.424346194235180 | 1.423488923998000 | 8.5727023718E-04 |
| 0.85 | 1.690394850516520 | 1.689427107399360 | 9.6774311716E-04 |
| 0.9 | 1.992278520037130 | 1.991193603937770 | 1.0849160994E-03 |
| 0.95 | 2.333602967532550 | 2.332394156699420 | 1.2088108331E-03 |
| 1 | 2.718281828459040 | 2.716942359130200 | 1.3394693288E-03 |

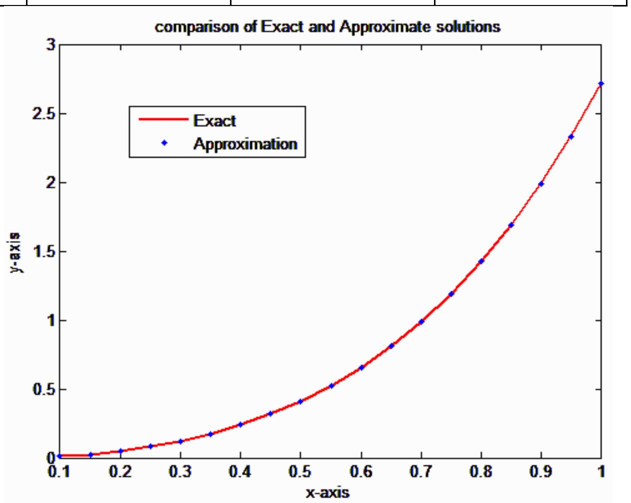


**Figure 3**: Graphical comparison of Exact and Approximate solutions of example 3.

## 4. CONCLUSION

In this paper, we have successfully employed Modified Adomian Decomposition method (MADM) for solving non-linear boundary value problems. It is demonstrated that the presented approach can be well suited to attain an accurate solution to the higher-order singular boundary value problems. The difficulty in problems due to the existence of singular point is overcome by (MADM). He,s polynomials are used to overcome the difficult calculation of (ADM).we applied a different implementation of the Homotopy Perturbation Method deduced from He,s HPM, it is clear that HPM provides us with a freedom of choice for construction of homotopy. The choice of homotopy for the problem plays a significant role for the accuracy of solution, while the He's HPM overcome the complex and problematic calculation of problems. After obtaining the approximate solutions of higher-order non-linear boundary value problems, we compared the results. It is examined that MADM is a time consuming method as compare to HPM. The comparison of these two methods shows that results are almost equivalent.

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