MULTIVARIATE CONDITIONAL & MARGINAL NORMAL DISTRIBUTION: A PROBABILITY APPROACH

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ABSTRACT: *Suppose that vector random variable* $X = (X1, X_2, X_3, \ldots\ldots X_p)^T$ *has a multivariate normal distribution then the probability distribution function (Pdf) is given by*

$$f(X)= \frac{e^{-\frac{1}{2}[X-\mu]\Sigma^{-1}[X-\mu]'}}{(2\pi)^{p/2}\,|\Sigma|^{\frac{1}{2}}}$$

where variance-covariance matrix $=\Sigma$ *and is of order p x p and the expected value=* μ *is of order Px1 vector.*
Now by considering the partitioning X into two components X_1 *and* X_2 *of orders q and p -q respectively where* $q \leq p$*, that is*

$$x = \begin{bmatrix} X^{(1)} \\ X^{(2)} \end{bmatrix}$$

Then we can deduce that
(a) The marginal distribution of X_1 *, and*
(b) The conditional distribution of X_2 *given that* $X_1 = x_1$
.Both (a) and (b) are also separately multivariate's normal distributions

Proof: Since we know that $X \sim N_p(\mu, \Sigma)$, then Pdf of X is $f(X) = \frac{e^{-\frac{1}{2}[X-\mu]\Sigma^{-1}[X-\mu]'}}{(2\pi)^{p/2}\,|\Sigma|^{\frac{1}{2}}}$ and if X *is partitioned so that X=*

$$\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_q \\ \cdots\cdots \\ X_{q+1} \\ \vdots \\ \vdots \\ X_P \end{bmatrix} = \begin{bmatrix} X^{(1)} \\ X^{(2)} \end{bmatrix} = X \quad and \quad \mu = \begin{bmatrix} \mu^{(1)} \\ \mu^{(2)} \end{bmatrix} \text{ and } \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} f(X^{(1)}, X^{(2)}) = \frac{e^{-\frac{1}{2}\left[X^{(1)}-\mu^{(1)} X^{(2)}-\mu^{(2)}\right]\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}^{-1}\left[X^{(1)}-\mu^{(1)} X^{(2)}-\mu^{(2)}\right]'}}{(2\pi)^{P/2} |\Sigma_{11}|^{\frac{1}{2}} |\Sigma_{22}|^{\frac{1}{2}} \left(1-\rho^2\right)^{\frac{1}{2}}}$$

$$\Sigma^{-1} = \begin{pmatrix} \Sigma_{22} & -\Sigma_{12} \\ -\Sigma_{21} & \Sigma_{11} \end{pmatrix} \Big/ \Sigma_{22}\Sigma_{11} - \Sigma_{12}\Sigma_{21} \qquad \Sigma_{22}\Sigma_{11} - \Sigma_{12}\Sigma_{21} = \Sigma_{22}\Sigma_{11}((1-\rho^2)^{\frac{1}{2}} \text{ and } f(X^{1)}) = \int f(X^{(1)}, X^{(2)}) dX^{(2)} = \frac{e^{-\frac{1}{2}\left[X^{(1)}-\mu^{(1)}\right]\Sigma^{-1}{}_{11}\left[X^{(1)}-\mu^{(1)}\right]'}}{(2\pi)^{q/2} |\Sigma_{11}|^{\frac{1}{2}}}$$

$$f(X^{(1)}, X^{(2)}) = \frac{e^{-\frac{1}{2\Sigma_{11}\Sigma_{11}(1-\rho^2)}\left[\left[X^{(1)}-\mu^{(1)}\right]\Sigma_{22}\left[X^{(1)}-\mu^{(1)}\right]'\right]-2\left[\left[X^{(1)}-\mu^{(1)}\right]\Sigma_{12}\left[X^{(2)}-\mu^{(2)}\right]'\right]+\left[\left[X^{(2)}-\mu^{(2)}\right]\Sigma_{11}\left[X^{(2)}-\mu^{(2)}\right]'\right]}}{(2\pi)^{q/2} (2\pi)^{p-q/2} |\Sigma_{11}|^{\frac{1}{2}} |\Sigma_{22}|^{\frac{1}{2}} \left(1-\rho^2\right)^{\frac{1}{2}}} \text{ and}$$

$$f(X^{(1)}, X^{(2)}) = \frac{e^{-\frac{1}{2(1-\rho^2)}\left[\left[X^{(1)}-\mu^{(1)}\right]\Sigma^{-1}{}_{11}\left[X^{(1)}-\mu^{(1)}\right]'\right]-2\left[\left[X^{(1)}-\mu^{(1)}\right]\Sigma_{12}/\Sigma_{11}\Sigma_{22}\left[X^{(2)}-\mu^{(2)}\right]'\right]+\left[\left[X^{(2)}-\mu^{(2)}\right]\Sigma^{-1}{}_{22}\left[X^{(2)}-\mu^{(2)}\right]'\right]}}{(2\pi)^{q/2} (2\pi)^{p-q/2} |\Sigma_{11}|^{\frac{1}{2}} |\Sigma_{22}|^{\frac{1}{2}} \left(1-\rho^2\right)^{\frac{1}{2}}}$$

$$f(X^{(1)}, X^{(2)}) = \frac{e^{-\frac{1}{2(1-\rho^2)}\left[\left[X^{(1)}-\mu^{(1)}\right]\Sigma^{-1}{}_{11}\left[X^{(1)}-\mu^{(1)}\right]'\right]-2\left[\left[X^{(1)}-\mu^{(1)}\right]\Sigma_{12}/\Sigma_{11}\Sigma_{22}\left[X^{(2)}-\mu^{(2)}\right]'\right]+\left[\left[X^{(2)}-\mu^{(2)}\right]\Sigma^{-1}{}_{22}\left[X^{(2)}-\mu^{(2)}\right]'\right]+\left[(X_1-\mu_1)\,\rho^2\Sigma^{-1}{}_{11}((X_1-\mu_1)'-(X_1-\mu_1)\,\rho^2\Sigma^{-1}{}_{11}((X_1-\mu_1)'\right]}}{(2\pi)^{q/2} (2\pi)^{p-q/2} |\Sigma_{11}|^{\frac{1}{2}} |\Sigma_{22}|^{\frac{1}{2}} \left(1-\rho^2\right)^{\frac{1}{2}}}$$

$$f(X^{(1)}, X^{(2)}) = \frac{e^{-\frac{1}{2(1-\rho^2)\Sigma_{22}}\left[\left[X^{(2)}-\mu^{(2)}-\rho\Sigma^{-1/2}{}_{22}\Sigma^{-1/2}{}_{11}(X^{(1)}-\mu^{(1)})\right]\left[\left[X^{(2)}-\mu^{(2)}-\rho\Sigma^{-1/2}{}_{22}\Sigma^{-1/2}{}_{11}(X^{(1)}-\mu^{(1)})\right]\right]'\Sigma^{-1}{}_{11}\left[X^{(1)}-\mu^{(1)}\right]'\right]}}{(2\pi)^{q/2} (2\pi)^{p-q/2} |\Sigma_{11}|^{\frac{1}{2}} |\Sigma_{22}|^{\frac{1}{2}} \left(1-\rho^2\right)^{\frac{1}{2}}} e^{\frac{-1}{2}\left[X^{(1)}-\mu^{(1)}\right]\Sigma_{11}{}^{-1}\left[X^{(1)}-\mu^{(1)}\right]'}$$

$$\left(X_1 - \mu_1\right) \rho^2\Sigma^{-1}{}_{11}((X_1 - \mu_1)'$$

$$\frac{f(X^{(1)}, X^{(2)})}{f(X^{(1)})} = \frac{\dfrac{e^{-\frac{1}{2(1-\rho^2)\Sigma_{22}}\left[\left[X^{(2)}-\mu^{(2)}-\rho\Sigma^{-1/2}{}_{22}\Sigma^{-1/2}{}_{11}(X^{(1)}-\mu^{(1)})\right]\left[\left[X^{(2)}-\mu^{(2)}-\rho\Sigma^{-1/2}{}_{22}\Sigma^{-1/2}{}_{11}(X^{(1)}-\mu^{(1)})\right]\right]'\Sigma^{-1}{}_{11}\left[X^{(1)}-\mu^{(1)}\right]'\right]}}{(2\pi)^{q/2} (2\pi)^{p-q/2} |\Sigma_{11}|^{\frac{1}{2}} |\Sigma_{22}|^{\frac{1}{2}} \left(1-\rho^2\right)^{\frac{1}{2}}} e^{\frac{-1}{2}\left[X^{(1)}-\mu^{(1)}\right]\Sigma_{11}{}^{-1}\left[X^{(1)}-\mu^{(1)}\right]'}}{\dfrac{e^{-\frac{1}{2}\left[X^{(1)}-\mu^{(1)}\right]\Sigma^{-1}{}_{11}\left[X^{(1)}-\mu^{(1)}\right]'}}{(2\pi)^{q/2} |\Sigma_{11}|^{\frac{1}{2}}}}$$

$$f(X^{(2)}/X^{(1)}) = \frac{\dfrac{e^{-\frac{1}{2(1-\rho^2)\Sigma_{22}}\left[\left[X^{(2)}-\mu^{(2)}-\rho\Sigma^{-1/2}{}_{22}\Sigma^{-1/2}{}_{11}(X^{(1)}-\mu^{(1)})\right]\left[\left[X^{(2)}-\mu^{(2)}-\rho\Sigma^{-1/2}{}_{22}\Sigma^{-1/2}{}_{11}(X^{(1)}-\mu^{(1)})\right]\right]'^{-1}\right]}}{(2\pi)^{p-q/2} |\Sigma_{22}|^{\frac{1}{2}} \left(1-\rho^2\right)^{\frac{1}{2}}}}{}$$

The marginal distributions of X_1 and X_2 are also normal with mean vector μ_i and covariance matrix (Σ_{ii} , i=1, 2), respectively.
Then
of order qx1, (p-q)x1 i.e. with a similar partition of μ and Σ

$$\mu = \begin{bmatrix} \mu^{(1)} \\ \mu^{(2)} \end{bmatrix} \text{and } \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \text{ of order pxp where}$$

The Joint Probability Distribution of f(X) is

$$f(X^{(1)}, X^{(2)}) = \frac{e^{-\frac{1}{2}\left[X^{(1)}-\mu^{(1)} X^{(2)}-\mu^{(2)}\right]\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}^{-1}\left[X^{(1)}-\mu^{(1)} X^{(2)}-\mu^{(2)}\right]}}{(2\pi)^{P/2} |\Sigma_{11}|^{\frac{1}{2}} |\Sigma_{22}|^{\frac{1}{2}} \left(1-\rho^2\right)^{\frac{1}{2}}}$$

Where $\Sigma^{-1} = \begin{pmatrix} \Sigma_{22} & -\Sigma_{12} \\ -\Sigma_{21} & \Sigma_{11} \end{pmatrix} \Big/ \Sigma_{22}\Sigma_{11} - \Sigma_{12}\Sigma_{21} \;=\; \Sigma_{22}\Sigma_{11} - \Sigma_{12}\Sigma_{21} = \Sigma_{22}\Sigma_{11}((1-\rho^2)^{\frac{1}{2}}$ *and marginal probability of* $X^{(1)}$*is*

$$f(X^{1)}) = \int f(X^{(1)}, X^{(2)})dX^{(2)} == \frac{e^{-\frac{1}{2}\left[X^{(1)}-\mu^{(1)}\right]\Sigma^{-1}{}_{11}\left[X^{(1)}-\mu^{(1)}\right]'}}{(2\pi)^{q/2}\,|\Sigma_{11}|^{\frac{1}{2}}} \quad \textit{where}$$

$$f(X^{(1)}, X^{(2)}) = \frac{e^{-\frac{1}{2\Sigma_{11}\Sigma_{11}(1-\rho^2)}\left[\left[X^{(1)}-\mu^{(1)}\right]\Sigma_{22}\left[X^{(1)}-\mu^{(1)}\right]'\right]-2\left[\left[X^{(1)}-\mu^{(1)}\right]\Sigma_{12}\left[X^{(2)}-\mu^{(2)}\right]'\right]+\left[\left[X^{(2)}-\mu^{(2)}\right]\Sigma_{11}\left[X^{(2)}-\mu^{(2)}\right]'\right]}}{(2\pi)^{q/2}(2\pi)^{p-q/2}\,|\Sigma_{11}|^{\frac{1}{2}}\,|\Sigma_{22}|^{\frac{1}{2}}\left(1-\rho^2\right)^{\frac{1}{2}}} \quad \textit{by simplifying}$$

we get

$$f(X^{(1)}, X^{(2)}) = \frac{e^{-\frac{1}{2(1-\rho^2)}\left[\left[X^{(1)}-\mu^{(1)}\right]\Sigma^{-1}{}_{11}\left[X^{(1)}-\mu^{(1)}\right]'\right]-2\left[\left[X^{(1)}-\mu^{(1)}\right]\Sigma_{12}/\Sigma_{11}\Sigma_{22}\left[X^{(2)}-\mu^{(2)}\right]'\right]+\left[\left[X^{(2)}-\mu^{(2)}\right]\Sigma^{-1}{}_{22}\left[X^{(2)}-\mu^{(2)}\right]'\right]}}{(2\pi)^{q/2}(2\pi)^{p-q/2}\,|\Sigma_{11}|^{\frac{1}{2}}\,|\Sigma_{22}|^{\frac{1}{2}}\left(1-\rho^2\right)^{\frac{1}{2}}}$$

for completing square we add and subtract $(X_1 - \mu_1)\,\rho^2\Sigma^{-1}{}_{11}((X_1 - \mu_1)'$

$$f(X^{(1)}, X^{(2)}) = \frac{e^{-\frac{1}{2(1-\rho^2)}\left[\left[X^{(1)}-\mu^{(1)}\right]\Sigma^{-1}{}_{11}\left[X^{(1)}-\mu^{(1)}\right]'\right]-2\left[\left[X^{(1)}-\mu^{(1)}\right]\Sigma_{12}/\Sigma_{11}\Sigma_{22}\left[X^{(2)}-\mu^{(2)}\right]'\right]+\left[\left[X^{(2)}-\mu^{(2)}\right]\Sigma^{-1}{}_{22}\left[X^{(2)}-\mu^{(2)}\right]'\right]+\left[(X_1-\mu_1)\,\rho^2\Sigma^{-1}{}_{11}((X_1-\mu_1)'-(X_1-\mu_1)\,\rho^2\Sigma^{-1}{}_{11}((X_1-\mu_1)'\right]}}{(2\pi)^{q/2}(2\pi)^{p-q/2}\,|\Sigma_{11}|^{\frac{1}{2}}\,|\Sigma_{22}|^{\frac{1}{2}}\left(1-\rho^2\right)^{\frac{1}{2}}}$$

By simplifying, we get

$$f(X^{(1)}, X^{(2)}) = \frac{e^{-\frac{1}{2(1-\rho^2)\Sigma_{22}}\left[\left[X^{(2)}-\mu^{(2)}-\rho\Sigma^{-1/2}{}_{22}\Sigma^{-1/2}{}_{11}(X^{(1)}-\mu^{(1)})\right]\left[\left[X^{(2)}-\mu^{(2)}-\rho\Sigma^{-1/2}{}_{22}\Sigma^{-1/2}{}_{11}(X^{(1)}-\mu^{(1)})\right]\right]'\Sigma^{-1}{}_{11}\left[X^{(1)}-\mu^{(1)}\right]'\right]}}{(2\pi)^{q/2}(2\pi)^{p-q/2}\,|\Sigma_{11}|^{\frac{1}{2}}\,|\Sigma_{22}|^{\frac{1}{2}}\left(1-\rho^2\right)^{\frac{1}{2}}}\, e^{\frac{-1}{2}\left[X^{(1)}-\mu^{(1)}\right]\Sigma_{11}{}^{-1}\left[X^{(1)}-\mu^{(1)}\right]'}$$

As we know that $f(x_2/x_1) = \dfrac{f(X^{(1)}, X^{(2)})}{f(X^{(1)})} =$

$$\frac{\dfrac{e^{-\frac{1}{2(1-\rho^2)\Sigma_{22}}\left[\left[X^{(2)}-\mu^{(2)}-\rho\Sigma^{-1/2}{}_{22}\Sigma^{-1/2}{}_{11}(X^{(1)}-\mu^{(1)})\right]\left[\left[X^{(2)}-\mu^{(2)}-\rho\Sigma^{-1/2}{}_{22}\Sigma^{-1/2}{}_{11}(X^{(1)}-\mu^{(1)})\right]\right]'\Sigma^{-1}{}_{11}\left[X^{(1)}-\mu^{(1)}\right]'\right]}}{(2\pi)^{q/2}(2\pi)^{p-q/2}\,|\Sigma_{11}|^{\frac{1}{2}}\,|\Sigma_{22}|^{\frac{1}{2}}\left(1-\rho^2\right)^{\frac{1}{2}}}\, e^{\frac{-1}{2}\left[X^{(1)}-\mu^{(1)}\right]\Sigma_{11}{}^{-1}\left[X^{(1)}-\mu^{(1)}\right]'}}{\dfrac{e^{-\frac{1}{2}\left[X^{(1)}-\mu^{(1)}\right]\Sigma^{-1}{}_{11}\left[X^{(1)}-\mu^{(1)}\right]'}}{(2\pi)^{q/2}\,|\Sigma_{11}|^{\frac{1}{2}}}}$$

$$f(X^{(2)}/X^{(1)}) = \frac{e^{-\frac{1}{2(1-\rho^2)\Sigma_{22}}\left[\left[X^{(2)}-\mu^{(2)}-\rho\Sigma^{-1/2}{}_{22}\Sigma^{-1/2}{}_{11}(X^{(1)}-\mu^{(1)})\right]\left[\left[X^{(2)}-\mu^{(2)}-\rho\Sigma^{-1/2}{}_{22}\Sigma^{-1/2}{}_{11}(X^{(1)}-\mu^{(1)})\right]\right]'^{-1}\right]}}{(2\pi)^{p-q/2}\,|\Sigma_{22}|^{\frac{1}{2}}\left(1-\rho^2\right)^{\frac{1}{2}}}$$

$$\frac{X_2}{X_1} \sim N_{p-q}\left(\mu_2 + \rho\Sigma_{22}{}^{-\frac{1}{2}}\Sigma_{11}{}^{-\frac{1}{2}}(X_1-\mu_1), (1-\rho^2)\Sigma_{22}\right)$$

$$\frac{X_2}{X_1} \sim N_{p-q}\left(\mu_2 + \frac{\Sigma_{12}\Sigma_{22}{}^{-\frac{1}{2}}\Sigma_{11}{}^{-\frac{1}{2}}}{\Sigma_{11}{}^{\frac{1}{2}}\Sigma_{22}{}^{\frac{1}{2}}}(X_1-\mu_1), (1-\frac{\Sigma_{12}\Sigma_{21}}{\Sigma_{11}\Sigma_{22}})\Sigma_{22}\right)$$

$$\frac{X_2}{X_1} \sim N_{p-q}\left[\left(\mu_2 + \Sigma_{12}\Sigma_{11}{}^{-1}(X_1 - \mu_1\right), \left(\Sigma_{22} - \Sigma_{12}\Sigma_{11}{}^{-1}\Sigma_{21}\right)\right]$$

Which fulfill the requirement of multivariate normal distribution.

These result can be verified for if X_1 and X_2 are independent i.e. $\rho=0$ ($\Sigma_{12}=0$) and X_2 / X_1 are independent.

$X_2/X_1=x_1 \sim N_1(\mu_2 , \Sigma_{22})$

This result also verify if X is bidimensional random variable then the resulting distribution is bivariate conditional normal distribution i.e. if q=1 and p=2 then $\Sigma_{11}=\sigma_1{}^2$, $\Sigma_{12}= \sigma_{12}$ and $\Sigma_{22}=\sigma_2{}^2$.

$X_2/X_1=x_1 \sim N_1[(\mu_2 +\sigma_{12} (X_1 - \mu_1)/ \sigma_1{}^2, \sigma_2{}^2 - (\sigma_{12}\, \sigma_{21})/\sigma_1{}^2]$ and again if X_2 is independent of X_1, then $\sigma_{12}=\sigma_{21} = 0$ and the result fulfill univariate normal distribution

$X_2/X_1=x_1 \sim N_1[\mu_2, \sigma_2{}^2]$

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